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k-BITRANSITIVE AND COMPOUND OPERATORS ON BANACH SPACES

In this this paper, we introduce new classes of operators in complex Banach spaces, which we call k-bitransitive operators and compound operators to study the direct sum of diskcyclic operators. We create a set of sufficient conditions for an operator to be k-bitransitive or compound. We give a relation between topologically mixing operators and compound operators. Also, we extend the Godefroy-Shapiro Criterion for topologically mixing operators to compound operators.

Key words and phrases: hypercyclic operators, diskcyclic operators, weakly mixing operators, direct sums.

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INTRODUCTION

A bounded linear operator T on a separable Banach space X is hypercyclic if there is a vector $x \in X$ such that $Orb(T, x) = \{T^n x : n \geq 0\}$ is dense in X, such a vector x is called hypercyclic for T. Similarly, an operator T is called diskcyclic if there is a vector $x \in X$ such that the disk orbit $\mathbb{D}Orb(T, x) = \{\alpha T^n x : \alpha \in \mathbb{C}, |\alpha| \leq 1, n \in \mathbb{N}\}$ is dense in X, such a vector x is called diskcyclic for T. In Banach spaces, hypercyclic (or diskcyclic) operators are identical to topological transitive (or disk transitive, respectively) [3, 4].

Definition 1. *A bounded linear operator $T : X \to X$ is called*

1. *topological transitive, if for any two non empty open sets U and V, there exists a positive integer n such that $T^n U \cap V \neq \varnothing$;*
2. *disk transitive, if for any two non empty open sets U and V, there exist a positive integer n and $\alpha \in \mathbb{C}, 0 < |\alpha| \leq 1$, such that $T^n \alpha U \cap V \neq \varnothing$.*

For more information on hypercyclic and diskcyclic operators the reader may refer to [2, 3, 4, 11].

A sufficient condition for hypercyclicity, the well known Hypercyclicity Criterion, independently discovered by Kitai [13] and Gethner and Shapiro [9]. Latter on, Godefroy and Shapiro [10] created another hypercyclic criterion which is called Godefroy-Shapiro Criterion, that is a set of sufficient condition in terms of the eigenvalues of an operator to be hypercyclic.

In 1982, Kitai [13] showed that if $T_1 \oplus T_2$ is hypercyclic, then T_1 and T_2 are hypercyclic. However, for the converse, Salas constructed an operator T such that both it and its adjoint T^* are hypercyclic, and so that their direct sum $T \oplus T^*$ is not. Moreover, Herrero asked in [12]

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whether $T \oplus T$ is hypercyclic whenever T is. De la Rosa and Read [7] showed that the Herrero's question is not true by giving a hypercyclic operator T such that $T \oplus T$ is not. On the other hand, if T satisfies hypercyclic criterion, then $T \oplus T$ is hypercyclic [4]. In 1999, Bés and Peris [5] proved that the converse is also true; that is, if $T \oplus T$ is hypercyclic, then T satisfies hypercyclic criterion.

For diskcyclic operators, Zeana proved that if the direct sum of k operators is diskcyclic then every operator is diskcyclic [14]. However, the converse is unknown. Particularly, we have the following question:

Question 1. *If there are k diskcyclic operators, what about their direct sum?*

The main purpose of this paper is to give a partial answer to this question by defining and studying a new class of operators, namely k-bitransitive operators. We determine conditions that ensure a linear operator to be k- bitransitive which is called k-bitransitive criterion. We use this criterion to show that in some cases the direct sum of k diskcyclic operators is k-bitransitive. Then, we define compound operators as a general form of mixing operators [6] to show that under certain conditions the direct sum of k diskcyclic operators is k-bitransitive. Then, we studied some properties of compound operators. In particular, we give some sufficient conditions for an operator to be compound which is refer to compound criterion. We use this criterion to show that not every compound operator is mixing. Finally, we extend Godefroy-Shapiro Criterion [1, Theorem 1.3] for mixing operators to compound operators. In particular, a special case of Theorem 3 is when $p = 1$ which is Godefroy-Shapiro Criterion.

1 MAIN RESULTS

In this this paper, all Banach spaces are separable over the field $\mathbb{C}$ of complex numbers. We denote by $\mathbb{D}$ the closed unit disk in $\mathbb{C}$, by $\mathbb{N}$ the set of all positive integers and by $\mathcal{B}(X)$ the set of all bounded linear operators on a Banach space X.

Let k be a positive integer and $T_i \in \mathcal{B}(X)$ for all $1 \leq i \leq k$, and let $T = \bigoplus_{i=1}^{k} T_i : \bigoplus_{i=1}^{k} X \to \bigoplus_{i=1}^{k} X$ then we call each operator T_i a component of T.

Definition 2. *An operator T is called k-bitransitive if there exist $T_1, T_2, \cdots T_k \in \mathcal{B}(X)$ such that $T = \bigoplus_{i=1}^{k} T_i$ and for any $2k$-tuples $U_1, \cdots, U_k, V_1, \cdots, V_k \subset X$ of nonempty open sets, there exist some $n \in \mathbb{N}$ and $\alpha_1, \cdots, \alpha_k \in \mathbb{D}\setminus\{0\}$ such that*

$$T^n\Big(\bigoplus_{i=1}^{k} \alpha_i U_i\Big) \cap \Big(\bigoplus_{i=1}^{k} V_i\Big) \neq \varnothing.$$

It is clear from Definition 2 above that 1-bitransitive is identical to disk transitive which in turn identical to diskcyclic.

To simplify Definition 2 above, we provide the following definition.

Definition 3. *Let $r \in \mathbb{N}$ be fixed. For each $1 \leq i \leq r$, let T_i be a bounded linear operator on a Banach space X, and A_i, B_i be nonempty subsets of X. Assume that $T = \bigoplus_{i=1}^{r} T_i$, $A = \bigoplus_{i=1}^{r} A_i$ and $B = \bigoplus_{i=1}^{r} B_i$. The junction set from the set A to the set B under T is defined as $J_T(A, B) = \{(n, \alpha_1, \cdots, \alpha_r) \in \mathbb{N} \times \mathbb{D}^r \setminus \{(0, \cdots, 0)\} : T^n(\bigoplus_{i=1}^{r} \alpha_i A_i) \cap (\bigoplus_{i=1}^{r} B_i) \neq \varnothing\}$.*

In Definition 3 above, we sometimes write $J_T(A,B)$ as $J(A,B)$. The next proposition gives an equivalent definition to k-bitransitivity in terms of junction set.

Proposition 1. *Let $T=\bigoplus_{i=1}^k T_i$. Then T is k-bitransitive if and only if for each $1\le i\le k$ and any nonempty open sets U_i and V_i, there exist $\alpha_i\in\mathbb{D}\setminus\{0\}$ and $n\in\mathbb{N}$ such that*

$$(n,\alpha_i)\in J_{T_i}(U_i,V_i).$$

The proof follows immediately by applying the definition of junction sets to Definition 2.

To answer Question 1, we need the following proposition, which gives a set of sufficient conditions for k-bitransitivity.

Proposition 2 (k-bitransitive criterion)**.** *Let $T=\bigoplus_{i=1}^k T_i$, and let $\{n_r\}_{r\in\mathbb{N}}$ be an increasing sequence of positive integers. Suppose that for each $1\le i\le k$ there exist a sequence $\left\{\lambda_{n_r}^{(i)}\right\}\subset\mathbb{D}\setminus\{0\}$, dense sets $X_i,Y_i\subset X$, and a map $S_i:Y_i\to X$ such that for all $(x_1,\cdots,x_k)\in\bigoplus_{i=1}^k X_i$ and $(y_1,\cdots,y_k)\in\bigoplus_{i=1}^k Y_i$, we have*

(i) $\left\|\bigoplus_{i=1}^k\lambda_{n_r}^{(i)}T_i^{n_r}(x_1,\cdots,x_k)\right\|\to 0,$

(ii) $\left\|\bigoplus_{i=1}^k\frac{1}{\lambda_{n_r}^{(i)}}S_i^{n_r}(y_1,\cdots,y_k)\right\|\to 0,$

(iii) $\bigoplus_{i=1}^k T_i^{n_r}S_i^{n_r}(y_1,\cdots,y_k)\to(y_1,\cdots,y_k)$

as $r\to\infty$. Then T is k-bitransitive.

Proof. Let U_i,V_i be open subsets of X for all $1\le i\le k$, then $\bigoplus_{i=1}^k U_i$ and $\bigoplus_{i=1}^k V_i$ are open in $\bigoplus_{i=1}^k X$. Also $\bigoplus_{i=1}^k X_i$ and $\bigoplus_{i=1}^k Y_i$ are dense in $\bigoplus_{i=1}^k X$. Let

$$(x_1,\cdots,x_k)\in\bigoplus_{i=1}^k U_i\cap\bigoplus_{i=1}^k X_i$$

and

$$(y_1,\cdots,y_k)\in\bigoplus_{i=1}^k V_i\cap\bigoplus_{i=1}^k Y_i.$$

Suppose that $z_r=(x_1,\cdots,x_k)+\bigoplus_{i=1}^k\frac{1}{\lambda_{n_r}^{(i)}}S_i^{n_r}(y_1,\cdots,y_k)$. By (ii), as $r\to\infty$ we have

$$\|z_r-(x_1,\cdots,x_k)\|=\left\|\bigoplus_{i=1}^k\frac{1}{\lambda_{n_r}^{(i)}}S_i^{n_r}(y_1,\cdots,y_k)\right\|\to 0. \tag{1}$$

Since

$$\bigoplus_{i=1}^k\lambda_{n_r}^{(i)}T_i^{n_r}(z_r)=\bigoplus_{i=1}^k\lambda_{n_r}^{(i)}T_i^{n_r}\left((x_1,\cdots,x_k)+\bigoplus_{i=1}^k\frac{1}{\lambda_{n_r}^{(i)}}S_i^{n_r}(y_1,\cdots,y_k)\right),$$

then by (i) and (iii), we have

$$\left\|\bigoplus_{i=1}^k\lambda_{n_r}^{(i)}T_i^{n_r}(z_r)-(y_1,\cdots,y_k)\right\|=\left\|\bigoplus_{i=1}^k\lambda_{n_r}^{(i)}T_i^{n_r}(x_1,\cdots,x_k)\right\|\to 0, \tag{2}$$

as $r \to \infty$. From Equations (1) and (2), there exists $N \in \mathbb{N}$ such that $z_N \in \bigoplus_{i=1}^{k} U_i$ and $\bigoplus_{i=1}^{k} \lambda_{n_r}^{(i)} T_i^{n_r}(z_N) \in \bigoplus_{i=1}^{k} V_i$, that is,

$$\bigoplus_{i=1}^{k} \lambda_{n_r}^{(i)} T_i^{n_r} \left(\bigoplus_{i=1}^{k} U_i \right) \cap \bigoplus_{i=1}^{k} V_i \neq \varnothing \text{ for all } r \geq N,$$

which is equivalent to

$$(T_1 \oplus \cdots \oplus T_k)^{n_r} (\lambda_{n_r}^{(1)} U_1 \oplus \cdots \oplus \lambda_{n_r}^{(k)} U_k) \cap (V_1 \oplus \cdots \oplus V_k) \neq \varnothing \quad \text{for all} \quad r \geq N.$$

That is,

$$(n_r, \lambda_{n_r}^{(i)}) \in J_{T_i}(U_i, V_i) \quad \text{for all} \quad 1 \leq i \leq k.$$

By Proposition 1, T is k-bitransitive. □

The following theorem gives a partial answer to Question 1.

Theorem 1. *If k operators satisfy diskcyclic criterion for the same increasing sequence of positive integers $\{n_r\}_{r \in \mathbb{N}}$, then their direct sum is a k-bitransitive operator.*

Proof. Let $T_i \in \mathcal{B}(X)$ satisfies diskcyclic criterion with respect to the same increasing sequence of positive integers $\{n_r\}_{r \in \mathbb{N}}$ for all $1 \leq i \leq k$ [2, Theorem 2.6]. Then for each $1 \leq i \leq k$, there exists a sequence $\left\{ \lambda_{n_r}^{(i)} \right\}_{r \in \mathbb{N}} \in \mathbb{D} \backslash \{0\}$, two dense sets D_i, D_i' and a map S_i such that for all $x_i \in D_i$ and $y_i \in D_i'$, we have

$$\left\| \lambda_{n_r}^{(i)} T_i^{n_r} x_i \right\| \to 0, \tag{3}$$

$$\left\| \frac{1}{\lambda_{n_r}^{(i)}} S_i^{n_r} y_i \right\| \to 0, \tag{4}$$

$$T_i^{n_r} S_i^{n_r} y_i \to y_i \tag{5}$$

as $r \to \infty$. By Equation (3), we get $\sum_{i=1}^{k} \left\| \lambda_{n_r}^{(i)} T_i^{n_r} x_i \right\| \to 0$; that is,

$$\left\| \bigoplus_{i=1}^{k} \lambda_{n_r}^{(i)} T_i^{n_r} (x_1, \cdots, x_k) \right\| \to 0 \tag{6}$$

as $r \to \infty$. Also by Equation (4), we get $\sum_{i=1}^{k} \left\| \frac{1}{\lambda_{n_r}^{(i)}} S_i^{n_r} y_i \right\| \to 0$; that is,

$$\left\| \bigoplus_{i=1}^{k} \frac{1}{\lambda_{n_r}^{(i)}} S_i^{n_r} (y_1, \cdots, y_k) \right\| \to 0 \tag{7}$$

as $r \to \infty$. Finally, by Equation (5), we get $(T_1^{n_r} S_1^{n_r} y_1, \cdots, T_k^{n_r} S_k^{n_r} y_k) \to (y_1, \cdots, y_k)$; that is,

$$\bigoplus_{i=1}^{k} T_i^{n_r} S_i^{n_r} (y_1, \cdots, y_k) \to (y_1, \cdots, y_k) \tag{8}$$

as $r \to \infty$. By Proposition 2, we get $T = \bigoplus_{i=1}^{k} T_i$ is k-bitransitive. □

To give another partial answer to Question 1, we define another class of operators which is called compound operators.

Definition 4. *Let $T \in \mathcal{B}(X)$. Then T is called compound if for any nonempty open sets U and V, there exist some $N \in \mathbb{N}$ and a sequence $\{\alpha_n\}_{n\in\mathbb{N}} \in \mathbb{D}\setminus\{0\}$ such that*

$$T^n(\alpha_n U) \cap V \neq \varnothing$$

for all $n \geq N$.

The following theorem gives another partial answer to Question 1. First, we need the following lemma.

Lemma 1. *If $T \in \mathcal{B}(X)$ is diskcyclic, then there exist an increasing sequence of positive integers $\{m_j\}_{j\in\mathbb{N}}$ and a sequence $\left\{\gamma_{m_j}\right\} \subset \mathbb{D}\setminus\{0\}$ such that $\left\{(m_j, \gamma_{m_j}) : j \in \mathbb{N}\right\} \subseteq J(U,V)$ for any two nonempty open sets $U, V \subset X$.*

Proof. Let $(n_1, \alpha_1) \in J(U,V)$, and let $W = U \cap T^{-n_1}\frac{1}{\alpha_1}V$. Since W is open set, then there exist $n_2 \in \mathbb{N}$ and $\alpha_2 \in \mathbb{D}$ such that $(n_2, \alpha_2) \in J(W,W)$, that is,

$$T^{n_2}\alpha_2 U \cap T^{n_2-n_1}\frac{\alpha_2}{\alpha_1}V \cap U \cap T^{-n_1}\frac{1}{\alpha_1}V \neq \varnothing.$$

It follows that

$$T^{n_2}\alpha_1\alpha_2 U \cap T^{-n_1}V \neq \varnothing.$$

Now, we have

$$T^{n_1+n_2}\alpha_1\alpha_2 U \cap V = T^{n_1}(T^{n_2}\alpha_1\alpha_2 U \cap T^{-n_1}V) \neq \varnothing,$$

that is,

$$(n_1+n_2, \alpha_1\alpha_2) \in J(U,V).$$

By continuing the same process, we get $(\sum_{i=1}^{j} n_i, \prod_{i=1}^{j}\alpha_i) \in J(U,V)$ for any $j, n_i \in \mathbb{N}$ and $\alpha_i \in \mathbb{D}$. Let $m_j = \sum_{i=1}^{j} n_i$ and $\gamma_{m_j} = \prod_{i=1}^{j}\alpha_i$ for all $j \in \mathbb{N}$, then

$$\left\{(m_j, \gamma_{m_j}) : j \in \mathbb{N}\right\} \subseteq J(U,V).$$

□

Theorem 2. *Let $T = \bigoplus_{i=1}^{k} T_i$. If every component of T is disk transitive and at least $(k-1)$ of them are compound, then T is k-bitransitive.*

Proof. Without loss of generality, we suppose that $k = 2$ and T_1 is compound. Let U_1, U_2, V_1, V_2 be nonempty open sets, by hypothesis there exist $N_1, N_2 \in \mathbb{N}$, $\alpha_1 \in \mathbb{D}\setminus\{0\}$ and a sequence $\{\beta_n : n \in \mathbb{N}\} \subset \mathbb{D}\setminus\{0\}$ such that

$$T_2^{N_1}\alpha_1 U_1 \cap U_2 \neq \varnothing \text{ and } T_1^n\beta_n V_1 \cap V_2 \neq \varnothing$$

for all $n \geq N_2$. By Lemma 1, there exist $N \in \mathbb{N}$ and $\alpha \in \mathbb{D}\setminus\{0\}$ such that

$$T_2^N\alpha U_1 \cap U_2 \neq \varnothing \text{ and } T_1^N\beta_N V_1 \cap V_2 \neq \varnothing.$$

It follows that

$$(T_1 \oplus T_2)^N(\alpha U_1 \oplus \beta_N V_1) \cap (U_2 \oplus V_2) \neq \varnothing.$$

Hence T is 2-bitransitive. □

It is clear that every compound operator is diskcyclic. A special case of compound operator is when $\alpha_n = 1$ for all $n \geq N$, and it is called mixing operators (see [6]). Therefore every mixing operator is compound. However, not every compound operator is mixing as shown in the following example. First, we need the following proposition which give sufficient conditions for an operator to be compound.

Proposition 3. *Let $T \in \mathcal{B}(X)$, suppose that there exist a sequence $\{\lambda_n\}_{n\in\mathbb{N}} \subset \mathbb{D}\backslash\{0\}$, two dense sets D_1 and D_2 in X, and a sequence of maps $S_n : D_2 \to X$ and such that*

(i) $\|\lambda_n T^n x\| \to 0$ *for any* $x \in D_1$,

(ii) $\left\|\frac{1}{\lambda_n} S_n y\right\| \to 0$ *for any* $y \in D_2$,

(iii) $T^n S_n y \to y$ *for any* $y \in D_2$

as $n \to \infty$. Then T is compound and it is called compound with respect to the sequence $\{\lambda_n\}$.

Proof. Suppose that U and V be two nonempty open sets. Let $x \in U \cap D_1$ and $y \in V \cap D_2$. Let N be a large positive integer such that $z = x + \frac{1}{\lambda_N} S_N y$, then by hypothesis we get

$$\|z - x\| = \left\|\frac{1}{\lambda_N} S_N y\right\| \to 0 \quad \text{and} \quad \left\|\lambda_N T^N z - y\right\| = \left\|\lambda_N T^N x\right\| \to 0.$$

Thus $T^n \lambda_n U \cap V \neq \varnothing$ for all $n \geq N$. So, T is compound. $\square$

The following proposition gives another criterion for compound operators without the need of the scalar sequence.

Proposition 4. *Let $T \in \mathcal{B}(X)$. If there exist two dense sets D_1 and D_2 in X, and a sequence of maps $S_n : D_2 \to X$ such that*

(i) $\|T^n x\| \, \|S_n y\| \to 0$ *for all* $x \in D_1$ *and* $y \in D_2$,

(ii) $\|S_n y\| \to 0$ *for all* $y \in D_2$,

(iii) $T^n S_n y \to y$ *for all* $y \in D_2$

as $n \to \infty$. Then T is compound.

The proof of Proposition 4 is followed by showing that both compound criteria in Propositions 3 and 4 are equivalent by using the same lines in [2, Proposition 2.8].

Example 1. *Let T be a bilateral forward weighted shift on $\ell_p, 1 \leq p < \infty$, with the weight sequence*

$$w_n = \begin{cases} R_1, & \text{if } n \geq 0, \\ R_2, & \text{if } n < 0, \end{cases}$$

where $R_1, R_2 \in \mathbb{R}^+; 1 < R_1 < R_2$. Then T is compound not mixing.

Proof. By applying [3, Corollary 2.15] and taking $\{n_r\}_{r\in\mathbb{N}} = \{n\}_{n\in\mathbb{N}}$, we get

$$\lim_{n\to\infty}\prod_{k=1}^{n}\frac{1}{w_{-k}} = \lim_{n\to\infty}\prod_{k=1}^{n}\frac{1}{R_2} = \lim_{n\to\infty}\frac{1}{R_2^n} = 0$$

and

$$\lim_{n\to\infty}\left(\prod_{k=1}^{n} w_k\right)\left(\prod_{k=1}^{n}\frac{1}{w_{-k}}\right) = \lim_{n\to\infty}\left(\prod_{k=1}^{n} R_1\right)\left(\prod_{k=1}^{n}\frac{1}{R_2}\right) = \lim_{n\to\infty}(R_1^n)(\frac{1}{R_2^n}) = 0.$$

It follows that T satisfies diskcyclic criterion with respect to the sequence $\{n\}_{n\in\mathbb{N}}$. Then, by Proposition 4, T is compound. Now, since

$$\lim_{n\to\infty}\left(\prod_{k=1}^{n} w_k\right) = \infty,$$

then by [8, Theorem 3.2] T is not topological transitive and so not mixing. □

The following theorem extends the Godefroy-Shapiro Criterion [1, Theorem 1.3] for mixing operators to compound operators.

Theorem 3. *Let $T \in \mathcal{B}(X)$. If there exists a positive integer $p \geq 1$ such that*

$$A = \text{span}\{x \in X : Tx = \alpha x \text{ for some } \alpha \in \mathbb{C}; |\alpha| < p\}$$

and

$$B = \text{span}\{y \in X : Ty = \lambda y \text{ for some } \lambda \in \mathbb{C}; |\lambda| > p\}$$

are dense in X, then T is compound.

Proof. Let U and V be nonempty open sets in X. Since A and B are dense, then there exist $x \in A \cap U$ and $y \in B \cap V$. Then $x = \sum_{i=1}^{k} a_i x_i$ and $y = \sum_{i=1}^{k} b_i y_i$, where $a_i, b_i \in \mathbb{C}$ for all $1 \leq i \leq k$. Also, $Tx_i = \alpha_i x_i$ and $Ty_i = \lambda_i y_i$, where $|\alpha_i| < p$ and $|\lambda_i| > p$ for all $1 \leq i \leq k$. Let $c \in \mathbb{C}$ be a scalar such that $p \leq |c| < |\lambda_i|$ for all $1 \leq i \leq k$, and let

$$z_n = \sum_{i=1}^{k} b_i(\frac{c}{\lambda_i})^n y_i \quad \text{for all} \quad n \geq 0.$$

Then

$$\frac{1}{c^n}T^n x = \sum_{i=1}^{k} a_i(\frac{\alpha_i}{c})^n x_i \to 0 \quad \text{and} \quad z_n \to 0 \quad \text{as} \quad n \to \infty.$$

Also, $\frac{1}{c^n}T^n z_n = y$ for all $n \geq 0$. It follows that there is a positive integer k such that for all $n \geq k$, we have

$$x + z_n \in U \quad \text{and} \quad \frac{1}{c^n}T^n(x+z_n) = \frac{1}{c^n}T^n x + \frac{1}{c^n}T^n z_n \in V \quad \text{for all} \quad n \geq k.$$

Therefore, $\frac{1}{c^n}T^n U \cap V \neq \varnothing$ for all $n \geq k$. It follows that T is compound. □

Note that in the above theorem, if $p = 1$, then it will be a Godefroy-Shapiro criterion for mixing operators [1, Theorem 1.3].

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В цій статті ми вводимо нові класи операторів у комплексних банахових просторах, які ми називаємо k-бітранзитивними операторами і операторами сполучення для вивчення прямих сум дисциклічних операторів. Запропоновано набір достатніх умов для того, щоб оператор був k-бітранзитивним чи оператором сполучення. Також встановлено зв'язок між операторами топологічного змішування і операторами сполучення. Також розширено критерій Ґодефруа-Шапіро для операторів топологічного змішування на випадок операторів сполучення.

Ключові слова і фрази: гіперциклічні оператори, дисциклічні оператори, оператори слабкого змішування, прямі суми.

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PERIODIC WORDS CONNECTED WITH THE FIBONACCI WORDS

In this paper we introduce two families of periodic words (FLP-words of type 1 and FLP-words of type 2), that are connected with the Fibonacci words. The properties of the families are investigated.

Key words and phrases: Fibonacci number, Fibonacci word.

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INTRODUCTION

The Fibonacci numbers F_n are defined by the recurrence relation $F_n = F_{n-1} + F_{n-2}$, for all integer $n > 1$, and with initial values $F_0 = 0$ and $F_1 = 1$. These numbers and their generalizations have interesting properties. Different kinds of the Fibonacci sequence and their properties have been presented in the literature, see, e.g., [1, 6, 11].

Many properties of Fibonacci numbers require the full ring structure of the integers. However, generalizations to the ring $\mathbb{Z}_m$ and groups have been considered, see, e.g., [3, 5, 14, 16]. The sequence $F_n \pmod{m}$ is periodic and it repeats by returning to its starting values because there are only a finite number m^2 of pairs of possible terms. Therefore, we obtain the repeating of all the sequence elements.

In analogy to the definition of the Fibonacci numbers, one defines the Fibonacci finite words as the concatenation of the two previous terms $f_n = f_{n-1}f_{n-2}$, $n > 1$, with initial values $f_0 = 1$ and $f_1 = 0$ and defines the infinite Fibonacci word f, $f = \lim f_n$ [2]. It is the archetype of a Sturmian word [7]. The properties of the Fibonacci infinite word have been studied extensively by many authors, see, e.g., [7, 8, 9, 10, 12, 15].

Using Fibonacci words, in the present article we shall introduce some new kinds of the infinite words, namely FLP-words, and investigate some of their properties.

For any notations not explicitly defined in this article we refer to [4, 6, 7].

1 FIBONACCI SEQUENCE MODULO m

The letter p, $p > 2$, is reserved to designate a prime, m may be arbitrary integer, $m > 2$.

Let $F_n^*(m)$ denote the n-th member of the sequence of integers $F_n \equiv F_{n-1} + F_{n-2} \pmod{m}$, for all integer $n > 1$, and with initial values $F_0 = 0$ and $F_1 = 1$. We reduce F_n modulo m taking the least nonnegative residues, and let $k(m)$ denote the length of the period of the repeating sequence $F_n^*(m)$.

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The problem of determining the length of the period of the recurring sequence arose in connection with a method for generating random numbers. A few properties of the function $k(m)$ are in the following theorem [14].

Theorem 1. *In $\mathbb{Z}_m$ the following statements hold.*

1. *Any Fibonacci sequence modulo m is periodic.*
2. *If $p \equiv \pm 1 \pmod{10}$, then $k(p)|(p-1)$. If $p \equiv \pm 3 \pmod{10}$, then $k(p)|2(p+1)$.*
3. *If m has prime factorization $m = \prod_{i=1}^{n} p_i^{e_i}$, then $k(m) = \operatorname{lcm}(k(p_1^{e_1}), \dots, k(p_n^{e_n}))$.*
4. *If $k(p^2) \neq k(p)$, then $k(p^i) = p^{i-1}k(p)$ for $i > 1$.*

The results in Theorem 1 give upper bounds for $k(p)$ but there are primes for which $k(p)$ is less than the given upper bound.

Let $h(m)$ denote the length of the period of the repeating sequence $2^{F_n} \pmod{m}$ and $\varphi(m)$ be Euler's totient function.

Theorem 2. *Let m be odd and $m > 1$. Then $h(m)|k(\varphi(m))$.*

Proof. This follows from Euler's theorem: if m and a are coprime positive integers, then $a^{\varphi(m)} \equiv 1 \pmod{m}$. When reducing the power of a a modulo m, one needs to work modulo $\varphi(m)$ in the exponent of a: if $x \equiv y \pmod{\varphi(m)}$ then $a^x \equiv a^y \pmod{m}$. $\square$

Corollary 1. *Let $p \geq 3$. Then $h(p)|k(p-1)$.*

2 FIBONACCI WORDS

Let $f_0 = 1$ and $f_1 = 0$. Now $f_n = f_{n-1}f_{n-2}$, $n > 1$, the concatenation of the two previous terms. The successive initial finite Fibonacci words are:

$$\begin{aligned} &f_0 = 1, \quad f_1 = 0, \quad f_2 = 01, \quad f_3 = 010, \\ &f_4 = 01001, \quad f_5 = 01001010, \quad f_6 = 0100101001001, \\ &f_7 = 010010100100101001010, \quad f_8 = 0100101001001010010100100101001001, \dots \end{aligned} \tag{1}$$

The infinite Fibonacci word f is the limit $f = \lim f_n$. It is referenced A003849 in the On-line Encyclopedia of Integer Sequences [13] and is certainly one of the most studied examples in the combinatorial theory of infinite words. The combinatorial properties of the Fibonacci infinite word are of great interest in some aspects of mathematics and physics, such as number theory, fractal geometry, cryptography, formal language, computational complexity, quasicrystals etc. (see [7]).

We denote as usual by $|f_n|$ the length (the number of symbols) of f_n (see [7]). The following proposition summarizes basic properties of the Fibonacci words [7, 10].

Theorem 3. *The infinite Fibonacci word and the finite Fibonacci words satisfy the following properties.*

1. *The words 11 and 000 are not subwords of the infinite Fibonacci word.*

2. *For all $n > 1$ let ab be the last two symbols of f_n, then we have $ab = 01$ if n is even and $ab = 10$ if n is odd.*

3. *The concatenation of two successive Fibonacci words is "almost commutative", i.e., $f_n f_{n-1}$ and $f_{n-1} f_n$ differ only by their last two symbols for all $n > 1$.*

4. *For all n $|f_n| = F_{n+1}$.*

5. *The number of 0 and 1 in f_n equals F_n and F_{n-1}, respectively.*

3 PERIODIC FLP-WORDS

Let us start with the classical definition of periodicity on words over arbitrary alphabet $\{a_0, a_1, a_2, \dots\}$ (see [4]).

Definition 1. *Let $w = a_0 a_1 a_2 \dots$ be an infinite word. We say that w is*

1) *a periodic word if there exists a positive integer t such that $a_i = a_{i+t}$ for all $i \geq 0$. The smallest t satisfying the previous condition is called the period of w;*

2) *an eventually periodic word if there exist two positive integers k, p such that $a_i = a_{i+p}$, for all $i > k$;*

3) *an aperiodic word if it is not eventually periodic.*

Theorem 4. *The infinite Fibonacci word is aperiodic.*

This statement is proved in [10]. We consider the finite Fibonacci words f_n (1) as numbers written in the binary system and denote them by b_n. Denote by d_n the value of the number b_n in usual decimal numeration system. We write $b_n = d_n$ meaning that b_n and d_n are writing of the same number in different numeration systems.

Example.

$$f_0 = 1,\ f_1 = 0,\ f_2 = 01,\ f_3 = 010,\ f_4 = 01001,\ f_5 = 01001010,\ f_6 = 0100101001001,\ \dots,$$
$$b_0 = 1,\ b_1 = 0,\ b_2 = 1,\ b_3 = 10,\ b_4 = 1001,\ b_5 = 1001010,\ b_6 = 100101001001,\ \dots,$$
$$d_0 = 1,\ d_1 = 0,\ d_2 = 1,\ d_3 = 2,\ d_4 = 9,\ d_5 = 74,\ d_6 = 2377, \dots.$$

Formally, for arbitrary $n > 1$ f_n coincide with the b_n, taken with prefix 0: $f_n = 0b_n$.

Theorem 5. *For any finite Fibonacci word f_n, $n > 1$, in decimal numeration system we have*

$$d_n = d_{n-1} 2^{F_{n-1}} + d_{n-2}, \text{ where } d_0 = 1 \text{ and } d_1 = 0. \tag{2}$$

Proof. One can easily verify (2) for the first few n : $d_2 = b_2 = 1 = 0 + 1 = d_1 + d_0$, $d_3 = b_3 = 10 = 10 + 0 = d_2 2^1 + d_1$, $d_4 = b_4 = 1001 = 1000 + 01 = d_3 2^2 + d_2$, $d_5 = b_5 = 1001010 = 1001000 + 010 = d_4 2^3 + d_3$. Statement (2) follows from Theorem 3 (statement 4) and the equality $d_n = b_n = b_{n-1} \underbrace{0 \dots 0}_{F_{n-1}} + b_{n-2} = d_{n-1} 2^{F_{n-1}} + d_{n-2}$. $\square$

Theorem 6. *Let $p > 3$. The sequence $d_n \pmod{p}$ has period $T(p) = p \cdot h(p)$.*

Proof. By Theorem 1 we have $\gcd(k(p-1), p) = 1$. By Corollary 1 we have $h(p)|k(p-1)$. Therefore $\gcd(h(p), p) = 1$. From (2) it follows that for arbitrary integer i, $0 \leq i < h(p)$, if j runs from 0 to $p-1$ then numbers $d_{i+jh(p)} \pmod p$ runs all residues $\mod p$ or stationary. Then sequence $d_n \pmod p$ has period $p \cdot h(p)$. □

Let $d_0(m) = 1$, $w_0(m) = 1$ and for arbitrary integer n, $n \geq 1$, $d_n(m) = d_n \pmod m$ in binary numeration system, $w_n(m) = w_{n-1}(m)d_n(m)$. Denote by $w(m)$ the limit $w(m) = \lim_{n\to\infty} w_n(m)$.

Definition 2. *We say that*

1. $w_n(m)$ *is a finite FLP-word of type 1 by modulo* m;
2. $w(m)$ *is a infinite FLP-word of type 1 by modulo* m.

Theorem 7. *The infinite FLP-word of type 1* $w(m)$ *is periodic.*

Proof. The statement follows from (2) and Theorem 2 because there are only a finite number of $d_n \pmod m$ and $2^{F_{n-1}} \pmod{\varphi(m)}$ possible, and the recurrence of the first few terms of sequence $d_n \pmod m$ gives recurrence of all subsequent terms. □

Theorem 8. *Let* $p > 3$. *The sequence subwords* $d_n(p)$ *of the infinite FLP-word* $w(p)$ *of type 1 has period* $T(p) = p \cdot h(p)$.

Proof. The proof is a direct corollary of Theorem 6. □

Using Fibonacci words (1) we define periodic FLP-word $w^*(m)$ (infinite FLP-word of type 2 by modulo m). We denote as usual by ε the empty word [7]. First we define words $w_n^*(m)$. Let $w_n^*(m)$ be the last $F_{n+1}^*(m)$ symbols of the word f_n. If $F_{n+1}^*(m) = 0$ for some n, then $w_n^*(m) = \varepsilon$. Since $F_n^*(m)$ is periodic sequence with period $k(m)$, the sequence $|w_n^*(m)|$ is periodic with the same period.

Theorem 9. *The word length* $|w_n^*(m)|$ *coincides with* $F_{n+1}^*(m)$.

Proof. This is clear by construction of $w^*(m)$. □

Theorem 10. *The word* $w_n^*(m)$ *coincides with the word* $w_{n+k(m)}^*(m)$.

Proof. Since $f_n = f_{n-1}f_{n-2}$, the last F_{n-1} symbols of the word f_n coincide with the word f_{n-2}, and therefore the last F_n elements of the word f_{n+2k} coincide with the word f_{n-2} for any natural number k. The period $k(m)$ is an even number [14], so the last $F_{n+1}^*(m)$ elements of the words f_n and $f_{n+k(m)}$ are equivalent. □

Let $f_0^*(m) = 1$ and for arbitrary integer n, $n \geq 1$, $f_n^*(m) = f_{n-1}^*(m)w_n^*(m)$. Denote by $w^*(m)$ the limit $w^*(m) = \lim_{n\to\infty} f_n^*(m)$.

Definition 3. *We say that*

1) $f_n^*(m)$ *is a finite FLP-word of type 2 by modulo* m;

2) $w^*(m)$ *is a infinite FLP-word of type 2 by modulo* m.

Theorem 11. *The infinite FLP-word* $w^*(m)$ *of type 2 is a periodic word and sequence subwords* $w_n^*(m)$ *of* $w^*(m)$ *has period* $k(m)$.

Proof. The proof is a direct corollary of Theorem 10. □

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Барабаш Г.М., Холявка Я.М., Титар І.В. *Періодичні слова, пов'язані зі словами Фібоначчі* // Карпатські матем. публ. — 2016. — Т.8, №1. — С. 11–15.

У цій статті означено два види періодичних слів (FLP-слова типу 1 та FLP-слова типу 2), які пов'язані зі словами Фібоначчі, та досліджено їх властивості.

Ключові слова і фрази: число Фібоначчі, слово Фібоначчі.

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BASIUK Y.V., TARASYUK S.I.

FOURIER COEFFICIENTS ASSOCIATED WITH THE RIEMANN ZETA-FUNCTION

We study the Riemann zeta-function $\zeta(s)$ by a Fourier series method. The summation of $\log|\zeta(s)|$ with the kernel $1/|s|^6$ on the critical line $\mathrm{Re}\, s = \frac{1}{2}$ is the main result of our investigation. Also we obtain a new restatement of the Riemann Hypothesis.

Key words and phrases: Fourier coefficients, the Riemann zeta-function, Riemann Hypothesis.

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INTRODUCTION

It is known that the integral $\int\limits_{-\infty}^{\infty} \log\left|\zeta\left(\frac{1}{2}+it\right)\right| dt$, where $\zeta(s)$ is the Riemann zeta-function, diverges. M. Balazard, E.Saias, M. Yor [1] summed $\log|\zeta(s)|$ on the critical line with the kernel $1/|s|^2$. Using the fact that $f(z) = \frac{z}{1-z}\,\zeta\left(\frac{1}{1-z}\right)$, $|z|<1$, belongs to the Hardy space $H^{\frac{1}{3}}$ and the result of Bercovici and Foias [2] on the factorization of $f(z)$, they have proved the following theorem.

Theorem ([1]).
$$\frac{1}{2\pi}\int_{Re\, s=\frac{1}{2}} \frac{\log|\zeta(s)|}{|s|^2}\,|ds| = \sum_{Re\,\rho_j>\frac{1}{2}} \log\left|\frac{\rho_j}{1-\rho_j}\right|,$$
where $\{\rho_j\}$ is the sequence of non-trivial zeroes of $\zeta(s)$.

In particular, the Riemann Hypothesis holds if and only if
$$\frac{1}{2\pi}\int_{Re\, s=\frac{1}{2}} \frac{\log|\zeta(s)|}{|s|^2}\,|ds| = 0.$$

A. Kondratyuk, P. Yatsulka [6], using the method of Fourier series, have established the following fact.

Theorem ([6]). *Let $\{\rho_j\}$ be the sequence of non-trivial zeroes of $\zeta(s)$. Then*
$$\frac{1}{2\pi}\int_{Re\, s=\frac{1}{2}} \frac{\log|\zeta(s)|}{|s|^4}\,|ds| = 1-\gamma+2\sum_{Re\,\rho_j>\frac{1}{2}} \log\left|\frac{\rho_j}{1-\rho_j}\right| + \sum_{Re\,\rho_j>\frac{1}{2}} \frac{(|\rho_j|^2 - Re\rho_j)(2Re\rho_j-1)}{|\rho_j(\rho_j-1)|^2},$$

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where γ is the Euler constant. The Riemann Hypothesis holds if and only if

$$\frac{1}{2\pi}\int_{Re\,s=\frac{1}{2}} \frac{\log|\zeta(s)|}{|s|^4}\,|ds| = 1-\gamma.$$

We make the next step studying the behaviour of the Riemann zeta-function on the critical line. The summation of $\log|\zeta(s)|$ with the kernel $1/|s|^6$ on the critical line $\mathrm{Re}\ s = \frac{1}{2}$ is the main result of our research.

1 SECTION WITH RESULTS

Our result is the following.

Theorem 1. *Let $\{\rho_j\}$ be the sequence of non-trivial zeroes of $\zeta(s)$. Then*

$$\begin{aligned}\frac{1}{2\pi}\int_{Re\,s=\frac{1}{2}} \frac{\log|\zeta(s)|}{|s|^6}\,|ds| = \frac{7}{2} - 4\gamma + \frac{\gamma_1-\gamma^2}{2} + 6\sum_{Re\,\rho_j>\frac{1}{2}} \log\left|\frac{\rho_j}{1-\rho_j}\right| \\ + 4\sum_{Re\,\rho_j>\frac{1}{2}} \frac{(|\rho_j|^2 - Re\rho_j)(2Re\rho_j-1)}{|\rho_j(\rho_j-1)|^2} \\ + \frac{1}{2}\sum_{Re\,\rho_j>\frac{1}{2}} \frac{Re(|\rho_j|^2-\overline{\rho}_j)^2(2Re\rho_j-1)(2|\rho_j|^2-2Re\rho_j+1)}{|\rho_j(\rho_j-1)|^4},\end{aligned} \tag{1}$$

where γ is the Euler constant,

$$\gamma_1 = -\lim_{N\to\infty}\left(\sum_{m\le N}\frac{1}{m}\log m - \frac{\log^2 N}{2}\right).$$

Also we obtain a new restatement of the Riemann Hypothesis.

Theorem 2. *The Riemann Hypothesis holds if and only if*

$$\frac{1}{2\pi}\int_{Re\,s=\frac{1}{2}} \frac{\log|\zeta(s)|}{|s|^6}\,|ds| = \frac{7}{2} - 4\gamma + \frac{\gamma_1-\gamma^2}{2}. \tag{2}$$

Proof of Theorem 1. Observe that the conformal map $z = 1 - 1/s$ transforms the domain $\left\{s : \mathrm{Re}\ s > \frac{1}{2}\right\}$ onto the unit disc $\{z : |z| < 1\}$. Consider the function

$$f(z) = (s-1)\,\zeta(s) = \frac{z}{1-z}\,\zeta\left(\frac{1}{1-z}\right).$$

We have

$$(s-1)\,\zeta(s) = 1 + \gamma(s-1) + \gamma_1(s-1)^2 + \cdots + \gamma_k(s-1)^{k+1} + \ldots, \tag{3}$$

where

$$\gamma_k = \frac{(-1)^k}{k!}\lim_{N\to\infty}\left(\sum_{m\le N}\frac{1}{m}\log^k m - \frac{\log^{k+1} N}{k+1}\right),\ k\in\mathbb{N},$$

([5, p.4]). Therefore $f(z)$ is holomorphic in the unit disk. It was showed in [3] that the function $f(z)$ belongs to the Hardy class $H^p,\ 0 < p < 1$. Earlier it was established in [1] and [2] that the function $f(z)$ belongs to the Hardy class $H^{\frac{1}{3}}$ and $\sigma = 0$, where σ is the singular measure from the factorization (see [4])

$$f(z) = B(z) \cdot \exp(iC) \cdot \exp\left(-\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{i\varphi}+z}{e^{i\varphi}-z}d\sigma(\varphi)\right)\exp\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{i\varphi}+z}{e^{i\varphi}-z}\log|f(e^{i\varphi})|\,d\varphi\right), \tag{4}$$

where

$$B(z) = \prod_j \frac{|a_j|}{a_j}\frac{a_j - z}{1-\overline{a}_j z}$$

is the Blaschke product, $\{a_j\}$ is the sequence of zeros of $f(z)$ and $C = \operatorname{Im} f(0)$ is a real constant.

Consider the Fourier coefficient of $\log|f(re^{i\theta})|$:

$$c_k(r,f) = \frac{1}{2\pi}\int_0^{2\pi} e^{-ik\theta}\log|f(re^{i\theta})|d\theta,\ r \le 1.$$

Note that $c_{-k}(r,f) = \overline{c_k(r,f)}$.

It follows from (3) that $f(0) = 1$, and (4) yields

$$c_0(1,f) = -\log|B(0)| = \sum_j \log\frac{1}{|a_j|}$$

and

$$\log|f(re^{i\theta})| = \log|B(re^{i\theta})| + \frac{1}{2\pi}\int_0^{2\pi}\operatorname{Re}\frac{e^{i\varphi}+re^{i\theta}}{e^{i\varphi}-re^{i\theta}}\log|f(e^{i\varphi})|\,d\varphi. \tag{5}$$

In some neighborhood of the origin, the function $F(z) = \log f(z)$, $\log f(0) = 0$, is holomorphic. Let $F(z) = A_1 z + A_2 z^2 + \dots$ be its Maclaurin expansion. According to (3)

$$A_1 = \gamma;\ A_2 = \frac{\gamma_1 - \gamma^2}{2}.$$

On the other hand,

$$\log|f(re^{i\varphi})| = \operatorname{Re}\log f(re^{i\varphi}) = \frac{F+\overline{F}}{2} = \frac{\gamma r(e^{i\varphi}+e^{-i\varphi})}{2} + \frac{(\gamma_1-\gamma^2)r^2(e^{2i\varphi}+e^{-2i\varphi})}{4} + \dots,$$

where r is sufficiently small.

The relation (5) implies, for small r,

$$\frac{\gamma_1 - \gamma^2}{4}\, r^2 = c_{-2}(r,B) + r^2\, c_{-2}(1,f).$$

In [7], the expression for the Fourier coefficient of the Blaschke product was obtained

$$c_{-2}(r,B) = \frac{r^2}{4}\sum_{j=1}^{\infty}\frac{1}{\overline{a}_j^{\,2}}\left(|a_j|^4 - 1\right)$$

for $r < |a_1|$. Thus,

$$c_{-2}(1,f) = \frac{\gamma_1-\gamma^2}{4} - \frac{1}{4}\sum_{j=1}^{\infty}\frac{1}{\overline{a}_j^{\,2}}\left(|a_j|^4 - 1\right). \tag{6}$$

Note that
$$c_{-2}(1,f)=\frac{1}{4}+\frac{1}{2\pi}\int_0^{2\pi} e^{2i\theta}\log\left|\zeta\left(\frac{1}{1-e^{i\theta}}\right)\right|\,d\theta. \tag{7}$$
Return to the variable s. Taking (6) and (7) into account, we obtain

$$\begin{aligned}&\frac{1}{4}+\frac{1}{2\pi}\int\limits_{\operatorname{Re}s=\frac{1}{2}}\left(1-\frac{1}{s}\right)^2\frac{\log|\zeta(s)|}{|s|^2}\,|ds|\\&=\frac{\gamma_1-\gamma^2}{4}+\frac{1}{4}\sum_{\operatorname{Re}\rho_j>\frac{1}{2}}\frac{\overline{\rho}_j^{\,2}(2\operatorname{Re}\rho_j-1)(2|\rho_j|^2-2\operatorname{Re}\rho_j+1)}{(\overline{\rho}_j-1)^2\,|\rho_j|^4}.\end{aligned} \tag{8}$$

Taking the real parts of both sides (8), we get

$$\begin{aligned}&\frac{1}{4}+\frac{1}{2\pi}\int_{\operatorname{Re}s=\frac{1}{2}}\frac{\log|\zeta(s)|}{|s|^2}\,|ds|-\frac{1}{2\pi}\int_{\operatorname{Re}s=\frac{1}{2}}\frac{\log|\zeta(s)|}{|s|^4}\,|ds|+\frac{1}{2\pi}\int_{\operatorname{Re}s=\frac{1}{2}}\operatorname{Re}\left(\overline{s}^2\right)\frac{\log|\zeta(s)|}{|s|^6}\,|ds|\\&=\frac{\gamma_1-\gamma^2}{4}+\frac{1}{4}\sum_{\operatorname{Re}\rho_j>\frac{1}{2}}\frac{\operatorname{Re}(|\rho_j|^2-\overline{\rho}_j)^2(2\operatorname{Re}\rho_j-1)(2|\rho_j|^2-2\operatorname{Re}\rho_j+1)}{|\rho_j(\rho_j-1)|^4}.\end{aligned}$$

Note that

$$\begin{aligned}\int\limits_{\operatorname{Re}s=\frac{1}{2}}\operatorname{Re}\left(\overline{s}^2\right)\frac{\log|\zeta(s)|}{|s|^6}\,|ds|&=2\int_0^\infty\left(\frac{1}{4}-t^2\right)\frac{\log\left|\zeta\left(\frac{1}{2}+it\right)\right|}{\left(\frac{1}{4}+t^2\right)^3}dt\\&=-2\int_0^\infty\frac{\log\left|\zeta\left(\frac{1}{2}+it\right)\right|}{\left(\frac{1}{4}+t^2\right)^2}dt+\int_0^\infty\frac{\log\left|\zeta\left(\frac{1}{2}+it\right)\right|}{\left(\frac{1}{4}+t^2\right)^3}dt\\&=-\int_{\operatorname{Re}s=\frac{1}{2}}\frac{\log|\zeta(s)|}{|s|^4}\,|ds|+\frac{1}{2}\int_{\operatorname{Re}s=\frac{1}{2}}\frac{\log|\zeta(s)|}{|s|^6}\,|ds|.\end{aligned}$$

Using the results from [1] and [6], we obtain (1). The proof is completed. □

Proof of Theorem 2. If the Riemann Hypothesis is true, then the series at the right hand side of (1) are absent, and we have (2)

$$\frac{1}{2\pi}\int_{\operatorname{Re}s=\frac{1}{2}}\frac{\log|\zeta(s)|}{|s|^6}\,|ds|=\frac{7}{2}-4\gamma+\frac{\gamma_1-\gamma^2}{2}.$$

Now assume that relation (2) holds. If the Riemann Hypothesis is not true, then in (1)

$$6\sum_{\operatorname{Re}\rho_j>\frac{1}{2}}\log\left|\frac{\rho_j}{1-\rho_j}\right|+4\sum_{\operatorname{Re}\rho_j>\frac{1}{2}}\frac{(|\rho_j|^2-\operatorname{Re}\rho_j)(2\operatorname{Re}\rho_j-1)}{|\rho_j(\rho_j-1)|^2}>0.$$

Examine carefully the series

$$\sum_{\operatorname{Re}\rho_j>\frac{1}{2}}\frac{\operatorname{Re}(|\rho_j|^2-\overline{\rho}_j)^2(2\operatorname{Re}\rho_j-1)(2|\rho_j|^2-2\operatorname{Re}\rho_j+1)}{|\rho_j(\rho_j-1)|^4}.$$

We are interested in when all terms of this series are positive. The following conditions appear

$$\mathrm{Re}(|\rho_j|^2 - \overline{\rho}_j)^2 > 0.$$

If $0 < \mathrm{Re}\,\rho_j < 1$ and $|\ \mathrm{Im}\,\rho_j\ | > \frac{1}{2} + \frac{1}{\sqrt{2}}$, then $\mathrm{Re}(|\rho_j|^2 - \overline{\rho}_j)^2 > 0$.

It is known (see [8]) that the first $10^{22} + 1$ non-trivial zeros of the Riemann zeta-function lie on the critical line. In particular, $\mathrm{Im}\,\rho_1 = 14,1347\ldots$.

These facts imply $\mathrm{Re}(|\rho_j|^2 - \overline{\rho}_j)^2 > 0$ for all non-trivial zeros ρ_j that lie inside the critical strip $0 < \mathrm{Re}\,s < 1$.

Hence, if the Riemann Hypothesis is not true, then

$$\frac{1}{2\pi} \int_{\mathrm{Re}\,s=\frac{1}{2}} \frac{\log|\zeta(s)|}{|s|^6}\,|ds| > \frac{7}{2} - 4\gamma + \frac{\gamma_1 - \gamma^2}{2}.$$

This is a contradiction with (2) which finishes the proof. □

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Басюк Ю.В., Тарасюк С.І. *Коефіцієнти Фур'є, асоційовані з дзета-функцією Рімана* // Карпатські матем. публ. — 2016. — Т.8, №1. — С. 16–20.

Ми вивчаємо дзета-функцію Рімана $\zeta(s)$, використовуючи метод коефіцієнтів Фур'є. Підсумовування $\log|\zeta(s)|$ з ядром $1/|s|^6$ на критичній прямій $\mathrm{Re}\ s = \frac{1}{2}$ є головним результатом нашого дослідження. Також отримали твердження, рівносильне гіпотезі Рімана.

Ключові слова і фрази: коефіцієнти Фур'є, дзета-функція Рімана, гіпотеза Рімана.

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OPTIMAL CONTROL PROBLEM FOR SYSTEMS GOVERNED BY NONLINEAR PARABOLIC EQUATIONS WITHOUT INITIAL CONDITIONS

An optimal control problem for systems described by Fourier problem for nonlinear parabolic equations is studied. Control functions occur in the coefficients of the state equations. The existence of the optimal control in the case of final observation is proved.

Key words and phrases: optimal control, problem without initial conditions, nonlinear parabolic equation.

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INTRODUCTION

Optimal control of determined systems governed by partial differential equations (PDEs) is currently of much interest. Optimal control problems for PDEs are most completely studied for the case in which the control functions occur either on the right-hand sides of the state equations, or the boundary or initial conditions [8, 22, 26]. So far, problems in which control functions occur in the coefficients of the state equations are less studied.

The main ideas and methods of solving different optimal control problems for systems governed by evolutionary equations and variational inequalities are considered in monograph [18]. Problem, where control functions occur in the coefficients of the state equations, is given as only one among many other problems which were considered there by author.

A lot of various generalizations of this problem were investigated in many papers, including [1, 2, 4, 5, 10–13, 15, 20, 21, 24, 25], where the state of controlled system is described by the initial-boundary value problems for parabolic equations.

In [1, 21, 24, 25] the state of controlled system is described by linear parabolic equations and systems, while in [1] and [21] control functions appears as coefficients at lower derivatives, and in [24, 25] the control functions are coefficients at higher derivatives. In [21] the existence and uniqueness of optimal control in the case of final observation was shown and a necessary optimality condition in the form of the generalized rule of Lagrange multipliers was obtained. In paper [1] authors proved the existence of at least one optimal control for system governed by a system of general parabolic equations with degenerate discontinuous parabolicity coefficient. In papers [24, 25] the authors consider cost function in general form, and as special case it includes different kinds of specific practical optimization problems. The well-posedness of the problem statement is investigated and a necessary optimality condition in the form of the generalized principle of Lagrange multiplies is established in this papers.

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In papers [2, 10–13, 15, 20] authors investigate optimal control of systems governed by nonlinear PDEs. In particular, in [2] the problem of allocating resources to maximize the net benefit in the conservation of a single species is studied. The population model is an equation with density dependent growth and spatial-temporal resource control coefficient. The existence of an optimal control and the uniqueness and the characterization of the optimal control are established. Numerical simulations illustrate several cases with Dirichlet and Neumann boundary conditions. In [11] the optimal control problem is converted to an optimization problem which is solved using a penalty function technique. The existence and uniqueness theorems are investigated. The derivation of formula for the gradient of the modified function is explained by solving the adjoint problem. Paper [15] presents analytical and numerical solutions of an optimal control problem for quasilinear parabolic equations. The existence and uniqueness of the solution are shown. The derivation of formula for the gradient of the modified cost function by solving the conjugated boundary value problem is explained. In [16] the authors consider the optimal control of a degenerate parabolic equation governing a diffusive population with logistic growth terms. The optimal control is characterized in terms of the solution of the optimality system, which is the state equation coupled with the adjoint equation. Uniqueness for the solutions of the optimality system is valid for a sufficiently small time interval due to the opposite time orientations of the two equations involved. In paper [20] optimal control for semilinear parabolic equations without Cesari-type conditions is investigated.

In this paper, we study an optimal control problem for systems whose states are described by problems without initial conditions or, other words, Fourier problems for nonlinear parabolic equations.

The problem without initial conditions for evolution equations describes processes that started a long time ago and initial conditions do not affect on them in the actual time moment. Such problem were investigated in the works of many mathematicians (see [3, 7, 23] and bibliography there).

As we know among numerous works devoted to the optimal control problems for PDEs, only in papers [4, 5] the state of controlled system is described by the solution of Fourier problem for parabolic equations. In the current paper, unlike the above two, we consider optimal control problem in case when the control functions occur in the coefficients of the state equation. The main result of this paper is existence of the solution of this problem.

The outline of this paper is as follows. In Section 1, we give notations, definitions of function spaces and auxiliary results. In Section 2, we prove existence and uniqueness of the solutions for the state equations. Furthermore, we construct a priori estimates for the weak solutions of the state equations. In Section 3, we formulate the optimal control problem. Finally, the existence of the optimal control is presented in Section 4.

1 PRELIMINARIES

Let n be a natural number, $\mathbb{R}^n$ be the linear space of ordered collections $x = (x_1, \dots, x_n)$ of real numbers with the norm $|x| := (|x_1|^2 + \dots + |x_n|^2)^{1/2}$. Suppose that Ω is a bounded domain in $\mathbb{R}^n$ with piecewise smooth boundary Γ. Set $S := (-\infty, 0]$, $Q := \Omega \times S$, $\Sigma := \Gamma \times S$.

Denote by $L^\infty_{\mathrm{loc}}(\overline{Q})$ the linear space of measurable functions on Q such that their restrictions to any bounded measurable set $Q' \subset Q$ belong to the space $L^\infty(Q')$.

Let X be an arbitrary Hilbert space with the scalar product $(\cdot,\cdot)_X$ and the norm $\|\cdot\|_X$.

Denote by $L^2_{\rm loc}(S;X)$ the linear space of measurable functions defined on S with values in X, whose restrictions to any segment $[a,b]\subset S$ belong to the space $L^2(a,b;X)$.

Let $\omega\in\mathbb{R}$, $\alpha\in C(S)$ be such that $\alpha(t)>0$ for all $t\in S$, $\gamma=\alpha$ or $\gamma=1/\alpha$, and let X be as above. Put by definition

$$L^2_{\omega,\gamma}(S;X):=\Big\{f\in L^2_{\rm loc}(S;X)\ \Big|\ \int\limits_S\gamma(t)e^{2\omega\int\limits_0^t\alpha(s)ds}\|f(t)\|^2_X dt<\infty\Big\}.$$

This space is a Hilbert space with respect to the scalar product

$$(f,g)_{L^2_{\omega,\gamma}(S;X)}=\int\limits_S\gamma(t)\,e^{2\omega\int\limits_0^t\alpha(s)\,ds}(f(t),g(t))_X\,dt$$

and the norm

$$\|f\|_{L^2_{\omega,\gamma}(S;X)}:=\Big(\int\limits_S\gamma(t)\,e^{2\omega\int\limits_0^t\alpha(s)\,ds}\|f(t)\|^2_X\,dt\Big)^{1/2}.$$

Denote by $C^1_c(a,b)$, where $-\infty\le a<b\le+\infty$, the linear space of continuously differentiable functions on (a,b) with compact supports.

Let $H^1(\Omega):=\{v\in L_2(\Omega)\mid v_{x_i}\in L_2(\Omega)\ (i=\overline{1,n})\}$ be a Sobolev space, which is a Hilbert space with respect to the scalar product $(v,w)_{H^1(\Omega)}:=\int\limits_\Omega\{\sum\limits_{i=1}^n v_{x_i}w_{x_i}+vw\}\,dx$ and the corresponding norm $\|v\|_{H^1(\Omega)}:=\Big(\int\limits_\Omega\{\sum\limits_{i=1}^n|v_{x_i}|^2+|v|^2\}\,dx\Big)^{1/2}$. Under $H^1_0(\Omega)$ we mean the closure in $H^1(\Omega)$ of the space $C^\infty_c(\Omega)$ consisting of infinitely differentiable functions on Ω with compact supports. Denote by

$$K:=\inf_{v\in H^1_0(\Omega),\ v\neq0}\frac{\int\limits_\Omega|\nabla v|^2\,dx}{\int\limits_\Omega|v|^2\,dx},\tag{1}$$

where $\nabla v=(v_{x_1},\dots,v_{x_n})$, $|\nabla v|^2=\sum\limits_{i=1}^n|v_{x_i}|^2$.

It is well known that the constant K is finite and coincides with the first eigenvalue of the following eigenvalue problem:

$$-\Delta v=\lambda v,\quad v|_{\partial\Omega}=0.\tag{2}$$

From (1) it clearly follows the Friedrichs inequality

$$\int\limits_\Omega|\nabla v|^2\,dx\ge K\int\limits_\Omega|v|^2\,dx\ \text{for all}\ \ v\in H^1_0(\Omega).\tag{3}$$

Also define $\partial_0 z=z,\quad \partial_j z=z_{x_j}$ if $j\in\{1,\dots,n\}$. Further, an important role will be played by the following statement.

Lemma 1. *Suppose that a function $z\in L^2(t_1,t_2;H^1_0(\Omega))$, where $t_1,t_2\in\mathbb{R}$ $(t_1<t_2)$, satisfies the identity*

$$\int\limits_{t_1}^{t_2}\int\limits_\Omega\{-z\psi\varphi'+\sum_{i=0}^n g_i\partial_i\psi\varphi\}\,dxdt=0,\quad \psi\in H^1_0(\Omega),\ \varphi\in C^1_c(t_1,t_2),\tag{4}$$

for some $g_i \in L^2(t_1,t_2;L^2(\Omega))$ $(i=\overline{0,n})$. *Then*

(i) the function z *belongs to the space* $C([t_1,t_2];L^2(\Omega))$ *and for every* $\theta \in C^1([t_1,t_2])$ *and for all* $\tau_1,\tau_2 \in [t_1,t_2]$ $(\tau_1<\tau_2)$ *we have*

$$\frac{1}{2}\theta(t)\int\limits_{\Omega}|z(x,t)|^2\,dx\Big|_{t=\tau_1}^{t=\tau_2} - \frac{1}{2}\int\limits_{\tau_1}^{\tau_2}\int\limits_{\Omega}|z|^2\theta'\,dxdt + \int\limits_{\tau_1}^{\tau_2}\int\limits_{\Omega}\{\sum_{i=0}^{n} g_i\partial_i z\}\theta\,dxdt = 0; \tag{5}$$

(ii) the derivative z_t *of the function* z *in the sense* $D'(t_1,t_2;H^{-1}(\Omega))$ *(the distributions space) belongs to* $L^2(t_1,t_2;H^{-1}(\Omega))$*, furthermore*

$$\int\limits_{t_1}^{t_2}\|z_t(\cdot,t)\|^2_{H^{-1}(\Omega)}\,dt \le \sum_{i=0}^{n}\|g_i\|^2_{L^2(\Omega\times(t_1,t_2))}. \tag{6}$$

Proof. The first statement follows directly from Lemma 2 of [6]. Let us prove the second statement. Firstly note that the following continuous and dense embeddings hold

$$H_0^1(\Omega)\subset L^2(\Omega)\subset H^{-1}(\Omega). \tag{7}$$

Let $C_c^\infty(t_1,t_2)$ be the space of functions on (t_1,t_2) which are infinitely continuously differentiable and have compact supports. Under $D'(t_1,t_2;H^{-1}(\Omega))$ we mean the space of distributions which are defined on $C_c^\infty(t_1,t_2)$ with values in $H^{-1}(\Omega)$ (see, for example, [14]). Since the spaces $L^2(t_1,t_2;H_0^1(\Omega))$, $L^2(t_1,t_2;H^{-1}(\Omega))$ can be identified with subspaces of the space of distributions $D'(t_1,t_2;H^{-1}(\Omega))$, then it allows us to speak about derivatives of functions from $L^2(t_1,t_2;H_0^1(\Omega))$ in the sense $D'(t_1,t_2;H^{-1}(\Omega))$ and their belonging to the space $L^2(t_1,t_2;H^{-1}(\Omega))$.

Let us rewrite equality (4) in the form

$$-\int\limits_{t_1}^{t_2}\int\limits_{\Omega} z\psi\varphi'\,dxdt = -\int\limits_{t_1}^{t_2}\int\limits_{\Omega}\sum_{i=0}^{n} g_i\partial_i\psi\varphi\,dxdt, \quad \psi\in H_0^1(\Omega),\ \varphi\in C_c^1(t_1,t_2). \tag{8}$$

According to the definition of the derivative of distributions from $D'(t_1,t_2;H^{-1}(\Omega))$, (8) implies that z_t belongs to the space $L^2(t_1,t_2;H^{-1}(\Omega))$, and for almost all $t\in(t_1,t_2)$

$$< z_t(\cdot,t),\psi(\cdot) >_{H_0^1(\Omega)} = -\int\limits_{\Omega}\sum_{i=0}^{n} g_i(x,t)\partial_i\psi(x)\,dx,$$

where $<\cdot,\cdot>_{H_0^1(\Omega)}$ denotes the canonical scalar product in $H^{-1}(\Omega)\times H_0^1(\Omega)$. From this, using the Cauchy-Schwarz inequality, for almost all $t\in(t_1,t_2)$ we obtain

$$\begin{aligned}|< z_t(\cdot,t),\psi(\cdot) >_{H_0^1(\Omega)}| &\le \sum_{i=0}^{n}\|g_i(\cdot,t)\|_{L^2(\Omega)}\|\partial_i\psi(\cdot)\|_{L^2(\Omega)}\\ &\le \Big(\sum_{i=0}^{n}\|g_i(\cdot,t)\|^2_{L^2(\Omega)}\Big)^{1/2}\|\psi(\cdot)\|_{H^1(\Omega)}.\end{aligned} \tag{9}$$

From (9) it follows that for almost all $t\in(t_1,t_2)$ the following estimate is valid

$$\|z_t(\cdot,t)\|^2_{H^{-1}(\Omega)} \le \sum_{i=0}^{n}\|g_i(\cdot,t)\|^2_{L^2(\Omega)},$$

which easily implies (6). $\square$

2 WELL-POSEDNESS OF THE PROBLEM WITHOUT INITIAL CONDITIONS FOR NONLINEAR PARABOLIC EQUATIONS

Consider the equation

$$y_t - \sum_{i=1}^{n} \frac{d}{dx_i} a_i(x,t,y,\nabla y) + a_0(x,t,y,\nabla y) = f(x,t), \quad (x,t) \in Q, \tag{10}$$

where $y : \overline{Q} \to \mathbb{R}$ is an unknown function and data-in satisfies following conditions:

$(\mathcal{A}_1)$ for every $i \in \{0,1,\dots,n\}$

$$Q \times \mathbb{R} \times \mathbb{R}^n \ni (x,t,s,\xi) \mapsto a_i(x,t,s,\xi) \in \mathbb{R}$$

is the Caratheodory function, i.e., $a_i(x,t,\cdot,\cdot) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is the continuous function for a.e. $(x,t) \in Q$, and $a_i(\cdot,\cdot,s,\xi) : Q \to \mathbb{R}$ is the measurable function for every $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$; moreover, $a_i(x,t,0,0) = 0$ for a. e. $(x,t) \in Q$;

$(\mathcal{A}_2)$ for every $i \in \{0,1,\dots,n\}$, for every $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$, and for a.e. $(x,t) \in Q$ the following estimate is valid $|a_i(x,t,s,\xi)| \le C_1\big(|s| + |\xi|\big) + h_i(x,t)$, where $C_1 = \text{const} > 0$, $h_i \in L^2_{\text{loc}}(S; L^2(\Omega))$;

$(\mathcal{A}_3)$ for every $(s_1,\xi^1), (s_2,\xi^2) \in \mathbb{R} \times \mathbb{R}^n$ and for a.e. $(x,t) \in Q$ the following inequality holds

$$\sum_{i=1}^{n} \big(a_i(x,t,s_1,\xi^1) - a_i(x,t,s_2,\xi^2)\big)(\xi_i^1 - \xi_i^2) \\ + \big(a_0(x,t,s_1,\xi^1) - a_0(x,t,s_2,\xi^2)\big)(s_1 - s_2) \ge \alpha(t)|\xi^1 - \xi^2|^2,$$

where $\alpha \in C(S)$ such that $\alpha(t) > 0$ for all $t \in S$;

$(\mathcal{F})$ $f \in L^2_{\text{loc}}(S; L^2(\Omega))$.

Additionally, we impose the boundary condition

$$y\big|_{\Sigma} = 0 \tag{11}$$

on a solution of equation (10).

Definition 1. *The function y is called a weak solution of equation* (10) *satisfying boundary condition* (11) *if it belongs to $L^2_{\text{loc}}(S; H^1_0(\Omega)) \cap C(S; L^2(\Omega))$ and the following integral equality holds*

$$\iint_Q \Big\{ -y\psi\varphi' + \sum_{i=0}^{n} a_i(x,t,y,\nabla y)\partial_i\psi\varphi \Big\}\, dxdt \\ = \iint_Q f\psi\varphi\, dxdt, \quad \psi \in H^1_0(\Omega),\ \varphi \in C^1_c(-\infty,0). \tag{12}$$

There may exist many weak solutions of equation (10) satisfying boundary condition (11). To ensure uniqueness of the weak solution of equation (10) satisfying condition (11), we have to impose some additional conditions on solutions, for instance, some restrictions on their behavior as $t \to -\infty$. We will consider the problem of finding a weak solution of equation (10) satisfying boundary condition (11) and the analogue of the initial condition

$$\lim_{t\to-\infty} e^{\omega\int_0^t \alpha(s)ds} \|y(\cdot,t)\|_{L^2(\Omega)} = 0, \tag{13}$$

where $\omega \in \mathbb{R}$. We will briefly call this problem by problem (10), (11), (13), and the function y is called the weak solution of problem (10), (11), (13).

Lemma 2. *Let $\omega < K$, where K is a constant defined in* (1)*, and conditions $(\mathcal{A}_1)$–$(\mathcal{A}_3)$ are satisfied. Then two following statements are true.*

(i) If y is a weak solution of problem (10), (11), (13) *and*

$$f \in L^2_{\omega,1/\alpha}(S;L^2(\Omega)), \tag{14}$$

then $y \in L^2_{\omega,\alpha}(S;H^1_0(\Omega))$ and the following estimates hold:

$$e^{2\omega\int_0^\tau \alpha(s)\,ds} \|y(\cdot,\tau)\|^2_{L^2(\Omega)} \le C_1 \int_{-\infty}^{\tau} [\alpha(t)]^{-1} e^{2\omega\int_0^t \alpha(s)\,ds} \|f(\cdot,t)\|^2_{L^2(\Omega)}\,dt,\ \tau \in S, \tag{15}$$

$$\|y\|_{L^2_{\omega,\alpha}(S;H^1_0(\Omega))} \le C_2 \|f\|_{L^2_{\omega,1/\alpha}(S;L^2(\Omega))}, \tag{16}$$

where C_1, C_2 are positive constants depending on K and ω only.

(ii) If y_1 and y_2 are two weak solutions of problem (10), (11), (13) *with $f = f_1$ and $f = f_2$ correspondingly, and*

$$f_k \in L^2_{\omega,1/\alpha}(S;L^2(\Omega)) \quad (k=1,2), \tag{17}$$

then the following estimates hold:

$$\begin{aligned} &e^{2\omega\int_0^\tau \alpha(s)\,ds} \|y_1(\cdot,\tau) - y_2(\cdot,\tau)\|^2_{L^2(\Omega)} \\ &\quad\le C_1 \int_{-\infty}^{\tau} [\alpha(t)]^{-1} e^{2\omega\int_0^t \alpha(s)\,ds} \|f_1(\cdot,t) - f_2(\cdot,t)\|^2_{L^2(\Omega)}\,dt,\ \tau \in S, \end{aligned} \tag{18}$$

$$\|y_1 - y_2\|_{L^2_{\omega,\alpha}(S;H^1_0(\Omega))} \le C_2 \|f_1 - f_2\|_{L^2_{\omega,1/\alpha}(S;L^2(\Omega))}, \tag{19}$$

where C_1, C_2 are positive constants such as in (15) *and* (16).

Proof. First we prove statement *(ii)*. For function $z : Q \to \mathbb{R}$ let us denote

$$a_i(z)(x,t) := a_i(x,t,z(x,t),\nabla z(x,t)), \quad (x,t) \in Q,\ i = \overline{0,n}. \tag{20}$$

From (12) for difference $y_{12} := y_1 - y_2$ we get such an integral identity

$$\begin{aligned} &\iint_Q \Big\{ -y_{12}\psi\varphi' + \sum_{i=0}^{n} \big(a_i(y_1) - a_i(y_2)\big)\partial_i\psi\varphi \Big\}\,dxdt \\ &\quad= \iint_Q f_{12}\psi\varphi\,dxdt, \quad \psi \in H^1_0(\Omega),\ \varphi \in C^1_c(-\infty,0), \end{aligned} \tag{21}$$

where $f_{12} := f_1 - f_2$. According to Lemma 1, (21) implies that

$$\begin{aligned} &\frac{1}{2}\theta(t)\int\limits_{\Omega}|y_{12}(x,t)|^2\,dx\Big|_{t=\tau_1}^{t=\tau_2} - \frac{1}{2}\int\limits_{\tau_1}^{\tau_2}\int\limits_{\Omega}|y_{12}|^2\theta'\,dxdt \\ &+\int\limits_{\tau_1}^{\tau_2}\int\limits_{\Omega}\Big[\sum_{i=0}^{n}\big(a_i(y_1)-a_i(y_2)\big)\partial_i y_{12}\Big]\theta\,dxdt = \int\limits_{\tau_1}^{\tau_2}\int\limits_{\Omega} f_{12}y_{12}\theta\,dxdt, \end{aligned} \tag{22}$$

where $\theta \in C^1(S)$ and $\tau_1, \tau_2 \in S$ $(\tau_1 < \tau_2)$ are arbitrary. Using Cauchy inequality with ε:

$$ab \le \frac{\varepsilon}{2}a^2 + \frac{1}{2\varepsilon}b^2, \quad a,b\in\mathbb{R},\ \varepsilon>0, \tag{23}$$

let us estimate the right side of equality (22) as follows:

$$\Big|\int\limits_{\tau_1}^{\tau_2}\int\limits_{\Omega} f_{12}y_{12}\theta\,dxdt\Big| \le \frac{\varepsilon}{2}\int\limits_{\tau_1}^{\tau_2}\int\limits_{\Omega}\alpha|y_{12}|^2\theta\,dxdt + \frac{1}{2\varepsilon}\int\limits_{\tau_1}^{\tau_2}\int\limits_{\Omega}[\alpha]^{-1}|f_{12}|^2\theta\,dxdt, \tag{24}$$

where $\varepsilon > 0$ is arbitrary. From condition $(\mathcal{A}_3)$ we obtain following

$$\int\limits_{\tau_1}^{\tau_2}\int\limits_{\Omega}\Big[\sum_{i=0}^{n}\big(a_i(y_1)-a_i(y_2)\big)\partial_i y_{12}\Big]\theta\,dxdt \ge \int\limits_{\tau_1}^{\tau_2}\int\limits_{\Omega}\alpha|\nabla y_{12}|^2\theta\,dxdt, \tag{25}$$

where $\nabla y := (y_{x_1},\ldots,y_{x_n})$. According to (24) and (25), (22) implies the inequality

$$\begin{aligned} &\frac{1}{2}\theta(\tau_2)\int\limits_{\Omega}|y_{12}(x,\tau_2)|^2\,dx - \frac{1}{2}\theta(\tau_1)\int\limits_{\Omega}|y_{12}(x,\tau_1)|^2\,dx - \frac{1}{2}\int\limits_{\tau_1}^{\tau_2}\int\limits_{\Omega}|y_{12}|^2\theta'\,dxdt \\ &+\int\limits_{\tau_1}^{\tau_2}\int\limits_{\Omega}\alpha|\nabla y_{12}|^2\theta\,dxdt \le \frac{\varepsilon}{2}\int\limits_{\tau_1}^{\tau_2}\int\limits_{\Omega}\alpha|y_{12}|^2\theta\,dxdt + \frac{1}{2\varepsilon}\int\limits_{\tau_1}^{\tau_2}\int\limits_{\Omega}[\alpha]^{-1}|f_{12}|^2\theta\,dxdt, \end{aligned}$$

where $\varepsilon > 0$ is arbitrary.

From this taking $\theta(t) = 2e^{2\omega\int_0^t \alpha(s)\,ds}$, $t \in S$, we obtain

$$\begin{aligned} &e^{2\omega\int_0^{\tau_2}\alpha(s)\,ds}\int\limits_{\Omega}|y_{12}(x,\tau_2)|^2dx - e^{2\omega\int_0^{\tau_1}\alpha(s)\,ds}\int\limits_{\Omega}|y_{12}(x,\tau_1)|^2dx \\ &\quad - 2\omega\int\limits_{\tau_1}^{\tau_2}\int\limits_{\Omega}\alpha(t)e^{2\omega\int_0^t\alpha(s)\,ds}|y_{12}|^2dxdt + 2\int\limits_{\tau_1}^{\tau_2}\int\limits_{\Omega}\alpha(t)e^{2\omega\int_0^t\alpha(s)\,ds}|\nabla y_{12}|^2\,dxdt \\ &\quad \le \varepsilon\int\limits_{\tau_1}^{\tau_2}\int\limits_{\Omega}\alpha(t)e^{2\omega\int_0^t\alpha(s)\,ds}|y_{12}|^2\,dxdt + \frac{1}{\varepsilon}\int\limits_{\tau_1}^{\tau_2}\int\limits_{\Omega}[\alpha(t)]^{-1}e^{2\omega\int_0^t\alpha(s)\,ds}|f_{12}|^2\,dxdt. \end{aligned} \tag{26}$$

Due to (26) using (3) we obtain

$$\begin{aligned} &e^{2\omega\int_0^{\tau_2}\alpha(s)\,ds}\int\limits_{\Omega}|y_{12}(x,\tau_2)|^2\,dx - e^{2\omega\int_0^{\tau_1}\alpha(s)\,ds}\int\limits_{\Omega}|y_{12}(x,\tau_1)|^2\,dx \\ &+\chi(K,\omega,\varepsilon)\int\limits_{\tau_1}^{\tau_2}\int\limits_{\Omega}\alpha(t)e^{2\omega\int_0^t\alpha(s)\,ds}|\nabla y_{12}|^2\,dxdt \le \frac{1}{\varepsilon}\int\limits_{\tau_1}^{\tau_2}\int\limits_{\Omega}[\alpha(t)]^{-1}e^{2\omega\int_0^t\alpha(s)\,ds}|f_{12}|^2\,dxdt, \end{aligned} \tag{27}$$

where $\chi(K,\omega,\varepsilon) := (2(K-\omega)-\varepsilon)/K$ if $0<\omega<K$, and $\chi(K,\omega,\varepsilon) := (2K-\varepsilon)/K$ if $\omega \le 0$.

Taking $\varepsilon = K$ if $\omega \le 0$, and $\varepsilon = K-\omega$ if $0<\omega<K$ in (27), we obtain

$$e^{2\omega\int_0^{\tau_2}\alpha(s)\,ds}\int_\Omega |y_{12}(x,\tau_2)|^2\,dx - e^{2\omega\int_0^{\tau_1}\alpha(s)\,ds}\int_\Omega |y_{12}(x,\tau_1)|^2\,dx$$
$$+\,C_3\int_{\tau_1}^{\tau_2}\int_\Omega \alpha(t)e^{2\omega\int_0^{t}\alpha(s)\,ds}|\nabla y_{12}|^2\,dxdt \le C_4\int_{\tau_1}^{\tau_2}\int_\Omega [\alpha(t)]^{-1}e^{2\omega\int_0^{t}\alpha(s)\,ds}|f_{12}|^2\,dxdt, \tag{28}$$

where C_3, C_4 are positive constants depending on K and ω only.

From (13) it easily follows the condition

$$e^{2\omega\int_0^{t}\alpha(s)ds}\int_\Omega |y_{12}(x,t)|^2\,dx \to 0 \quad \text{as} \quad t\to-\infty. \tag{29}$$

Taking into account (29) and (17), we let $\tau_1 \to -\infty$ in (28). As a result, adopting $\tau_2 = \tau \in S$, we obtain

$$e^{2\omega\int_0^{\tau}\alpha(s)\,ds}\int_\Omega |y_{12}(x,\tau)|^2\,dx + C_3\int_{-\infty}^{\tau}\int_\Omega \alpha(t)e^{2\omega\int_0^{t}\alpha(s)\,ds}|\nabla y_{12}|^2\,dxdt$$
$$\le C_4\int_{-\infty}^{\tau}\int_\Omega [\alpha(t)]^{-1}e^{2\omega\int_0^{t}\alpha(s)\,ds}|f_{12}|^2\,dxdt. \tag{30}$$

Hence, using inequality (3), we easily obtain estimates (18) and (19).

Now let us prove statement *(i)*. Using the condition $(\mathcal{A}_1)$ one can easily see that $y=0$ is a weak solution of problem (10), (11), (13) with $f=0$, thus estimates (18) and (19) with $y_1 = y$, $f_1 = f$ and $y_2 = 0$, $f_2 = 0$ imply estimates (15) and (16). Estimate (16) implies that $y \in L^2_{\omega,\alpha}(S;H^1_0(\Omega))$. □

Lemma 3. *If $\omega \le K$, where K is a constant defined by* (1)*, then problem* (10), (11), (13) *has at most one weak solution.*

Proof. Assume the opposite. Let y_1, y_2 be two weak solutions of problem (10), (11), (13). In case $\omega < K$ according to Lemma 2 we obtain the equality

$$e^{2\omega\int_0^{\tau}\alpha(s)\,ds}\int_\Omega |y_1(x,\tau)-y_2(x,\tau)|^2\,dx = 0 \text{ for all } \tau\in S. \tag{31}$$

From proof of Lemma 2 it follows that estimate (31) is correct in case $\omega = K$ also. Indeed, if $\omega = K$, then in (27) and (30) we have $\chi(K,\omega,\varepsilon) = 0$ and $C_3 = 0$, correspondingly, and its easily follows from the proof that estimate (18) is correct.

Equality (31) implies equality $y_1(x,t) - y_2(x,t) = 0$ for a. e. $(x,t)\in Q$, that is, $y_1(x,t) = y_2(x,t) = 0$ for a. e. $(x,t)\in Q$. The resulting contradiction proves our statement. □

Remark 1. *Functions* $y_c(x,t) = cv(x)e^{-Kt}$, $(x,t) \in \overline{Q}$ $(c \in \mathbb{R})$, *where* v *is an eigenfunction of problem* (2) *corresponding to the first eigenvalue, are weak solutions of equation* (10) *satisfying condition* (11), *when* $a_i = \xi_i$ $(i = \overline{1,n})$, $a_0 = 0$ *and* $f = 0$. *In this case we have* $\alpha(t) = 1$, *therefore condition* (13) *takes on the form:* $e^{\omega t}\|y(\cdot,t)\|_{L^2(\Omega)} \underset{t\to-\infty}{\longrightarrow} 0$. *Obviously in this case for nonzero solutions we have* $e^{Kt}\|y_c(\cdot,t)\|_{L^2(\Omega)} \underset{t\to-\infty}{\longrightarrow} C = const \neq 0$, $e^{\omega t}\|y_c(\cdot,t)\|_{L^2(\Omega)} \underset{t\to-\infty}{\longrightarrow} +\infty$ *if* $\omega < K$, *and* $e^{\omega t}\|y_c(\cdot,t)\|_{L^2(\Omega)} \underset{t\to-\infty}{\longrightarrow} 0$ *if* $\omega > K$. *This means that the condition* $\omega \leq K$ *is essential for ensuring the uniqueness of the weak solution of problem* (10), (11), (13), *i.e., it cannot be simplified.*

Theorem 1. *Suppose that conditions* $(\mathcal{A}_1)$–$(\mathcal{A}_3)$ *hold, and* $\omega < K$, *where* K *is a constant defined in* (1), *and*

$$f \in L^2_{\omega,1/\alpha}(S; L^2(\Omega)). \tag{32}$$

Then there exists a unique weak solution of problem (10), (11), (13), *it belongs to the space* $L^2_{\omega,\alpha}(S; H^1_0(\Omega))$ *and estimates* (15) *and* (16) *are correct.*

Proof. Lemma 3 gives us a uniqueness of a weak solution of problem (10), (11), (13). It remains to prove the existence of a weak solution of this problem.

For each $m \in N$ we define $f_m(\cdot,t) := f(\cdot,t)$, if $-m < t \leq 0$, and $f_m(\cdot,t) := 0$, if $t \leq -m$, and consider the problem of finding a function $y_m \in L^2(-m,0; H^1_0(\Omega)) \cap C([-m,0]; L^2(\Omega))$ satisfying the initial condition

$$y_m(x,-m) = 0, \quad x \in \Omega, \tag{33}$$

(as an element of space $C([-m,0]; L^2(\Omega))$) and equation (10) in Q_m in the sense of the following integral identity

$$\iint\limits_{Q_m}\Big\{ -y_m\psi\varphi' + \sum_{i=0}^{n} a_i(y_m)\partial_i\psi\varphi\Big\}\,dxdt = \iint\limits_{Q_m} f_m\psi\varphi\,dxdt, \quad \psi \in H^1_0(\Omega), \varphi \in C^1_c(-m,0).$$

The existence and uniqueness of the solution of this problem easily follows from the known results (see, for example, [14]). For every $m \in \mathbb{N}$ we extend y_m by zero for the entire set Q and keep the same notation y_m for this extension. Note that for each $m \in N$, the function y_m belongs to $L^2(S; H^1_0(\Omega)) \cap C(S; L^2(\Omega))$ and satisfies integral identity (12) with f_m substituted for f, i.e.,

$$\iint\limits_{Q}\Big\{ -y_m\psi\varphi' + \sum_{i=0}^{n} a_i(y_m)\partial_j\psi\varphi\Big\}\,dxdt = \iint\limits_{Q} f_m\psi\varphi\,dxdt, \psi \in H^1_0(\Omega),\ \varphi \in C^1_c(-\infty,0). \tag{34}$$

Consequently, we have shown that y_m is a weak solution of problem (10), (11), (13) with f_m substituted for f. Then, in particular, statement *(i)* of Lemma 2 implies estimates

$$e^{2\omega\int_0^\tau \alpha(s)\,ds}\|y_m(\cdot,\tau)\|^2_{L^2(\Omega)} \leq C_1 \int_{-\infty}^{\tau} [\alpha(t)]^{-1} e^{2\omega\int_0^t \alpha(s)\,ds}\|f(\cdot,t)\|^2_{L^2(\Omega)}\,dt,\ \tau \in S, \tag{35}$$

$$\|y_m\|_{L^2_{\omega,\alpha}(S;H^1_0(\Omega))} \leq C_2\|f\|_{L^2_{\omega,1/\alpha}(S;L^2(\Omega))}, \tag{36}$$

where C_1, C_2 are positive constants such as in estimates (15), (16).

Let us take identity (34) with alternately $m = k$ and $m = l$, where k, l are arbitrary positive integers, $l > k$, and apply statement (ii) of Lemma 2. As a result, we obtain estimates similar to (18), (19), i.e.

$$e^{2\omega \int\limits_0^\tau \alpha(s)\,ds} \|y_k(\cdot,\tau) - y_l(\cdot,\tau)\|^2_{L^2(\Omega)} \le C_1 \int\limits_{-l}^{-k} [\alpha(t)]^{-1} e^{2\omega \int\limits_0^t \alpha(s)\,ds} \|f(\cdot,t)\|^2_{L^2(\Omega)}\,dt,\ \tau \in S, \tag{37}$$

$$\|y_k - y_l\|_{L^2_{\omega,\alpha}(S;H^1_0(\Omega))} \le C_2 \int\limits_{-l}^{-k} [\alpha(t)]^{-1} e^{2\omega \int\limits_0^t \alpha(s)\,ds} \|f(\cdot,t)\|^2_{L^2(\Omega)}\,dt. \tag{38}$$

Condition (32) implies that the right-hand sides of inequalities (37) and (38) tend to zero when k and l tend to $+\infty$. This means that the sequence $\{y_m\}_{m=1}^\infty$ is a Cauchy sequence in the space $L^2_{\omega,\alpha}(S; H^1_0(\Omega))$ and $C(S; L^2(\Omega))$. Consequently, we obtain the existence of the function $y \in L^2_{\omega,\alpha}(S; H^1_0(\Omega)) \cap C(S; L^2(\Omega))$ such that

$$y_m \underset{m\to\infty}{\longrightarrow} y \qquad \text{strongly in} \quad L^2_{\omega,\alpha}(S; H^1_0(\Omega)) \quad \text{and} \quad C(S; L^2(\Omega)). \tag{39}$$

Note that (39) implies

$$\partial_i y_m \underset{m\to\infty}{\longrightarrow} \partial_i y \quad \text{strongly in} \quad L^2_{\text{loc}}(S; L^2(\Omega)), \quad i = \overline{0,n}. \tag{40}$$

Condition $(\mathcal{A}_2)$ and estimate (36) gives us for each $t_1, t_2 \in S(t_1 < t_2)$ the following:

$$\int\limits_{t_1}^{t_2}\int\limits_\Omega |a_i(y_m)|^2\,dxdt \le C_5 \int\limits_{t_1}^{t_2}\int\limits_\Omega \left(|y_m|^2 + |\nabla y_m|^2 + |h_i|^2\right) dxdt \le C_6, \tag{41}$$

where C_5 and C_6 are positive constants independent on m.

Hence, from (41) we obtain that $a_i(y_m)$ is bounded in $L^2_{\text{loc}}(S; L^2(\Omega))$. This and (40) yield that there exists a subsequence of $\{y_m\}_{m=1}^\infty$ (still denoted by $\{y_m\}_{m=1}^\infty$) and functions $\chi_i \in L_{2,\text{loc}}(S; L_2(\Omega))$ $(i = \overline{0,n})$ such that

$$\partial_i y_m \underset{m\to\infty}{\longrightarrow} \partial_i y \quad \text{a.e. on} \quad Q, \quad i = \overline{0,n}, \tag{42}$$

$$a_i(y_m) \underset{m\to\infty}{\longrightarrow} \chi_i \quad \text{weakly in} \quad L_{2,\text{loc}}(S; L_2(\Omega)), \quad i = \overline{0,n}. \tag{43}$$

Condition $(\mathcal{A}_1)$ and (42) yield

$$a_i(y_m) \underset{m\to\infty}{\longrightarrow} a_i(y) \quad \text{a.e. on Q}, \quad i = \overline{0,n}. \tag{44}$$

According to [17, Lemma 1.3], from (43) and (44) we obtain

$$a_i(y_m) \underset{m\to\infty}{\longrightarrow} a_i(y) \quad \text{weakly in} \quad L_{2,\text{loc}}(S; L_2(\Omega)), \quad i = \overline{0,n}. \tag{45}$$

Let us show that the function y is a weak solution of problem (10), (11), (13). To do this, we let $m \to \infty$ in identity (34), taking into account (40), (45) and the definition of the function f_m. As a result we obtain identity (12). Now, taking into account (39), we let $m \to +\infty$ in (35). From the resulting inequality and condition (32), we obtain condition (13). Hence, we have proven that y is a weak solution of problem (10), (11), (13). $\square$

3 FORMULATION OF THE OPTIMAL CONTROL PROBLEM AND THE MAIN RESULT

Let $U := L^\infty(Q)$ be a space of controls and $U_\partial := \Big\{ v \in U \,\Big|\, v \geq 0 \quad \text{a. e. in} \quad Q \Big\}$ be the set of admissible controls. We assume that the state of the investigated evolutionary system for a given control $v \in U_\partial$ is described by a weak solution of the equation

$$y_t - \sum_{i=1}^{n} \frac{d}{dx_i} a_i(x,t,y,\nabla y) + a_0(x,t,y,\nabla y) + v(x,t)y = f(x,t), \quad (x,t) \in Q, \tag{46}$$

satisfying conditions (11) and (13) (this problem is similar to problem (10), (11), (13)). This means that y is a function belonging to the space $L^2_{\text{loc}}(S; H^1_0(\Omega)) \cap C(S; L^2(\Omega))$ and satisfying the integral identity

$$\begin{aligned} &\iint\limits_{Q} \Big\{ -y\psi\varphi' + \sum_{i=0}^{n} a_i(x,t,y,\nabla y)\partial_i\psi\varphi + vy\psi\varphi \Big\}\, dxdt \\ &\qquad = \iint\limits_{Q} f\psi\varphi \, dxdt, \quad \psi \in H^1_0(\Omega),\ \varphi \in C^1_c(-\infty, 0), \end{aligned} \tag{47}$$

and condition (13) under assumptions $(\mathcal{A}_1)$–$(\mathcal{A}_3)$, $(\mathcal{F})$.

A weak solution y of the specified problem will be called a weak solution of problem (46), (11), (13) for control v, and will be denoted by $y(v)$, or $y(x,t)$, $(x,t) \in Q$, or $y(x,t;v)$, $(x,t) \in Q$. Further, we assume that condition (32) and the inequality $\omega < K$ hold. From the previous section (see Theorem 1), we immediately obtain the existence and uniqueness of a weak solution of problem (46), (11), (13) (for a given $v \in U_\partial$) and its estimates (15), (16).

We assume that the cost functional has the form

$$J(v) = \|y(\cdot, 0; v) - z_0(\cdot)\|^2_{L^2(\Omega)} + \mu \|v\|_{L^\infty(Q)}, \quad v \in U, \tag{48}$$

where $z_0 \in L^2(\Omega)$, $\mu > 0$ are given.

We consider the following *optimal control problem*: find a control $u \in U_\partial$ such that

$$J(u) = \inf_{v \in U_\partial} J(v). \tag{49}$$

We briefly call this problem (49), and its solutions will be called *optimal controls*.

The main result of this paper is the following theorem.

Theorem 2. *Problem (49) has a solution.*

4 PROOF OF THE MAIN RESULT

Proof of Theorem 2. Since the cost functional J is bounded below, there exists a minimizing sequence $\{v_k\}$ for J in U_∂, i.e., $J(v_k) \underset{k\to\infty}{\longrightarrow} \inf\limits_{v \in U_\partial} J(v)$. This and (48) imply that the sequence $\{v_k\}$ is bounded in the space $L^\infty(Q)$, that is

$$\operatorname*{ess\,sup}_{(x,t)\in Q} |v_k(x,t)| \leq C_7 \text{ for all } k \in \mathbb{N}, \tag{50}$$

where C_7 is a constant, which does not depend on k.

Since for each $k \in \mathbb{N}$ the function $y_k := y(v_k)$ $(k \in \mathbb{N})$ is a weak solution of problem (46), (11), (13) for $v = v_k$, the following identity holds:

$$\iint_Q \Big\{ -y_k\psi\varphi' + \sum_{i=0}^n a_i(y_k)\partial_i\psi\varphi + v_k y_k\psi\varphi\Big\}\,dxdt \\ = \iint_Q f\psi\varphi\,dxdt, \quad \psi \in H_0^1(\Omega),\ \varphi \in C_c^1(-\infty,0). \tag{51}$$

According to Lemma 2 for each $k \in \mathbb{N}$ we have the estimates

$$e^{2\omega\int_0^\tau \alpha(s)\,ds}\|y_k(\cdot,\tau)\|^2_{L_2(\Omega)} \le C_1\int_{-\infty}^{\tau}[\alpha(t)]^{-1}e^{2\omega\int_0^t\alpha(s)\,ds}\|f(\cdot,t)\|^2_{L_2(\Omega)}\,dt, \quad \tau \in S, \tag{52}$$

$$\|y_k\|_{L^2_{\omega,\alpha}(S;H_0^1(\Omega))} \le C_2\|f\|_{L^2_{\omega,1/\alpha}(S;L^2(\Omega))}, \tag{53}$$

where constants C_1, C_2 are independent on $k \in \mathbb{N}$. From $(\mathcal{A}_2)$ and (53) it follows

$$\int_{\tau_1}^{\tau_2}\int_\Omega \sum_{i=0}^n |a_i(y_k)|^2\,dxdt \le C_8\int_{\tau_1}^{\tau_2}\int_\Omega \big(|y_k|^2 + |\nabla y_k|^2 + |h_i|^2\big)\,dxdt \le C_9, \tag{54}$$

where $\tau_1, \tau_2 \in S$ $(\tau_1 < \tau_2)$ are arbitrary, and C_8, C_9 are positive constants independent on k.

Taking into statement *(ii)* of Lemma 1, from (51) for arbitrary $\tau_1, \tau_2 \in S$ $(\tau_1 < \tau_2)$ and $k \in \mathbb{N}$ we obtain

$$\int_{\tau_1}^{\tau_2}\|y_{k,t}\|^2_{H^{-1}(\Omega)}\,dt \le \int_{\tau_1}^{\tau_2}\int_\Omega \Big(\sum_{i=0}^n |a_i(y_k)|^2 + |v_k y_k - f|^2\Big)dxdt. \tag{55}$$

Taking into account condition (32), (50) and (54), estimate (55) implies

$$\int_{\tau_1}^{\tau_2}\|y_{k,t}\|^2_{H^{-1}(\Omega)}\,dt \le C_{10} \text{ for all } k \in \mathbb{N}, \tag{56}$$

where $\tau_1, \tau_2 \in S$ $(\tau_1 < \tau_2)$ are arbitrary, $C_{10} > 0$ is a constant which depends on τ_1 and τ_2, but does not depend on k.

According to the Compactness Lemma (see [19, Proposition 4.2]), and the compactness of the embedding $H_0^1(\Omega) \subset L^2(\Omega)$ (see [18] c. 245), estimates (50), (53), (54), (56) yield that there exists a subsequence of the sequence $\{v_k, y_k\}$ (still denoted by $\{v_k, y_k\}$) and functions $u \in U_\partial$, $y \in L^2_{\omega,\alpha}(S;H_0^1(\Omega))$ and $\chi_i \in L^2_{\text{loc}}(S;L_2(\Omega))$ $(i = \overline{0,n})$ such that

$$v_k \underset{k\to\infty}{\longrightarrow} u \quad *\text{-weakly in} \quad L^\infty(Q), \tag{57}$$

$$y_k \underset{k\to\infty}{\longrightarrow} y \quad \text{weakly in} \quad L^2_{\omega,\alpha}(S;H_0^1(\Omega)), \tag{58}$$

$$y_k \underset{k\to\infty}{\longrightarrow} y \quad \text{strongly in} \quad L^2_{\text{loc}}(S;L^2(\Omega)), \tag{59}$$

$$a_i(y_k) \underset{k\to\infty}{\longrightarrow} \chi_i \quad \text{weakly in} \quad L_{2,\text{loc}}(S;L_2(\Omega)), \quad i = \overline{0,n}. \tag{60}$$

Note that (58) implies the following

$$\partial_i y_k \underset{k\to\infty}{\longrightarrow} \partial_i y \quad \text{weakly in} \quad L^2_{\mathrm{loc}}(S; L^2(\Omega)), \quad i = \overline{0,n}. \tag{61}$$

Let us show that (57) and (59) yield

$$\iint\limits_Q y_k v_k \psi\varphi \, dxdt \underset{k\to\infty}{\longrightarrow} \iint\limits_Q yu\psi\varphi \, dxdt \text{ for all } \psi \in H^1_0(\Omega),\ \varphi \in C^1_c(-\infty, 0). \tag{62}$$

Indeed, let $g := \psi\varphi$ and $t_1, t_2 \in S$ be such that $\operatorname{supp}\varphi \subset [t_1, t_2]$. Then we have

$$\iint\limits_Q y_k v_k g \, dxdt = \int\limits_{t_1}^{t_2}\int\limits_\Omega (y_k v_k - y v_k + y v_k) g \, dxdt = \int\limits_{t_1}^{t_2}\int\limits_\Omega y v_k g \, dxdt + \int\limits_{t_1}^{t_2}\int\limits_\Omega (y_k - y) v_k g \, dxdt. \tag{63}$$

From (50) and (59) it follows

$$\Big| \int\limits_{t_1}^{t_2}\int\limits_\Omega (y_k - y) v_k g \, dxdt \Big| \le \Big(\int\limits_{t_1}^{t_2}\int\limits_\Omega |v_k g|^2 \, dxdt \Big)^{1/2} \Big(\int\limits_{t_1}^{t_2}\int\limits_\Omega |y_k - y|^2 \, dxdt \Big)^{1/2} \underset{k\to\infty}{\longrightarrow} 0. \tag{64}$$

Thus, using (64) and (57), (63) implies (62). Similarly to (62) it can be easily shown that (57) and (59) yield

$$\iint\limits_Q |y_k|^2 v_k \varphi \, dxdt \underset{k\to\infty}{\longrightarrow} \iint\limits_Q |y|^2 u\varphi \, dxdt \text{ for all } \varphi \in C^1_c(-\infty, 0). \tag{65}$$

Using (61), (62), and letting $k \to \infty$ in (51), we obtain

$$\iint\limits_Q \Big\{ -y\psi\varphi' + \sum_{i=0}^n \chi_i \partial_i\psi\varphi + uy\psi\varphi \Big\} \, dxdt = \iint\limits_Q f\psi\varphi \, dxdt, \quad \psi \in H^1_0(\Omega),\ \varphi \in C^1_c(-\infty, 0). \tag{66}$$

According to Lemma 1, identity (66) implies that $y \in C(S; L^2(\Omega))$.

Now let us show that the equality

$$\int\limits_\Omega \Big\{ \sum_{i=0}^n \chi_i \partial_i \psi \Big\} \, dx = \int\limits_\Omega \Big\{ \sum_{i=0}^n a_i(y) \partial_i \psi \Big\} \, dx \tag{67}$$

is valid for every $\psi \in H^1_0(\Omega)$ and for a. e. $t \in S$. For this we use the monotonicity method (see [17]). Let us take an arbitrary functions $w \in L_{2,\mathrm{loc}}(S; H^1(\Omega))$ and $\theta \in C^1_c(-\infty, 0)$, $\theta(t) \ge 0$ for all $t \in (-\infty, 0)$. Using condition $(\mathcal{A}_3)$ for every $k \in \mathbb{N}$ we have

$$W_k := \iint\limits_Q \Big\{ \sum_{i=0}^n (a_i(y_k) - a_i(w))(\partial_i y_k - \partial_i w) \Big\} \theta \, dxdt \ge 0.$$

From this we obtain

$$W_k = \iint\limits_Q \sum_{i=0}^n a_i(y_k)\partial_i y_k \theta \, dxdt - \iint\limits_Q \sum_{i=0}^n \big[a_i(y_k)\partial_i w + a_i(w)(\partial_i y_k - \partial_i w) \big] \theta \ge 0, \ \ k \in \mathbb{N}. \tag{68}$$

According to Lemma 1, (51) implies

$$-\frac{1}{2}\iint\limits_{Q}|y_k|^2\theta'\,dxdt+\iint\limits_{Q}\Big\{\sum_{i=0}^{n}a_i(y_k)\partial_i y_k+v_k|y_k|^2\Big\}\theta\,dxdt=\iint\limits_{Q}fy_k\theta\,dxdt. \tag{69}$$

From (68), using (69), we obtain

$$\begin{aligned}W_k=&\iint\limits_{Q}\Big\{\frac{1}{2}|y_k|^2\theta'+\big(fy_k-v_k|y_k|^2\big)\theta\Big\}dxdt\\&-\iint\limits_{Q}\sum_{i=0}^{n}\big[a_i(y_k)\partial_i w+a_i(w)(\partial_i y_k-\partial_i w)\big]\theta\,dxdt\geq 0,\ k\in\mathbb{N}.\end{aligned}\tag{70}$$

Taking into account (59) and (65) we have

$$\lim_{k\to\infty}\iint\limits_{Q}\Big\{\frac{1}{2}|y_k|^2\theta'+\big(fy_k-v_k|y_k|^2\big)\theta\Big\}\,dxdt=\iint\limits_{Q}\Big\{\frac{1}{2}|y|^2\theta'+\big(fy-u|y|^2\big)\theta\Big\}\,dxdt. \tag{71}$$

By (60), (61) and (71) from (70) we get

$$\begin{aligned}0\leq\lim_{k\to\infty}W_k=&\iint\limits_{Q}\Big\{\frac{1}{2}|y|^2\theta'+\big(fy-u|y|^2\big)\theta\Big\}\,dxdt\\&-\iint\limits_{Q}\sum_{i=0}^{n}\big[\chi_i\partial_i w+a_i(w)(\partial_i y-\partial_i w)\big]\theta dxdt.\end{aligned}\tag{72}$$

From (66), using Lemma 1, we obtain

$$\iint\limits_{Q}\sum_{i=0}^{n}\chi_i\partial_i y\theta\,dxdt=\iint\limits_{Q}\Big\{\frac{1}{2}|y|^2\theta'+\big(fy-u|y|^2\big)\theta\Big\}\,dxdt. \tag{73}$$

Thus, (72) and (73) imply that

$$\iint\limits_{Q}\Big\{\sum_{i=0}^{n}(\chi_i-a_i(w))(\partial_i y-\partial_i w)\Big\}\theta\,dxdt\geq 0. \tag{74}$$

Substituting $w=y-\lambda\psi$ in the above inequality, where $\psi\in H_0^1(\Omega)$, $\lambda>0$ are arbitrary, and dividing the obtained inequality by λ we get

$$\iint\limits_{Q}\Big\{\sum_{i=0}^{n}(\chi_i-a_i(u-\lambda\psi))\partial_i\psi\Big\}\theta\,dxdt\geq 0. \tag{75}$$

Letting $\lambda\to 0+$ in (75), using condition $(\mathcal{A}_2)$ and the Dominated Convergence Theorem (see [9, p. 648]), we have

$$\iint\limits_{Q}\Big\{\sum_{i=1}^{n}(\chi_i-a_i(y))\partial_i\psi\Big\}\theta\,dxdt=0. \tag{76}$$

Since $\psi \in H_0^1(\Omega)$, $\theta \in C_c^1(-\infty,0)$ are arbitrary functions, then (76) impliest (67).

Therefore y is a weak solution of equation (46), satisfying boundary condition (11). Hence, the function y is a weak solution of equation (46) for $v = u$, satisfying boundary condition (11). Let us show that y satisfies condition (13). First, we prove the following convergence:

$$\text{for all } \tau \in S: \quad y_k(\cdot,\tau) \underset{k\to\infty}{\longrightarrow} y(\cdot,\tau) \quad \text{strongly in} \quad L^2(\Omega). \tag{77}$$

For this purpose, we subtract identity (51) from identity (47) with $v = u$, $\psi \in H_0^1(\Omega)$, $\varphi \in C_c^1(-\infty,0)$:

$$\iint\limits_Q \Big\{ -(y-y_k)\psi\varphi' + \sum_{i=0}^n \big(a_i(y) - a_i(y_k)\big)\partial_i\psi\varphi + uy - v_k y_k \Big\}\, dxdt = 0. \tag{78}$$

To the resulting identity (78), we apply Lemma 1 with $\theta(t) = 2(t-\tau+1)$, $\tau_1 = \tau - 1$, $\tau_2 = \tau$, where $\tau \in S$ is any fixed. Consequently, we get

$$\begin{aligned} &\int\limits_\Omega |y(x,\tau) - y_k(x,\tau)|^2\, dx - \int\limits_{\tau-1}^{\tau}\int\limits_\Omega |y-y_k|^2\, dxdt \\ &\quad + \int\limits_{\tau-1}^{\tau}\int\limits_\Omega \Big[\sum_{i=0}^n \big(a_i(y) - a_i(y_k)\big)\partial_i(y-y_k) + (uy - v_k y_k)(y-y_k)\Big]\theta\, dxdt = 0. \end{aligned} \tag{79}$$

From (79), taking into account condition $(\mathcal{A}_3)$ we obtain:

$$\int\limits_\Omega |y(x,\tau) - y_k(x,\tau)|^2\, dx \le \int\limits_{\tau-1}^{\tau}\int\limits_\Omega \big[|y-y_k|^2 - (uy - v_k y_k)(y-y_k)\theta\big]\, dxdt. \tag{80}$$

Inequality (80) implies

$$\int\limits_\Omega |y(x,\tau) - y_k(x,\tau)|^2\, dx \le 2\int\limits_{\tau-1}^{\tau}\int\limits_\Omega \big[(1+v_k)|y-y_k|^2 + |y||u-v_k||y-y_k|\big]\, dxdt. \tag{81}$$

Using (50) and Cauchy-Schwarz inequality, from (81) we obtain

$$\int\limits_\Omega |y(x,\tau) - y_k(x,\tau)|^2\, dx \le C_{11}\Big(\Big[\int\limits_{\tau-1}^{\tau}\int\limits_\Omega |y-y_k|^2\, dxdt\Big]^{1/2} + \int\limits_{\tau-1}^{\tau}\int\limits_\Omega |y-y_k|^2\, dxdt\Big), \tag{82}$$

where $C_{11} > 0$ is a constant which does not depend on k. From (82), according to (59), we get (77). Taking into account (77), let $k \to \infty$ in (52). The resulting inequality, according to condition (32), implies

$$\lim_{\tau\to-\infty} e^{2\omega\int_0^\tau \alpha(s)\, ds} \int\limits_\Omega |y(x,\tau|^2\, dx = 0, \tag{83}$$

that is condition (13) holds. Hence, we have shown that $y = y(u) = y(x,t;u)$, $(x,t) \in Q$, is the state of the controlled system for the control u.

It remains to prove that u is a minimizing element of the functional J. Indeed, (77) implies

$$\|y_k(\cdot,0) - z_0(\cdot)\|^2_{L^2(\Omega)} \underset{k\to\infty}{\longrightarrow} \|y(\cdot,0) - z_0(\cdot)\|^2_{L^2(\Omega)}. \tag{84}$$

Also, (57) and properties of $*$-weakly convergent sequences yield

$$\liminf_{k\to\infty} \|v_k\|_{L^\infty(Q)} \geq \|u\|_{L^\infty(Q)}. \tag{85}$$

From (48), (84) and (85), it easily follows that $\lim\limits_{k\to\infty} J(v_k) \geq J(u)$. Thus, we have shown that u is a solution of problem (49). $\square$

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Бокало М.М., Цебенко А.М. *Задача оптимального керування системами, стан яких описується задачею без початкових умов для нелінійних параболічних рівнянь* // Карпатські матем. публ. — 2016. — Т.8, №1. — С. 21–37.

Досліджено задачу оптимального керування системами, стан яких описується задачею Фур'є для нелінійних параболічних рівнянь. Керування входить як коефіцієнт в рівнянні стану системи. Доведено існування оптимального керування у випадку фінального спостереження.

Ключові слова і фрази: оптимальне керування, задача без початкових умов, нелінійне параболічне рівняння.

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CONTINUOUS BLOCK-SYMMETRIC POLYNOMIALS OF DEGREE AT MOST TWO ON THE SPACE $(L_\infty)^2$

We introduce block-symmetric polynomials on $(L_\infty)^2$ and prove that every continuous block-symmetric polynomial of degree at most two on $(L_\infty)^2$ can be uniquely represented by some "elementary" block-symmetric polynomials.

Key words and phrases: block-symmetric polynomial, symmetric function on L_∞.

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INTRODUCTION

Firstly symmetric functions of infinite number of variables were studied by Nemirovski and Semenov in [5]. Authors considered functions on ℓ_p and L_p spaces. Some of their results were generalized by González, Gonzalo and Jaramillo [2] to real separable rearrangement-invariant function spaces. In [3] Kravtsiv and Zagorodnyuk considered block-symmetric polynomials on ℓ_1-sum of copies of Banach space. In the joint paper of the author with Galindo and Zagorodnyuk [1] the algebra of symmetric analytic functions of bounded type on the complex space L_∞ is studied in detail and its spectrum is described.

A map $P : X \to \mathbb{C}$, where X is a complex Banach space, is called an n-homogeneous polynomial if there exists an n-linear symmetric form $A_P : X^n \to \mathbb{C}$, such that $P(x) = A_P(x, \overset{n}{\ldots}, x)$ for every $x \in X$. Here "symmetric" means that

$$A_P(x_{\tau(1)}, \ldots, x_{\tau(n)}) = A_P(x_1, \ldots, x_n)$$

for every permutation $\tau : \{1, \ldots, n\} \to \{1, \ldots, n\}$. Note that A_P is called the symmetric n-linear form *associated* with P. It is known (see e.g. [4], Theorem 1.10) that A_P can be recovered from P by means of the so-called Polarization Formula:

$$A_P(x_1, \ldots, x_n) = \frac{1}{n!2^n} \sum_{\varepsilon_1,\ldots,\varepsilon_n=\pm 1} \varepsilon_1 \ldots \varepsilon_n P(\varepsilon_1 x_1 + \ldots + \varepsilon_n x_n). \tag{1}$$

In the case $n = 2$ formula (1) can be written as

$$A_P(x_1, x_2) = \frac{1}{4}\Big(P(x_1 + x_2) - P(x_1 - x_2)\Big). \tag{2}$$

It is also convenient to define 0-homogeneous polynomials as constant mappings.

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A mapping $P : X \to \mathbb{C}$ is called a polynomial of degree at most m if it can be represented as

$$P = P_0 + P_1 + \ldots + P_m,$$

where P_j is a j-homogeneous polynomial for $j = 0, \ldots, m$.

Let L_∞ be the *complex* Banach space of all Lebesgue measurable essentially bounded complex-valued functions x on $[0,1]$ with norm

$$\|x\|_\infty = \operatorname{ess\,sup}_{t\in[0,1]} |x(t)|.$$

Let Ξ be the set of all measurable bijections of $[0,1]$ that preserve the measure. A function $F : L_\infty \to \mathbb{C}$ is called Ξ-*symmetric* (or just *symmetric* when the context is clear) if for every $x \in L_\infty$ and for every $\sigma \in \Xi$

$$F(x \circ \sigma) = F(x).$$

The functions $R_n : L_\infty \to \mathbb{C}$ defined by

$$R_n(x) = \int_0^1 x^n(t)\, dt$$

for every $n \in \mathbb{N} \cup \{0\}$ are called the *elementary symmetric polynomials*. In [1] it is shown that for each continuous Ξ-symmetric polynomial $P : L_\infty \to \mathbb{C}$ of degree at most m there is a unique finitely many variables polynomial q such that

$$P(x) = q(R_0(x), \ldots, R_m(x))$$

for every $x \in L_\infty$.

Let $(L_\infty)^2$ be the Cartesian square of the space L_∞, endowed with norm $\|(x,y)\| = \max\{\|x\|_\infty, \|y\|_\infty\}$. Clearly, $(L_\infty)^2$ is a complex Banach space. A function $F : (L_\infty)^2 \to \mathbb{C}$ we call *block-symmetric* if for every $(x,y) \in (L_\infty)^2$ and for every $\sigma \in \Xi$

$$F((x \circ \sigma, y \circ \sigma)) = F((x,y)).$$

We restrict our attention to continuous block-symmetric polynomials of degree at most two on $(L_\infty)^2$. In Section 1 we prove that every such a polynomial can be uniquely represented as an algebraic combination of the polynomials

$$R_0((x,y)) = 1, \quad R_{10}((x,y)) = R_1(x), \quad R_{01}((x,y)) = R_1(y),$$
$$R_{20}((x,y)) = R_2(x), \quad R_{11}((x,y)) = \int_0^1 x(t)y(t)\, dt, \quad R_{02}((x,y)) = R_2(y),$$

which we call *the elementary block-symmetric polynomials of degree at most two.*

1 THE MAIN RESULT

By $\mathbf{1}_E$ we denote the characteristic function of a set $E \subset [0,1]$. We also define functions $\mathbf{1} = \mathbf{1}_{[0,1]}$ and $\mathbf{r} = \mathbf{1}_{[0,\frac{1}{2}]} - \mathbf{1}_{[\frac{1}{2},1]}$.

Theorem 1. *Every continuous block-symmetric polynomial $P = P_0 + P_1 + P_2$, where P_j is a j-homogeneous polynomial for $j = 0, 1, 2$, can be represented as*

$$P = a_0 R_{00} + a_{10} R_{10} + a_{01} R_{01} + a_{20} R_{20} + a_{11} R_{11} + a_{02} R_{02} + a_{1010} R_{10}^2 + a_{1001} R_{10} R_{01} + a_{0101} R_{01}^2,$$

where

$$\begin{aligned}
&a_0 = P_0, \quad a_{10} = P_1((\mathbf{1}, 0)), \quad a_{01} = P_1((0, \mathbf{1})),\\
&a_{20} = P_2((\mathbf{r}, 0)), \quad a_{11} = A_{P_2}((\mathbf{r}, 0), (0, \mathbf{r})), \quad a_{02} = P_2((0, \mathbf{r})),\\
&a_{1010} = P_2((\mathbf{1}, 0)) - P_2((\mathbf{r}, 0)), \quad a_{1001} = A_{P_2}((\mathbf{1}, 0), (0, \mathbf{1})) - A_{P_2}((\mathbf{r}, 0), (0, \mathbf{r})),\\
&a_{0101} = P_2((0, \mathbf{1})) - P_2((0, \mathbf{r})).
\end{aligned}$$

Here we denote by A_{P_2} the symmetric bilinear form, associated with P_2.

Proof. It can be easily checked that

$$\begin{aligned}
&P_0((x, y)) = P((0, 0)), \quad P_1((x, y)) = \frac{1}{2}\Big(P((x, y)) - P((-x, -y))\Big),\\
&P_2((x, y)) = P((x, y)) - P_1((x, y)) - P_0((x, y))
\end{aligned}$$

for every $(x, y) \in (L_\infty)^2$. This implies that P_0, P_1 and P_2 are continuous and block-symmetric.

By the linearity of P_1

$$P_1((x, y)) = P_1((x, 0) + (0, y)) = P_1((x, 0)) + P_1((0, y)).$$

Let $f_1(x) = P_1((x, 0))$ for $x \in L_\infty$. Clearly, f_1 is a continuous linear Ξ-symmetric functional on L_∞. It is known (see [1, 6]) that every such a functional f can be represented as

$$f(x) = f(\mathbf{1}) R_1(x). \tag{3}$$

Therefore $f_1(x) = f_1(\mathbf{1}) R_1(x)$, i. e. $P_1((x, 0)) = P_1((\mathbf{1}, 0)) R_1(x)$. Analogously, $P_1((0, y)) = P_1((0, \mathbf{1})) R_1(y)$. Thus

$$P_1((x, y)) = P_1((\mathbf{1}, 0)) R_1(x) + P_1((0, \mathbf{1})) R_1(y) = a_{10} R_{10}((x, y)) + a_{01} R_{01}((x, y)).$$

Since A_{P_2} is bilinear and symmetric, it follows that

$$P_2((x, y)) = A_{P_2}((x, 0), (x, 0)) + 2A_{P_2}((x, 0), (0, y)) + A_{P_2}((0, y), (0, y)).$$

We define following bilinear forms:

$$\begin{aligned}
&B_I(x_1, x_2) = A_{P_2}((x_1, 0), (x_2, 0)), \quad B_{II}(x_1, x_2) = A_{P_2}((x_1, 0), (0, x_2)),\\
&B_{III}(x_1, x_2) = A_{P_2}((0, x_1), (0, x_2)),
\end{aligned} \tag{4}$$

where $x_1, x_2 \in L_\infty$. Note that B_I and B_{III} are symmetric. By the formula (2)

$$A_{P_2}((x_1, y_1), (x_2, y_2)) = \frac{1}{4}\Big(P_2((x_1 + x_2, y_1 + y_2)) - P_2((x_1 - x_2, y_1 - y_2))\Big).$$

Therefore by the symmetry of P_2

$$A_{P_2}((x_1 \circ \sigma, y_1 \circ \sigma), (x_2 \circ \sigma, y_2 \circ \sigma)) = A_{P_2}((x_1, y_1), (x_2, y_2)) \tag{5}$$

for every $\sigma \in \Xi$ and $(x_1, y_1), (x_2, y_2) \in (L_\infty)^2$. By (5) we have that

$$B_j(x_1 \circ \sigma, x_2 \circ \sigma) = B_j(x_1, x_2), \tag{6}$$

for every $j \in \{I, II, III\}$, $x_1, x_2 \in L_\infty$ and $\sigma \in \Xi$.

Let Q_I be the restriction of B_I to the diagonal. By the continuity of B_I and by (6) we have that Q_I is a continuous 2-homogeneous Ξ-symmetric polynomial. It is known (see [1]) that every continuous 2-homogeneous Ξ-symmetric polynomial Q on L_∞ can be represented as

$$Q = \alpha R_1^2 + \beta R_2. \tag{7}$$

It can be easily checked that $\alpha = Q(\mathbf{1}) - Q(\mathbf{r})$ and $\beta = Q(\mathbf{r})$. Note that

$$Q_I(x) = A_{P_2}((x,0),(x,0)) = P_2((x,0)).$$

Thus

$$\begin{aligned} A_{P_2}((x,0),(x,0)) &= \Big(P_2((\mathbf{1},0)) - P_2((\mathbf{r},0))\Big) R_1^2(x) + P_2((\mathbf{r},0))R_2(x) \\ &= a_{1010} R_{10}^2((x,y)) + a_{20} R_{20}((x,y)). \end{aligned}$$

Analogously

$$A_{P_2}((0,y),(0,y)) = a_{0101} R_{10}^2((x,y)) + a_{02} R_{20}((x,y)).$$

The bilinear form B_{II} can be represented as the sum of the symmetric and the antisymmetric forms

$$B_{II}^s(x_1,x_2) = \frac{1}{2}\Big(B_{II}(x_1,x_2) + B_{II}(x_2,x_1)\Big)$$

and

$$B_{II}^a(x_1,x_2) = \frac{1}{2}\Big(B_{II}(x_1,x_2) - B_{II}(x_2,x_1)\Big)$$

respectively. Let us prove that $B_{II}^a(x_1,x_2) = 0$ for every $x_1, x_2 \in L_\infty$.

Lemma 1. $B_{II}^a(\mathbf{1}_{[0,\frac{1}{2}]}, \mathbf{1}_{[\frac{1}{2},1]}) = 0$.

Proof. Let $\sigma(t) = 1 - t$. By (6) $B_{II}^a(\mathbf{1}_{[0,\frac{1}{2}]}, \mathbf{1}_{[\frac{1}{2},1]}) = B_{II}^a(\mathbf{1}_{[0,\frac{1}{2}]} \circ \sigma, \mathbf{1}_{[\frac{1}{2},1]} \circ \sigma) = B_{II}^a(\mathbf{1}_{[\frac{1}{2},1]}, \mathbf{1}_{[0,\frac{1}{2}]})$. On the other hand, since B_{II}^a is antisymmetric, it follows that

$$B_{II}^a(\mathbf{1}_{[0,\frac{1}{2}]}, \mathbf{1}_{[\frac{1}{2},1]}) = -B_{II}^a(\mathbf{1}_{[\frac{1}{2},1]}, \mathbf{1}_{[0,\frac{1}{2}]}).$$

Therefore $B_{II}^a(\mathbf{1}_{[0,\frac{1}{2}]}, \mathbf{1}_{[\frac{1}{2},1]}) = 0$. $\square$

Lemma 2. $B_{II}^a(\mathbf{1}_E, \mathbf{1}_F) = 0$ *for every measurable sets* $E \subset [0, \frac{1}{2}]$ *and* $F \subset [\frac{1}{2}, 1]$.

Proof. For every $x \in L_\infty$ we define $\widehat{x} \in L_\infty$ by

$$\widehat{x}(t) = \begin{cases} x(2t), & \text{if } t \in [0, \frac{1}{2}], \\ 0, & \text{if } t \in (\frac{1}{2}, 1]. \end{cases}$$

Let $z \in L_\infty$ be such that its restriction to $[0, \frac{1}{2})$ is constant. Let $f_z(x) = B_{II}^a(\widehat{x}, z)$, where $x \in L_\infty$. Clearly, f_z is a continuous linear functional on L_∞. Let us prove that f_z is Ξ-symmetric. For every $\sigma \in \Xi$ let

$$\widetilde{\sigma}(t) = \begin{cases} \frac{1}{2}\sigma(2t), & \text{if } t \in [0, \frac{1}{2}], \\ t, & \text{if } t \in (\frac{1}{2}, 1]. \end{cases}$$

Clearly, $\widetilde{\sigma} \in \Xi$ and $z \circ \widetilde{\sigma} = z$. It can be checked that $\widehat{x \circ \sigma} = \widehat{x} \circ \widetilde{\sigma}$. Therefore by (6)

$$f_z(x \circ \sigma) = B_{II}^a(\widehat{x \circ \sigma}, z) = B_{II}^a(\widehat{x} \circ \widetilde{\sigma}, z \circ \widetilde{\sigma}) = B_{II}^a(\widehat{x}, z) = f_z(x).$$

Thus f_z is Ξ-symmetric. By (3) $f_z(x) = f_z(\mathbf{1})R_1(x)$, i. e. $B_{II}^a(\widehat{x}, z) = B_{II}^a(\widehat{\mathbf{1}}, z)R_1(x)$. Since $\widehat{\mathbf{1}} = \mathbf{1}_{[0,\frac{1}{2}]}$, $\widehat{\mathbf{1}_{2E}} = \mathbf{1}_E$ and $R_1(\mathbf{1}_{2E}) = 2\mu(E)$, where $2E = \{2t : t \in E\}$, it follows that

$$B_{II}^a(\mathbf{1}_E, z) = B_{II}^a(\mathbf{1}_{[0,\frac{1}{2}]}, z)2\mu(E).$$

Analogously it can be proven that $B_{II}^a(u, \mathbf{1}_F) = B_{II}^a(u, \mathbf{1}_{[\frac{1}{2},1]})2\mu(F)$, where $u \in L_\infty$ such that its restriction to $(\frac{1}{2}, 1]$ is constant. Therefore

$$B_{II}^a(\mathbf{1}_E, \mathbf{1}_F) = B_{II}^a(\mathbf{1}_{[0,\frac{1}{2}]}, \mathbf{1}_F)2\mu(E) = B_{II}^a(\mathbf{1}_{[0,\frac{1}{2}]}, \mathbf{1}_{[\frac{1}{2},1]})4\mu(E)\mu(F) = 0$$

by Lemma 1. □

Lemma 3. *$B_{II}^a(\mathbf{1}_E, \mathbf{1}_F) = 0$ for disjoint measurable sets $E, F \subset [0,1]$ such that $\mu(E) \leq \frac{1}{2}$ and $\mu(F) \leq \frac{1}{2}$.*

Proof. By [1, Proposition 1.2] there exists $\sigma_{E,F} \in \Xi$ such that $\mathbf{1}_E = \mathbf{1}_{[0,a]} \circ \sigma_{E,F}$ and $\mathbf{1}_F = \mathbf{1}_{[a,a+b]} \circ \sigma_{E,F}$, where $a = \mu(E)$ and $b = \mu(F)$. Let

$$\sigma_1(t) = \begin{cases} t - a + \frac{1}{2}, & \text{if } t \in [a, a+b], \\ t - \frac{1}{2} + a, & \text{if } t \in [\frac{1}{2}, \frac{1}{2} + b], \\ t, & \text{otherwise}. \end{cases}$$

Clearly, $\sigma_1 \in \Xi$, $\mathbf{1}_{[0,a]} = \mathbf{1}_{[0,a]} \circ \sigma_1$ and $\mathbf{1}_{[a,a+b]} = \mathbf{1}_{[\frac{1}{2},\frac{1}{2}+b]} \circ \sigma_1$. Therefore $\mathbf{1}_E = \mathbf{1}_{[0,a]} \circ \sigma_1 \circ \sigma_{E,F}$ and $\mathbf{1}_F = \mathbf{1}_{[\frac{1}{2},\frac{1}{2}+b]} \circ \sigma_1 \circ \sigma_{E,F}$. By (6) and by Lemma 2

$$B_{II}^a(\mathbf{1}_E, \mathbf{1}_F) = B_{II}^a(\mathbf{1}_{[0,a]} \circ \sigma_1 \circ \sigma_{E,F}, \mathbf{1}_{[\frac{1}{2},\frac{1}{2}+b]} \circ \sigma_1 \circ \sigma_{E,F}) = B_{II}^a(\mathbf{1}_{[0,a]}, \mathbf{1}_{[\frac{1}{2},\frac{1}{2}+b]}) = 0.$$

□

Lemma 4. *$B_{II}^a(\mathbf{1}_E, \mathbf{1}_F) = 0$ for every disjoint measurable sets $E, F \subset [0,1]$.*

Proof. If $\mu(E) = \mu(F)$, then $\mu(E)$ and $\mu(F)$ cannot be greater than $\frac{1}{2}$ and $B_{II}^a(\mathbf{1}_E, \mathbf{1}_F) = 0$ by Lemma 3. Note that $B_{II}^a(\mathbf{1}_E, \mathbf{1}_F) = 0$ if $\mu(E) = 0$ or $\mu(F) = 0$. Let $\mu(E) > \mu(F) > 0$. Let $N = \left\lfloor \frac{\mu(E)}{\mu(F)} \right\rfloor$. We can choose disjoint measurable subsets $E_1, \ldots, E_N \subset E$ such that $\mu(E_1) = \ldots = \mu(E_N) = \mu(F)$. Set $E_0 = E \setminus \cup_{j=1}^N E_j$. Then

$$B_{II}^a(\mathbf{1}_E, \mathbf{1}_F) = \sum_{j=0}^N B_{II}^a(\mathbf{1}_{E_j}, \mathbf{1}_F) = B_{II}^a(\mathbf{1}_{E_0}, \mathbf{1}_F).$$

Since $\mu(E_0) < \mu(F) < \frac{1}{2}$, it follows that $B_{II}^a(\mathbf{1}_{E_0}, \mathbf{1}_F) = 0$ by Lemma 3. □

Lemma 5. *$B_{II}^a(\mathbf{1}_E, \mathbf{1}_F) = 0$ for every measurable sets $E, F \subset [0,1]$.*

Proof. Note that $E = (E \setminus F) \sqcup (E \cap F)$ and $F = (F \setminus E) \sqcup (E \cap F)$. Therefore

$$B_{II}^a(\mathbf{1}_E, \mathbf{1}_F) = B_{II}^a(\mathbf{1}_{E \setminus F}, \mathbf{1}_{F \setminus E}) + B_{II}^a(\mathbf{1}_{E \setminus F}, \mathbf{1}_{E \cap F}) + B_{II}^a(\mathbf{1}_{E \cap F}, \mathbf{1}_{F \setminus E}) + B_{II}^a(\mathbf{1}_{E \cap F}, \mathbf{1}_{E \cap F}) = 0$$

by Lemma 4 and by the antisymmetry of B_{II}^a. □

Proof of the Theorem 1 (continuation). For the simple measurable functions $x_1, x_2 \in L_\infty$ we have $B^a_{II}(x_1, x_2) = 0$ by the bilinearity of B^a_{II}. Since the set of simple measurable functions is dense in L_∞, the continuity of B^a_{II} leads to $B^a_{II}(x_1, x_2) = 0$ for every $x_1, x_2 \in L_\infty$. Thus $B_{II} = B^s_{II}$, i. e. B_{II} is symmetric. Let Q_{II} be the restriction of B_{II} to the diagonal. Q_{II} is a continuous 2-homogeneous Ξ-symmetric polynomial. Therefore by (7) $Q_{II}(x) = (Q_{II}(\mathbf{1}) - Q_{II}(\mathbf{r}))R_1^2(x) + Q_{II}(\mathbf{r})R_2(x)$.

By (2) $B_{II}(x,y) = \frac{1}{4}\Big(Q_{II}(x+y) - Q_{II}(x-y)\Big)$. Since

$$B_{II}(x,y) = A_{P_2}((x,0),(0,y)), \quad Q_{II}(\mathbf{1}) = A_{P_2}((\mathbf{1},0),(0,\mathbf{1})), \quad Q_{II}(\mathbf{r}) = A_{P_2}((\mathbf{r},0),(0,\mathbf{r})),$$
$$R_1^2(x+y) - R_1^2(x-y) = 4R_1(x)R_1(y), \quad R_2(x+y) - R_2(x-y) = 4\int_0^1 x(t)y(t)\,dt,$$

it follows that

$$\begin{aligned} A_{P_2}((x,0),(0,y)) &= \left(A_{P_2}((\mathbf{1},0),(0,\mathbf{1})) - A_{P_2}((\mathbf{r},0),(0,\mathbf{r}))\right) R_1(x)R_1(y) \\ &+ A_{P_2}((\mathbf{r},0),(0,\mathbf{r}))\int_0^1 x(t)y(t)\,dt = a_{1001}R_{10}((x,y))R_{01}((x,y)) + a_{11}R_{11}((x,y)). \end{aligned}$$

□

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Василишин Т.В. *Неперервні блочно-симетричні поліноми степеня щонайбільше два на просторі $(L_\infty)^2$* // Карпатські матем. публ. — 2016. — Т.8, №1. — С. 38–43.

Введено поняття блочно-симетричного полінома на просторі $(L_\infty)^2$ і показано, що кожен неперервний блочно-симетричний поліном степеня щонайбільше два на просторі $(L_\infty)^2$ можна єдиним чином виразити через деякі "елементарні" блочно-симетричні поліноми.

Ключові слова і фрази: блочно-симетричний поліном, симетрична функція на L_∞.

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CONTINUOUS APPROXIMATIONS OF CAPACITIES ON METRIC COMPACTA

A method of "almost optimal" continuous approximation of capacities on a metric compactum with possibility measures, necessity measures, or with capacities on a closed subspace, is presented.
Key words and phrases: capacity, metric compactum, approximation.

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INTRODUCTION

Capacities were introduced by Choquet [1] and found numerous applications in different theories. Spaces of upper semicontinuous capacities on compacta were systematically studied in [2]. In particular, in the latter paper functoriality of the construction of a space of capacities was proved and Prokhorov-style and Kantorovich-Rubinstein-style metrics on the set of capacities on a metric compactum were introduced. Needs of practice require that a capacity can be approximated with capacities of simpler structure or with some convenient properties. It was shown in[3] that each normalized capacity on a compactum is the value of a so-called $\cup$-capacity (or possibility measure) on the space of $\cap$-capacities (necessity measures) under the multiplication mapping of the capacity monad. Nevertheless it is impossible to represent every capacity in this manner using only capacities of one of the two mentioned classes. We can discuss only approximation of an arbitrary capacity with $\cup$- or $\cap$-capacities. A construction of the capacity from the class of $\cup$- or $\cap$-capacities that is the closest to the given one w.r.t. the Prokhorov metric was described in [4]. A method of optimal approximation of a capacity with a capacity on a closed subspace was also presented there. Although the proposed approximations are optimal (belong to the optimal ones), they does not depend continuously on the original capacity. In this paper we consider the problem of *continuous* approximation. It is proved that the space $\underline{M}X$ of subnormalized capacities on a metric compactum X is an I-convex compactum, hence all elements of $\underline{M}X$ can be approximated with "almost optimal" precision with elements of an arbitrary closed I-convex subset $X_0 \subset \underline{M}X$, in particular, with $\cup$-capacities, $\cap$-capacities, or capacities on a fixed closed subspace $X_0 \subset X$, so that the approximation is continuous w.r.t. the original capacity and the chosen "tolerance".

1 BASIC FACTS AND DEFINITIONS

We follow the terminology and notation of [2] and denote by $\exp X$ the set of all non-empty closed subsets of a compactum X. The set $\exp X$ is considered with the Vietoris topology. If

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a metric d on X is admissible, then the Hausdorff metric $\hat{d}$ is admissible on $\exp X$. For a point x in (X,d) and a non-empty subset $S \subset X$ we denote $d(x,S) = \inf\{d(x,x') \mid x' \in S\}$, and I is the unit segment $[0,1]$.

We call a function $c : \exp X \cup \{\varnothing\} \to I$ a *capacity* on a compactum X if the three following properties hold for all subsets $F, G \underset{\text{cl}}{\subset} X$:

1. $c(\varnothing) = 0$;
2. if $F \subset G$, then $c(F) \le c(G)$ (monotonicity);
3. if $c(F) < a$, then there is an open subset $U \supset F$ such that for all $G \subset U$ the inequality $c(G) < a$ is valid (upper semicontinuity).

If, additionally, $c(X) = 1$ (or $c(X) \le 1$) holds, then the capacity is called *normalized* (resp. *subnormalized*).

We denote by MX and $\underline{M}X$ the sets of all normalized and of all subnormalized capacities respectively. It was shown in [2] that MX carries a compact Hausdorff topology with the subbase of all sets of the form

$$O_-(F,a) = \{c \in MX \mid c(F) < a\}, \text{ where } F \underset{\text{cl}}{\subset} X, a \in I,$$

and

$$\begin{aligned} O_+(U,a) &= \{c \in MX \mid c(U) > a\} \\ &= \{c \in MX \mid \text{there is a compactum } F \subset U, c(F) > a\}, \text{where } U \underset{\text{op}}{\subset} X, a \in I. \end{aligned}$$

The same formulae determine a subbase of a compact Hausdorff topology on $\underline{M}X$ and therefore $MX \subset \underline{M}X$ is a subspace.

We consider the following subclasses of MX.

1. $M_\cap X$ is the set of the so-called $\cap$-*capacities* (or necessity measures) with the property: $c(A \cap B) = \min\{c(A), c(B)\}$ for all $A, B \underset{\text{cl}}{\subset} X$.
2. $M_\cup X$ is the set of the so-called $\cup$-*capacities* (or possibility measures) with the property: $c(A \cup B) = \max\{c(A), c(B)\}$ for all $A, B \underset{\text{cl}}{\subset} X$.
3. Class MX_0 of capacities defined *on a closed subspace* $X_0 \subset X$. We regard each capacity c_0 on X_0 as a capacity on X extended with the formula $c(F) = c_0(F \cap X_0)$, $F \underset{\text{cl}}{\subset} X$.

Analogous subclasses are defined in $\underline{M}X$, with the obvious denotations. It was proved in [3] that the subsets $M_\cap X$, $M_\cup X$, and MX_0 are closed in MX, hence for a compactum X they are compacta as well.

From now on we restrict to $\underline{M}X$, results for MX are quite analogous. We consider the metric on the set $\underline{M}X$ of subnormalized capacities on a metric compactum (X,d) :

$$\hat{d}(c,c') = \inf\{\varepsilon > 0 \mid c(\bar{O}_\varepsilon(F)) + \varepsilon \ge c'(F), c'(\bar{O}_\varepsilon(F)) + \varepsilon \ge c(F), \forall F \underset{\text{cl}}{\subset} X\}.$$

Here $\bar{O}_\varepsilon(F)$ is the closed ε-neighborhood of a subset $F \subset X$. This metric is admissible [2]. Recall some definitions and well-known facts on compact topological semilattices and compact idempotent semimodules.

A poset $(X, \leq)$ is called an *upper semilattice* is pairwise suprema $x \vee y$ exist for all $x, y \in X$. A subset Y of an upper semilattice Y is called an *upper subsemilattice* if the supremum of each two elements of Y is in Y. Then Y is an upper semilattice as well, and suprema of all finite non-empty subsets of Y in X and in Y exist and are equal.

An upper semilattice $(X, \leq)$ is called *topological* if a topology is fixed on X such that the pairwise supremum $x \vee y$ depends on $x, y \in X$ continuously.

A topological semilattice is called *Lawson* [7] if in each its point it possesses a local base consisting of subsemilattices.

An upper semilattice is *complete* if each it non-empty subset has the least upper bound. It is well-known that any compact topological upper semilattice is complete and contains agreatest element [6]. A compact Hausdorff topological upper semilattice X is Lawson if and only if the mapping $\sup : \exp X \to X$ that assigns the least upper bound to each non-empty closed subset $A \subset X$ is continuous w.r.t. the Vietoris topology.

We call $(X, \oplus, \circledast)$ a (left idempotent) $(I, \max, *)$*-semimodule* if X is a set with operations $\oplus : X \times X \to X$, $\circledast : I \times X \to X$ such that for all $x, y, z \in X$, $\alpha, \beta \in I$ the following holds:

1. $x \oplus y = y \oplus x$;
2. $(x \oplus y) \oplus z = x \oplus (y \oplus z)$;
3. there is a unique $\bar{0} \in X$ such that $x \oplus \bar{0} = x$ for all x;
4. $\alpha \circledast (x \oplus y) = (\alpha \circledast x) \oplus (\alpha \circledast y)$, $\max\{\alpha, \beta\} \circledast x = (\alpha \circledast x) \oplus (\beta \circledast x)$;
5. $(\alpha * \beta) \circledast x = \alpha \circledast (\beta \circledast x)$;
6. $1 \circledast x = x$;
7. $0 \circledast x = \bar{0}$.

In the sequel we use a shorter term "I-semimodule" for $(I, \max, *)$-semimodule.

A triple $(X, \oplus, \circledast)$ is called *a compact Hausdorff Lawson I-semimodule* if $(X, \oplus, \circledast)$ is an I-semimodule and a compact Hausdorff topology is fixed on X that makes it a compact Lawson upper semilattice with $\oplus$ being pairwise supremum (hence the partial order is defined as $x \leq y \Leftrightarrow x \oplus y = y$), and the multiplication $\circledast$ is continuous.

For all points $x_1, x_2, \ldots, x_n \in X$ and coefficients $\alpha_1, \alpha_2, \ldots, \alpha_n \in I$ such that $\max\{\alpha_1, \alpha_2, \ldots, \alpha_n\} = 1$ we define *the I-convex combination* of a finite number of elements $\alpha_1 \circledast x_1 \oplus \alpha_2 \circledast x_2 \oplus \ldots \oplus \alpha_n \circledast x_n$, which from now on is denoted simply as $\alpha_1 x_1 \oplus \alpha_2 x_2 \oplus \ldots \oplus \alpha_n x_n$. It can be calculated stepwise using pairwise convex combinations of the form $x \oplus \alpha y$, which in fact are values of a mapping $X \times I \times X \to X$.

If the mentioned pairwise I-convex combination is continuous, then $(X, \oplus, \circledast)$ is called an *I-convex compactum* [5]. Hence an I-convex compactum is a compact Hausdorff space X with a Lawson continuous pairwise I-convex combination $(x, \alpha, y) \mapsto x \oplus \alpha y$, $X \times I \times X \to X$, which (for $\alpha = 1$) makes X a compact Hausdorff Lawson upper semilattice.

In compact Hausdorff Lawson I-semimodules we can define an I-convex combination of an infinite number of elements using finite combinations as follows:

$$\bigoplus_{i \in \mathcal{I}} \alpha_i x_i = \inf\{\sup_{i \in \mathcal{I}_1} \alpha_i \circledast \sup_{i \in \mathcal{I}_1} x_i \oplus \ldots \oplus \sup_{i \in \mathcal{I}_n} \alpha_i \circledast \sup_{i \in \mathcal{I}_n} x_i \mid n \in \mathbb{N}, \mathcal{I} = \mathcal{I}_1 \cup \mathcal{I}_2 \cup \ldots \cup \mathcal{I}_n\}.$$

Observe that the above I-convex combination does not depend on $\alpha_i x_i$ such that the respective α_i are equal to zero. Theorem [5, 5.9.2] implies an important property of the mapping that sends each collection of elements with coefficients to their I-convex combination.

Lemma 1. *Let $(X, \oplus, \circledast)$ be an I-convex compactum and $\exp_1(X \times I) \subset \exp(X \times I)$ the subspace of all closed subsets of $X \times I$ that contain at least one pair of the form $(x, 1)$. Then the mapping $h : \exp_1(X \times I) \to X$ defined for $\mathcal{A} \underset{\mathrm{cl}}{\subset} X \times I$ by the formula*

$$h(\mathcal{A}) = \bigoplus_{i \in \mathcal{I}} \{\alpha_i x_i | (x_i, \alpha_i) \in \mathcal{A}\}$$

is continuous.

2 SOME MAPPINGS IN METRIC I-CONVEX COMPACTA

We need some auxilliary statements. Let $S \subset X$ be a non-empty closed I-convex subset of a metric I-convex compactum $(X, \oplus, \circledast)$, i.e. S contains all I-convex combinations of its elements. Then S is known [5] to be an I-convex compactum as well. For the product topology on $X \times \mathbb{R}$ the metric $\rho((x_1, a_1), (x_2, a_2)) = \max\{d(x_1, x_2), |a_1 - a_2|\}$ is admissible.

For an element $x \in X$ consider the set $\mathfrak{F}_x = \{(x', a) | x \in S, d(x, x') \le a \le \operatorname{diam} X\}$.

Proposition 1. *The set $\mathfrak{F}_x \subset S \times [0, \operatorname{diam} X]$ is closed and the mapping $f : X \to \exp(S \times [0, \operatorname{diam} X])$ that assigns $\mathfrak{F}_x$ to each $x \in X$ is continuous.*

The proof relies on the two following lemmas.

Lemma 2. *Let (X, d) be a metric compactum, then for all $x \in X$ the set $\mathcal{F}_x = \{(x', a) | x' \in X, d(x, x') \le a \le \operatorname{diam} X\}$ is non-empty and closed in $X \times [0, \operatorname{diam} X]$.*

Proof. Obviously $(x', \operatorname{diam} X) \in \mathfrak{F}_x$ for all $x' \in X$, hence the set in question is non-empty. We show that the complement $X \times [0, \operatorname{diam} X] \setminus \mathcal{F}_x$ is open. Let a point (x', a) belong to the complement, i.e. $d(x, x') > a$. Put $\varepsilon = \dfrac{d(x, x') - a}{2}$. Then $\varepsilon > 0$ and for any point (y, b) in the ε-neighborhood of (x', a), which is a ball $B_\varepsilon(x') \times (a - \varepsilon, a + \varepsilon)$, the inequalities $d(y, x) \ge d(x', x) - d(x', y) > (a + 2\varepsilon) - \varepsilon = a + \varepsilon > b$ are valid. Hence the ε-neighborhood of the point (x', a) is contained in the set $X \times [0, \operatorname{diam} X] \setminus \mathcal{F}_x$. □

Therefore the set $\mathfrak{F}_x = \mathcal{F}_x \cap (S \times I)$ is non-empty and closed in $S \times [0, \operatorname{diam} X]$ as well.

Lemma 3. *Let (X, d) be a metric compactum and S its non-empty closed subset, then the mapping f from X to the space $\exp(S \times [0, \operatorname{diam} X])$ of all non-empty closed subsets with the Hausdorff metric that sends each $x \in X$ to the set $\mathfrak{F}_x$, is non-expanding.*

Proof. Let $x, y \in X$, $x \ne y$, hence $r = d(x, y) > 0$. If $(x', a) \in \mathfrak{F}_x$, i.e. $d(x, x') \le a$, put $b = \min\{a + r, \operatorname{diam} X\}$. Thus $|b - a| = \rho((x', a), (x', b)) \le r$ and $d(y, x') \le d(x, y) + d(x, x') = d(x, x') + r$. Taking into account $d(y, x') \le \operatorname{diam} X$ we deduce $d(y, x') \le b$, hence $(x', b) \in \mathfrak{F}_y$.

Thus for each point $(x', a) \in \mathfrak{F}_x$ there is a point $(x', b) \in \mathfrak{F}_y$ at a distance $\le r$, and vice versa. Thus the Hausdorff distance ρ_H between $\mathfrak{F}_x$ and $\mathfrak{F}_y$ does not exceed $r = d(x, y)$, i.e. f is non-expanding. This completes the proof. □

Assign to all $x \in X$ and $\varepsilon > 0$ the set $\mathfrak{G}_x \subset S \times I$ of the form

$$\mathfrak{G}_x = \left\{ (x', \alpha) | x' \in S,\ \alpha \in I,\ \alpha \leq \max \left\{ 0, 1 - \frac{d(x,x') - d(x,S)}{\varepsilon} \right\} \right\}.$$

Observe that a point (x', α), with $\alpha > 0$, can belong to $\mathfrak{G}_x$ only if $x' \in S$, $d(x,x') < d(x,S) + \varepsilon$.

Proposition 2. *The following statements hold:*

(1) the set $\mathfrak{G}_x$ is closed in $S \times I$;

(2) the mapping $g : X \times (0, +\infty) \to \exp(S \times I)$ that assigns $\mathfrak{G}_x$ to each element $x \in X$ and $\varepsilon > 0$ is continuous;

(3) for all $x \in X$, $\varepsilon > 0$ the equality $\max\{\alpha \in I \mid (x', \alpha) \in \mathfrak{G}_x \text{ for some } x' \in S\} = 1$ is valid.

Proof. The set $\mathfrak{G}_x \subset S \times I$ is the image of the set $\mathfrak{F}_x \subset S \times [0, \operatorname{diam} X]$, namely $\mathfrak{G}_x = (1_X \times \theta_{x,\varepsilon})(\mathfrak{F}_x)$, where $\theta_{x,\varepsilon} : [0, \operatorname{diam} X] \to I$ is defined by the formula $\theta_{x,\varepsilon}(a) = \max\{1 - \dfrac{a - d(x,S)}{\varepsilon}, 0\}$. Hence $\mathfrak{G}_x$ is closed as the image of a closed set under a continuous mapping of compacta (1). Moreover $\mathfrak{F}_x$ and $\theta_{x,\varepsilon}$ depend on x and ε continuously, therefore the same holds for $\mathfrak{G}_x$ (2). Compactness of $S \subset X$ implies existence of $x' \in S$ such that $d(x,x') = d(x,S)$, hence $(x', 1) \in \mathfrak{G}_x$ (3). ☐

Proposition 3. *The mapping $\Phi : X \times (0, +\infty) \to S$ defined as*

$$\Phi(x, \varepsilon) = \bigoplus_{i \in \mathcal{I}} \{\alpha_i x_i | (x_i, \alpha_i) \in \mathfrak{G}_x\}$$

is continuous.

Proof. Continuity of Φ is a corollary of Proposition 2 and Lemma 1 because Φ is the composition of the continuous mappings g and h (cf. Lemma 1). ☐

3 CONSTRUCTION OF ALMOST OPTIMAL APPROXIMATIONS OF CAPACITIES

Consider the space $\underline{M}X$ of subnormalized capacities. For reader's convenience we present and prove properties of $\underline{M}X$ [5] in the following statement.

Proposition 4. *The triple $(\underline{M}X, \vee, \wedge)$ is a $(I, \max, \min)$-convex compactum, if the operations $\vee : \underline{M}X \times \underline{M}X \to \underline{M}X$ and $\wedge : I \times \underline{M}X \to \underline{M}X$ are defined by the formulae:*

$$c_1 \vee c_2(F) = \max\{c_1(F), c_2(F)\},\ \alpha \wedge c(F) = \min\{\alpha, c(F)\}$$

for $c_1, c_2 \in \underline{M}X$, $\alpha \in I$, $F \underset{\text{cl}}{\subset} X$.

Proof. It is almost obvious that the defined above functions $c_1 \vee c_2 : \exp X \to I$, $\alpha \wedge c : \exp X \cup \{\varnothing\} \to I$ are capacities on X. Put $\oplus = \vee$, $\circledast = \wedge$ and set the zero element $\bar{0} \in \underline{M}X$ to the "zero capacity" with the values $\bar{0}(F) = 0$ for all $F \underset{\text{cl}}{\subset} X$. It is easy to observe that axioms (1)—(7) from the definition of semimodule hold. Thus $(\underline{M}X, \vee, \wedge)$ is a (left idempotent) $(I, \max, \min)$-semimodule. Recall (see [2]) that the subbase of all sets of the form $O_-(F, a)$ and $O_+(U, a)$, for $A \underset{\text{cl}}{\subset} X$, $U \underset{\text{op}}{\subset} X$, $a \in I$, determines a compact Hausdorff topology τ on $\underline{M}X$.

It a partial order at $\underline{M}X$ is defined as

$$c_1 \le c_2 \Leftrightarrow c_1 \vee c_2 = c_2 \Leftrightarrow c_1(F) \le c_2(F), \text{ for all } F \underset{\text{cl}}{\subset} X,$$

then the pairwise suprema are calculated argumentwise: $c_1 \vee c_2(F) = \max\{c_1(F), c_2(F)\}$, and $\underline{M}X$ is an upper semilattice with the least element $\bar{0}$. It was proved in [5] that $(\underline{M}X, \le)$ is a topological (i.e. the pairwise supremum $c_1 \vee c_2$ depends on c_1 and c_2 continuously w.r.t. the topology τ) upper Lawson semilattice (because subbase elements $O_-(F, a)$ and $O_+(U, a)$ are subsemilattices), and τ is the Lawson topology.

The function $c_1 \vee \alpha c_2 : \exp X \cup \{\varnothing\} \to I$ defined by the formula

$$c_1 \vee \alpha c_2(F) = c_1 \vee (\alpha \wedge c_2)(F) = \max\{c_1(F), \min\{\alpha, c_2(F)\}\}$$

is a subnormalized capacity on X, and the mapping $\underline{M}X \times I \times \underline{M}X \to \underline{M}X$ that assigns $c_1 \vee \alpha c_2$ to (c_1, α, c_2) is continuous. Hence $\underline{M}X$ is a compact Hausdorff space with a Lawson continuous pairwise I-convex combination which makes it a compact Hausdorff Lawson upper semilattice, i.e. $(\underline{M}X, \vee, \wedge)$ is an I-convex compactum. □

If a compact topology on X is determined with an admissible metric d, then $(\underline{M}X, \hat{d})$ is a metric compactum and the defined above metric $\hat{d}$ on $\underline{M}X$ is admissible, i.e. $(\underline{M}X, \vee, \wedge)$ is a metric I-convex compactum. The following property of $\hat{d}$ is crucial.

Lemma 4. *Let (X, d) be a metric compactum, $c_0, c_i \in \underline{M}X$ for $i \in \mathcal{I}$ are capacities such that $\hat{d}(c_0, c_i) \le \varepsilon$ for some $\varepsilon \ge 0$ and all i. Then for arbitrary coeficients $\alpha_i \in I$ such that* $\sup_{i\in\mathcal{I}} \alpha_i = 1$ *the inequality $\hat{d}(c_0, \bigvee\limits_{i\in\mathcal{I}} \alpha_i c_i) \le \varepsilon$ is valid.*

For a finite number of c_i the inequality is straightforward, and by continuity we extend it to infinite combinations.

Remark. *Since $MX \subset \underline{M}X$ is a closed subsemimodule,* everything *said above on $\underline{M}X$ applies also to MX.*

Therefore the above statements can be used to approximate a capacity $c \in \underline{M}X$ (or $c \in MX$) with capacities from a closed I-convex subspace $S \subset \underline{M}X$ (resp. $S \subset MX$). The convexity means that S contains all I-convex combinations of the form $\bigvee\limits_{i\in\mathcal{I}} (\alpha_i \wedge c_i)$, where $c_i \in S$, $\alpha_i \in I$, $\max\{\alpha_i | i \in \mathcal{I}\} = 1$. For simplicity consider a more general case of $\underline{M}X$.

For a capacity $c \in \underline{M}X$ and a number $\varepsilon > 0$ construct the set

$$\mathfrak{G}_c = \left\{ (c', \alpha) | c' \in S, \alpha \in I,\ \alpha \le \max\left\{ 0, 1 - \frac{\hat{d}(c, c') - \hat{d}(c, S)}{\varepsilon} \right\} \right\},$$

which is closed in $S \times I$ due to Proposition 2.

Define a capacity $\tilde{c}_\varepsilon$ with the formula $\tilde{c}_\varepsilon = \bigvee\limits_{i\in\mathcal{I}} \{\alpha_i \wedge c_i | (c_i, \alpha_i) \in \mathfrak{G}_c\}$. Equivalently $\tilde{c}_\varepsilon$ can be defined as

$$\tilde{c}_\varepsilon(F) = \sup\left\{ (1 - \frac{\hat{d}(c, c') - \hat{d}(c, S)}{\varepsilon}) \wedge c'(F) | c' \in S, \hat{d}(c, c') \le \hat{d}(c, S) + \varepsilon \right\} \tag{1}$$

for all $F \underset{\text{cl}}{\subset} X$. Although $\tilde{c}_\varepsilon$ is not the closest to $c \in \underline{M}X$ in the subspace S, it is “almost the closest” in the sense of the following theorem.

Theorem 1. *For a capacity $c \in \underline{M}X$, a number $\varepsilon > 0$ and a closed I-convex subspace $S \subset \underline{M}X$ the capacity $\tilde{c}_\varepsilon$ belongs to S and satisfies the inequality $\hat{d}(c,\tilde{c}_\varepsilon) \leq \hat{d}(c,S)+\varepsilon$. The mapping $\Phi : \underline{M}X \times (0, \operatorname{diam} \underline{M}X] \to S$ defined as $\Phi(c,\varepsilon) = \tilde{c}_\varepsilon$ is continuous.*

Proof. Continuity of Φ and $\tilde{c}_\varepsilon \in S$ follow from Proposition 3. By the equality (1) the capacity $\tilde{c}_\varepsilon$ is an I-convex combination of capacities $c' \in S$ such that $\hat{d}(c,c') \leq \hat{d}(c,S)+\varepsilon$, hence by Lemma 4 the inequality $\hat{d}(c,\tilde{c}_\varepsilon) \leq \hat{d}(c,S)+\varepsilon$ is valid as well. $\square$

Remark. *Obviously an analogous theorem is valid for MX.*

It is easy to verify that the subspaces $M_\cap X$ and MX_0 are closed and I-convex subsets of the semimodule $(MX,\vee,\wedge)$ ($M_\cup X$ is I-convex if the I-convex combination on $(MX,\vee,\wedge)$ is defined in a dual manner, cf. [5]). Methods of calculating of the distances $\hat{d}(c,M_\cap X)$, $\hat{d}(c,M_\cup X)$, $\hat{d}(c,MX_0)$ were presented in [4]. Thus we can use the latter theorem to construct approximations of an arbitrary subnormalized capacity c on X with $\cup$-capacities, $\cap$-capacities or capacities on $X_0 \subset X$ that are ε-closed to optimal and depend on c, ε continuously.

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Представлено метод "майже оптимального" неперервного наближення ємностей на метричному компакті мірами можливості, мірами необхідності чи ємностями на замкненому підпросторі.

Ключові слова і фрази: ємність, метричний компакт, наближення.

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PROPERTIES OF DISTANCE SPACES WITH POWER TRIANGLE INEQUALITIES

Metric spaces provide a framework for analysis and have several very useful properties. Many of these properties follow in part from the triangle inequality. However, there are several applications in which the triangle inequality does not hold but in which we may still like to perform analysis. This paper investigates what happens if the triangle inequality is removed all together, leaving what is called a distance space, and also what happens if the triangle inequality is replaced with a much more general two parameter relation, which is herein called the "power triangle inequality". The power triangle inequality represents an uncountably large class of inequalities, and includes the triangle inequality, relaxed triangle inequality, and inframetric inequality as special cases. The power triangle inequality is defined in terms of a function that is herein called the power triangle function. The power triangle function is itself a power mean, and as such is continuous and monotone with respect to its exponential parameter, and also includes the operations of maximum, minimum, mean square, arithmetic mean, geometric mean, and harmonic mean as special cases.

Key words and phrases: metric space, distance space, semimetric space, quasi-metric space, triangle inequality, relaxed triangle inequality, inframetric, arithmetic mean, means square, geometric mean, harmonic mean, maximum, minimum, power mean.

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1 INTRODUCTION AND SUMMARY

Metric spaces provide a framework for analysis and have several very useful properties. Many of these properties follow in part from the triangle inequality. However, there are several applications[1] in which the triangle inequality does not hold but in which we would still like to perform analysis. So the questions that natually follow are:

Q1. What happens if we remove the triangle inequality all together?

Q2. What happens if we replace the triangle inequality with a generalized relation?

A distance space is a metric space without the triangle inequality constraint. Section 3 introduces distance spaces and demonstrates that some properties commonly associated with metric spaces also hold in any distance space:

D1. $\varnothing$ and X are open, (Theorem 1),

D2. the intersection of a finite number of open sets is open, (Theorem 1),

D3. the union of an arbitrary number of open sets is open, (Theorem 1),

D4. every Cauchy sequence is bounded, (Proposition 1),

D5. any subsequence of a Cauchy sequence is also Cauchy, (Proposition 2),

D6. the Cantor Intersection Theorem holds, (Theorem 4).

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[1] References for applications in which the triangle inequality may not hold: [21, 32–34, 65, 76, 80, 108, 114–116].

The following five properties (M1–M5) *do* hold in any metric space. However, the examples from Section 3 listed below demonstrate that the five properties do *not* hold in all distance spaces:

M1. the metric function is continuous — fails to hold in Examples 1–3,
M2. open balls are open — fails to hold in Examples 1 and 2,
M3. the open balls form a base for a topology — fails to hold in Examples 1 and 2,
M4. the limits of convergent sequences are unique — fails to hold in Example 1,
M5. convergent sequences are Cauchy — fails to hold in Example 2.

Hence, Section 3 answers question Q1.

Section 4 begins to answer question Q2 by first introducing a new function, called the power triangle function (see Definition 21) in a distance space (X, d), as

$$\tau(p, \sigma; x, y, z; \mathsf{d}) := 2\sigma\left[\frac{1}{2}\mathsf{d}^p(x,z) + \frac{1}{2}\mathsf{d}^p(z,y)\right]^{\frac{1}{p}}$$

for some $(p, \sigma) \in \mathbb{R}^* \times \mathbb{R}$. Section 4 then goes on to use this function to define a new relation, called the power triangle inequality in (X, d), and defined as

$$\textcircled{\scriptsize$\triangle$}(p, \sigma; \mathsf{d}) := \left\{(x, y, z) \in X^3 \mid \mathsf{d}(x, y) \le \tau(p, \sigma; x, y, z; \mathsf{d})\right\}.$$

The power triangle inequality is a generalized form of the triangle inequality in the sense that the two inequalities coincide at $(p, \sigma) = (1, 1)$. Other special values include $(1, \sigma)$ yielding the relaxed triangle inequality (and its associated near metric space) and (∞, σ) yielding the σ-inframetric inequality (and its associated σ-inframetric space). Collectively, a distance space with a power triangle inequality (see Definition 23) is herein called a power distance space (see Definition 24) and denoted $(X, \mathsf{d}, p, \sigma)$.[2]

The power triangle function, at $\sigma = \frac{1}{2}$, is a special case of the power mean (see Definition 32) with $N = 2$ and $\lambda_1 = \lambda_2 = \frac{1}{2}$. Power means have the elegant properties of being continuous and monontone with respect to a free parameter p. From this it is easy to show that the power triangle function is also continuous and monontone with respect to both p and σ. Special values of p yield operators coinciding with maximum, minimum, mean square, arithmetic mean, geometric mean, and harmonic mean. Power means are briefly described in Appendix B.2 (see also Corollaries 2, 3, 8 and Theorem 18).

Section 4.2 investigates the properties of power distance spaces. In particular, it shows for what values of (p, σ) the properties M1–M5 hold. Here is a summary of the results in a power distance space $(X, \mathsf{d}, p, \sigma)$, for all $x, y, z \in X$:

(M1) holds for any $(p, \sigma) \in (\mathbb{R}^* \backslash \{0\}) \times \mathbb{R}^+$ such that $2\sigma = 2^{\frac{1}{p}}$, (Theorem 9),
(M2) holds for any $(p, \sigma) \in (\mathbb{R}^* \backslash \{0\}) \times \mathbb{R}^+$ such that $2\sigma \le 2^{\frac{1}{p}}$, (Corollary 7),
(M3) holds for any $(p, \sigma) \in (\mathbb{R}^* \backslash \{0\}) \times \mathbb{R}^+$ such that $2\sigma \le 2^{\frac{1}{p}}$, (Corollary 6),
(M4) holds for any $(p, \sigma) \in \mathbb{R}^* \times \mathbb{R}^+$, (Theorem 10),
(M5) holds for any $(p, \sigma) \in \mathbb{R}^* \times \mathbb{R}^+$, (Theorem 7).

Appendix A briefly introduces topological spaces. The open balls of any metric space form a base for a topology. This is largely due to the fact that in a metric space, open balls are open. Because of this, in metric spaces it is convenient to use topological structure to define

[2] For examples of power distance spaces see Definition 24.

and exploit analytic concepts such as continuity, convergence, closed sets, closure, interior, and accumulation point. For example, in a metric space, the traditional definition of defining continuity using open balls and the topological definition using open sets, coincide with each other. Again, this is largely because the open balls of a metric space are open.

However, this is not the case for all distance spaces. In general, the open balls of a distance space are not open, and they are not a base for a topology. In fact, the open balls of a distance space are a base for a topology if and only if the open balls are open. While the open sets in a distance space do induce a topology, it's open balls may not (see Theorem 2, Corollary 1).

2 STANDARD DEFINITIONS

2.1 Standard sets

Definition 1. *Let $\mathbb{R}$ be the set of real numbers. Let $\mathbb{R}^{\vdash}$ (resp. $\mathbb{R}^{+}$) be the set of non-negative (resp. postive) real numbers. Let $\mathbb{R}^{*} := \mathbb{R} \cup \{-\infty, \infty\}$ be the set of extended real numbers [95]. Let $\mathbb{Z}$ be the set of integers. Let $\mathbb{N} := \{n \in \mathbb{Z} \,|\, n \geq 1\}$ be the set of natural numbers. Let $\mathbb{Z}^{*} := \mathbb{Z} \cup \{-\infty, \infty\}$ be the extended set of integers.*

Definition 2. *Let X be a set. The quantity 2^X (the set of all subsets of X) is the power set of X, i.e. $2^X := \{A \subseteq X\}$.*

2.2 Relations

Definition 3 ([12,13,29,57,67,78,106])**.** *Let X and Y be sets. The Cartesian product $X \times Y$ of X and Y is the set $X \times Y := \{(x, y) \,|\, x \in X \text{ and } y \in Y\}$. An ordered pair (x, y) on X and Y is any element in $X \times Y$. A relation ® on X and Y is any subset of $X \times Y$ such that $® \subseteq X \times Y$. The set 2^{XY} is the set of all relations in $X \times Y$. A relation $\mathsf{f} \in 2^{XY}$ is a function if $(x, y_1) \in \mathsf{f}$ and $(x, y_2) \in \mathsf{f}$ implies $y_1 = y_2$. The set Y^X is the set of all functions in 2^{XY}.*

Note, that the notation Y^X and 2^{XY} is motivated by the fact that for finite X and Y, $|\,Y^X\,| = |\,Y\,|^{|\,X\,|}$ and $|\,2^{XY}\,| = 2^{|\,X\,|\cdot|\,Y\,|}$.

2.3 Set functions

Definition 4 ([55,87,92])**.** *Let 2^X be the power set of a set X. A set $\mathcal{S}(X)$ is a set structure on X if $\mathcal{S}(X) \subseteq 2^X$. A set structure $\mathcal{Q}(X)$ is a paving on X if $\varnothing \in \mathcal{Q}(X)$.*

Definition 5 ([25,55,56,92])**.** *Let $\mathcal{Q}(X)$ be a paving on a set X. Let Y be a set containing the element 0. A function $\mathsf{m} \in Y^{\mathcal{Q}(X)}$ is a set function if $\mathsf{m}(\varnothing) = 0$.*

Definition 6. *The set function $|\,A\,| \in {\mathbb{Z}^{*}}^{2^X}$ is the cardinality of $A \in 2^X$ such that*

$$|\,A\,| := \begin{cases} \text{the number of elements in } A, & \text{for finite } A, \\ \infty, & \text{otherwise.} \end{cases}$$

Definition 7. *Let $|\,X\,|$ be the* cardinality *of a set X. The structure $\varnothing$ is the empty set, and is a set such that $|\,\varnothing\,| = 0$.*

2.4 Order

Definition 8 ([4,38,70,77]). *Let X be a set. A relation $\le$ is an order relation in 2^{XX} if*

1. $x \le x \qquad \forall x \in X$ (*reflexive*) *and*
2. $x \le y$ *and* $y \le z \implies x \le z \qquad \forall x, y \in X$ (*transitive*) *and*
3. $x \le y$ *and* $y \le x \implies x = y \qquad \forall x, y \in X$ (*anti-symmetric*).

An ordered set is the pair $(X, \le)$.[3] *A relation $\le$ is a preorder relation in 2^{XX} if only the first two conditions hold.*

We write $x < y$ if $x \le y$ and $x \ne y$ for any x, y from an ordered set $(X, \le)$.

Definition 9 ([2,91]). *In an ordered set $(X, \le)$ the set $[x : y] := \{z \in X \mid x \le z \le y\}$ is a closed interval, the sets $(x : y] := \{z \in X \mid x < z \le y\}$ and $[x : y) := \{z \in X \mid x \le z < y\}$ are half-open intervals, the set $(x : y) := \{z \in X \mid x < z < y\}$ is an open interval.*

Definition 10. *Let $(\mathbb{R}, \le)$ be the ordered set of real numbers. The absolute value $|\cdot| \in \mathbb{R}^{\mathbb{R}}$ is defined as*[4] $|x| := \begin{cases} -x, & \text{for } x \le 0, \\ x, & \text{otherwise.} \end{cases}$

3 BACKGROUND: DISTANCE SPACES

A distance space can be defined as a metric space without the triangle inequality constraint. Much of the material in this section about distance spaces is standard in metric spaces. However, this paper works through this material again to demonstrate "how far we can go", and can't go, without the triangle inequality.

3.1 Fundamental structure of distance spaces

3.1.1 Definitions

Definition 11 ([6,9,10,41,50,68,74,82,118]). *A function d in the set $\mathbb{R}^{X \times X}$ is a distance if*

1. $\mathsf{d}(x,y) \ge 0 \qquad \forall x, y \in X$ (*non-negative*) *and*
2. $\mathsf{d}(x,y) = 0 \iff x = y \qquad \forall x, y \in X$ (*nondegenerate*) *and*
3. $\mathsf{d}(x,y) = \mathsf{d}(y,x) \qquad \forall x, y \in X$ (*symmetric*).

The pair (X, d) is a distance space if d is a distance on a set X.

Definition 12. [5] *Let (X, d) be a distance space and 2^X be the power set of X. The diameter in (X, d) of a set $A \in 2^X$ is*

$$\operatorname{diam} A := \begin{cases} 0, & \text{for } A = \varnothing, \\ \sup\{\mathsf{d}(x,y) \mid x, y \in A\}, & \text{otherwise.} \end{cases}$$

Definition 13 ([16,110]). *A set $A \in 2^X$ is bounded in a distance space (X, d) if $\operatorname{diam} A < \infty$.*

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[3] An order relation is also called a partial order relation. An ordered set is also called a partially ordered set or poset.

[4] A more general definition for absolute value is available for any commutative ring [26]. Let R be a commutative ring. A function $|\cdot|$ in R^R is an absolute value, or modulus, on R if

1. $|x| \ge 0 \qquad x \in R$ (non-negative) and
2. $|x| = 0 \iff x = 0 \qquad x \in R$ (nondegenerate) and
3. $|xy| = |x| \cdot |y| \qquad x, y \in R$ (homogeneous/submultiplicative) and
4. $|x + y| \le |x| + |y| \qquad x, y \in R$ (subadditive/triangle inequality).

[5] For definition in metric space see [30,60,83,87].

3.1.2 Properties

Remark 1. *Let $\{x_n\}_{n\in\mathbb{Z}}$ be a sequence in a distance space (X, d). The distance space (X, d) does not necessarily have all the nice properties that a metric space has. In particular, note the following:*

1.	d *is a distance in* (X, d)	$\not\Rightarrow$	d *is continuous in* (X, d),	*(Example 3),*
2.	B *is an open ball in* (X, d)	$\not\Rightarrow$	B *is open in* (X, d),	*(Example 2),*
3.	B *is the set of all open balls in* (X, d)	$\not\Rightarrow$	B *is a base for a topology on* X,	*(Example 2),*[6]
4.	$\{x_n\}$ *is convergent in* (X, d)	$\not\Rightarrow$	*limit is unique,*	*(Example 1),*
5.	$\{x_n\}$ *is convergent in* (X, d)	$\not\Rightarrow$	$\{x_n\}$ *is Cauchy in* (X, d),	*(Example 2).*

3.2 Open sets in distance spaces

3.2.1 Definitions

Definition 14 ([1])**.** *Let (X, d) be a distance space. An open (resp. closed) ball centered at x with radius r is the set $\mathsf{B}(x, r) := \{y \in X \mid \mathsf{d}(x, y) < r\}$ $\left(\text{resp. } \overline{\mathsf{B}}(x, r) := \{y \in X \mid \mathsf{d}(x, y) \leq r\}\right)$.*

Definition 15. *Let (X, d) be a distance space. Let $X \backslash A$ be the* set difference *of X and a set A. A set U is open in (X, d) if $U \in 2^X$ and for every x in U there exists $r \in \mathbb{R}^+$ such that $\mathsf{B}(x, r) \subseteq U$. A set U is an open set in (X, d) if U is open in (X, d). A set D is closed in (X, d) if $X \backslash D$ is open. A set D is a closed set in (X, d) if D is closed in (X, d).*

3.2.2 Properties

Theorem 1 ([43, 97])**.** *Let (X, d) be a distance space. Let N be any (finite) positive integer. Let Γ be a set possibly with an uncountable number of elements. Then the following statements hold.*

1. *X is open.*
2. *$\varnothing$ is open.*
3. *Each element in $\{U_\gamma \in 2^X \mid \gamma \in \Gamma\}$ is open* $\Longrightarrow$ $\bigcup_{\gamma\in\Gamma} U_\gamma$ *is open.*
4. *Each element in $\{U_n | n = 1, 2, \ldots, N\}$ is open* $\Longrightarrow$ $\bigcap_{n=1}^{N} U_n$ *is open.*

Proof. 1. By definition of open set, X is open iff $\forall x \in X \,\exists\, r$ such that $\mathsf{B}(x, r) \subseteq X$. By definition of open ball, it is always true that $\mathsf{B}(x, r) \subseteq X$ in (X, d). Therefore, X is open in (X, d).

2. By definition of open set, $\varnothing$ is open iff $\forall x \in \varnothing \,\exists\, r$ such that $\mathsf{B}(x, r) \subseteq \varnothing$. By definition of empty set $\varnothing$, this is always true because no x is in $\varnothing$. Therefore, $\varnothing$ is open in (X, d).

3. By definition of open set, $\bigcup U_\gamma$ is open iff $\forall x \in \bigcup U_\gamma \,\exists\, r$ such that $\mathsf{B}(x, r) \subseteq \bigcup U_\gamma$. If $x \in \bigcup U_\gamma$, then there is at least one $U \in \bigcup U_\gamma$ that contains x. By the left hypothesis in statement 3, that set U is open and so for that x $\exists r$ such that $\mathsf{B}(x, r) \subseteq U \subseteq \bigcup U_\gamma$. Therefore, $\bigcup U_\gamma$ is open in (X, d).

4. Let us prove that if U_1 and U_2 are open, then $U_1 \cap U_2$ is open. By definition of open set, $U_1 \cap U_2$ is open iff $\forall x \in U_1 \cap U_2 \,\exists\, r$ such that $\mathsf{B}(x, r) \subseteq U_1 \cap U_2$. By the left hypothesis above, U_1 and U_2 are open, and by the definition of open sets, there exists r_1 and r_2 such that

[6] See [50, 61].

$\mathsf{B}(x, r_1) \subseteq U_1$ and $\mathsf{B}(x, r_2) \subseteq U_2$. Let $r := \min\{r_1, r_2\}$. Then $\mathsf{B}(x, r) \subseteq U_1$ and $\mathsf{B}(x, r) \subseteq U_2$. By definition of set intersection, $\mathsf{B}(x, r) \subseteq U_1 \cap U_2$. Hence, $U_1 \cap U_2$ is open.

Let us prove that $\bigcap_{n=1}^{N} U_n$ is open by induction. For $N = 1$ case: $\bigcap_{n=1}^{N} U_n = \bigcap_{n=1}^{1} U_n = U_1$ is open by hypothesis. By property of intersection $\bigcap_{n=1}^{N+1} U_n = \left(\bigcap_{n=1}^{N} U_n\right) \cap U_{N+1}$, therefore $\bigcap_{n=1}^{N+1} U_n$ is open via "N case" hypothesis and above proof for two sets. □

Corollary 1. *Let (X, d) be a distance space. The set $T := \{U \in 2^X \mid U \text{ is open in } (X, \mathsf{d})\}$ is a topology on X, and (X, T) is a topological space.*

Proof. This follows directly from the definition of an open set, Theorem 1, and the definition of topology. □

Of course it is possible to define a very large number of topologies even on a finite set with just a handful of elements;[7] and it is possible to define an infinite number of topologies even on a linearly ordered infinite set like the real line $(\mathbb{R}, \leq)$.[8] Be that as it may, Definition 16 defines a single but convenient topological space in terms of a distance space. Note that every metric space conveniently and naturally induces a topological space because the open balls of the metric space form a base for the topology. This is not the case for all distance spaces. But if the open balls of a distance space are all open, then those open balls induce a topology (next theorem).[9]

Definition 16. *Let (X, d) be a distance space. The set $T := \{U \in 2^X \mid U \text{ is open in } (X, \mathsf{d})\}$ is the topology induced by (X, d) on X. The pair (X, T) is called the topological space induced by (X, d).*

For any distance space (X, d), no matter how strange, there is guaranteed to be at least one topological space induced by (X, d) — and that is the indiscrete topological space (Example 9) because for any distance space (X, d), $\varnothing$ and X are open sets in (X, d) (Theorem 1).

Theorem 2. *Let B be the set of all open balls in a distance space (X, d). Then every open ball in B is open if and only if B is a base for a topology.*

Proof. Let every open ball in B be open. Then for every x in $B_y \in B$ there exists $r \in \mathbb{R}^+$ such that $\mathsf{B}(x, r) \subseteq B_y$ by Definition 15. It implies for every $x \in X$ and for every $B_y \in B$ containing x, there exists $B_x \in B$ such that $x \in B_x \subseteq B_y$, because $\forall (x, r) \in X \times \mathbb{R}^+, \mathsf{B}(x, r) \subseteq X$. Hence, B is a base for T by Theorem 11.

Vice versa. Let B is a base for a topology. Then for every $x \in X$ and for every $U \subseteq T$ containing x, there exists $B_x \in B$ such that $x \in B_x \subseteq U$ by Theorem 11. From Definition 26 it follows that for every $x \in X$ and for every $B_y \in B \subseteq T$ containing x, there exists $B_x \in B$ such that $x \in B_x \subseteq B_y$. Therefore for every $x \in B_y \in B \subseteq T$, there exists $B_x \in B$ such that $x \in B_x \subseteq B_y$. Hence, every open ball in B is open (see Definition 15). □

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[7] For a finite set X with n elements, there are 29 topologies on X if $n = 3$; 6942 topologies on X if $n = 5$; and and 8.977.053.873.043 (almost 9 trillion) topologies on X if $n = 10$. See [15, 24, 28, 29, 45, 71, 104].

[8] For examples of topologies on the real line see [27, 66, 90, 99].

[9] Metric space: Definition 24; open ball: Definition 14; base: Definition 26; topology: Definition 25; not all open balls are open in a distance space: Example 1 and Example 2.

3.3 Sequences in distance spaces

3.3.1 Definitions

Definition 17. [10] *Let $\{x_n\}_{n\in\mathbb{Z}} \subset X$ be a sequence in a distance space (X, d). The sequence $\{x_n\}$ converges to a limit x if for any $\varepsilon \in \mathbb{R}^+$, there exists $N \in \mathbb{Z}$ such that $\mathsf{d}(x_n, x) < \varepsilon$ for all $n > N$. This condition can be expressed in any of the following forms:*

1. *the limit of the sequence $\{x_n\}$ is x;*
2. *the sequence $\{x_n\}$ is convergent with limit x;*
3. $\lim\limits_{n\to\infty} \{x_n\} = x$;
4. $\{x_n\} \to x$.

A sequence that converges is convergent.

Definition 18. [11] *Let $\{x_n\}_{n\in\mathbb{Z}} \subset X$ be a sequence in a distance space (X, d). The sequence $\{x_n\}$ is a Cauchy sequence in (X, d) if for every $\varepsilon \in \mathbb{R}^+$, there exists $N \in \mathbb{Z}$ such that $\mathsf{d}(x_n, x_m) < \varepsilon$ for all $n, m > N$.*

Definition 19. [12] *Let $\{x_n\}_{n\in\mathbb{Z}} \subset X$ be a sequence in a distance space (X, d). The sequence $\{x_n\}$ is complete in (X, d) if the following implication holds: $\{x_n\}$ is Cauchy in $(X, \mathsf{d}) \implies \{x_n\}$ is convergent in (X, d).*

3.3.2 Properties

Proposition 1. *Let $\{x_n\}_{n\in\mathbb{Z}} \subset X$ be a sequence in a distance space (X, d). If $\{x_n\}$ is Cauchy in (X, d), then it is bounded in (X, d).*

Proof. Let $\{x_n\}$ be a Cauchy sequence. It means that for every $\varepsilon \in \mathbb{R}^+$ there exists $N \in \mathbb{Z}$ such that $\forall n, m > N$, $\mathsf{d}(x_n, x_m) < \varepsilon$. Let $\varepsilon = 1$. Then $\exists N \in \mathbb{Z}$ such that $\mathsf{d}(x_n, x_m) < 1$ for all $n, m > N$. It implies $\mathsf{d}(x_n, x_{m+1}) < \max\left\{\{1\} \cup \left\{\mathsf{d}\left(x_p, x_q\right) \mid p, q \not> N\right\}\right\}$. Hence, the sequence $\{x_n\}$ is bounded by Definition 13. ☐

Proposition 2. *Let $\{x_n\}_{n\in\mathbb{Z}} \subset X$ be a sequence in a distance space (X, d). Let $\mathsf{f} \in \mathbb{Z}^{\mathbb{Z}}$ be a strictly monotone function such that $\mathsf{f}(n) < \mathsf{f}(n+1)$. Then if $\{x_n\}_{n\in\mathbb{Z}}$ is a Cauchy sequence, then subsequence $\left\{x_{\mathsf{f}(n)}\right\}_{n\in\mathbb{Z}}$ is also Cauchy.*

Proof. Let $\{x_n\}_{n\in\mathbb{Z}}$ be a Cauchy sequence. It means that for any given $\varepsilon > 0$, $\exists N$ such that $\forall n, m > N$, $\mathsf{d}(x_n, x_m) < \varepsilon$. Therefore there exists N' such that $\mathsf{d}\left(x_{\mathsf{f}(n)}, x_{\mathsf{f}(m)}\right) < \varepsilon$ for all $\mathsf{f}(n), \mathsf{f}(m) > N'$. So, $\left\{x_{\mathsf{f}(n)}\right\}_{n\in\mathbb{Z}}$ is Cauchy sequence. ☐

Theorem 3. [13] *Let (X, d) be a distance space. Let A^- be the closure of a A in a topological space induced by (X, d). If limits are unique in (X, d) and (A, d) is complete in (X, d), then A is closed in (X, d), i.e. $A = A^-$.*

Proof. By Lemma 3 we have $A \subseteq A^-$. Let us prove that $A^- \subseteq A$.

Let x be a point in A^-. Define a sequence of open balls $\left\{\mathsf{B}\left(x, \frac{1}{1}\right), \mathsf{B}\left(x, \frac{1}{2}\right), \mathsf{B}\left(x, \frac{1}{3}\right), \ldots\right\}$. Define a sequence of points $\{x_1, x_2, x_3, \ldots\}$ such that $x_n \in \mathsf{B}\left(x_n, \frac{1}{n}\right) \cap A$. Then $\{x_n\}$ is convergent in X with limit x by Definition 17 and $\{x_n\}$ is Cauchy in A by Definition 18. Since (A, d) is

[10] For definition in metric space see [53, 68, 75, 97].
[11] For definition in metric space see [2, 97].
[12] For definition in metric space see [97].
[13] For theorem in metric space see [18, 54, 72, 107].

complete in (X, d), $\{x_n\}$ is therefore also convergent in A. Let this limit be y. Note that $y \in A$. From uniqueness of limits it follows $y = x$, and therefore $x \in A$. Hence $A^- \subseteq A$. $\square$

Proposition 3. *Let $\{x_n\}_{n\in\mathbb{Z}}$ be a sequence in a distance space (X, d). Let $\mathsf{f} : \mathbb{Z} \to \mathbb{Z}$ be a strictly increasing function such that $\mathsf{f}(n) < \mathsf{f}(n+1)$. If the sequence $\{x_n\}_{n\in\mathbb{Z}}$ converges to limit x, then a subsequence $\left\{x_{\mathsf{f}(n)}\right\}_{n\in\mathbb{Z}}$ converges to the same limit x.*

Proof. By Theorem 6 we have $\forall\varepsilon > 0$, $\exists N$ such that $\forall n > N$, $\mathsf{d}(x_n, x) < \varepsilon$. Therefore $\forall\varepsilon > 0$, $\exists\mathsf{f}(N)$ such that $\forall\mathsf{f}(n) > \mathsf{f}(N)$, $\mathsf{d}\left(x_{\mathsf{f}(n)}, x\right) < \varepsilon$. So, $\left\{x_{\mathsf{f}(n)}\right\}_{n\in\mathbb{Z}} \to x$ via Theorem 6. $\square$

Theorem 4 (Cantor intersection theorem)**.** *Let (X, d) be a distance space, $\{A_n\}_{n\in\mathbb{Z}}$ a sequence with each $A_n \in 2^X$, and $|\,A\,|$ the number of elements in A. If (X, d) is complete, A_n is closed for all $n \in \mathbb{N}$, $\operatorname{diam} A_n \geq \operatorname{diam} A_{n+1}$ for all $n \in \mathbb{N}$, and $\operatorname{diam}\{A_n\}_{n\in\mathbb{Z}} \to 0$, then $\left|\bigcap_{n\in\mathbb{N}} A_n\right| = 1$.*

Proof. Let us prove that $|\bigcap A_n| < 2$. Let $A := \cap A_n$. For any $x \neq y$ and $\{x, y\} \in A$ we have $\mathsf{d}(x, y) > 0$ and $\{x, y\} \subseteq A_n$ for all n. Since $\operatorname{diam}\{A_n\}_{n\in\mathbb{Z}} \to 0$, there exists n such that $\operatorname{diam} A_n < \mathsf{d}(x, y)$. It implies $\exists n$ such that $\sup\{\mathsf{d}(x, y) \,|\, x, y \in A_n\} < \mathsf{d}(x, y)$. This is a contradiction, so $\{x, y\} \notin A$ and $|\bigcap A_n| < 2$.

Let us prove that $|\bigcap A_n| \geq 1$. Let $x_n \in A_n$ and $x_m \in A_m$. Since $\operatorname{diam}\{A_n\}_{n\in\mathbb{Z}} \to 0$, for all ε there exists $N \in \mathbb{N}$ such that $\operatorname{diam} A_N < \varepsilon$. Therefore $\forall m, n > N$, $x_n \in A_n \subseteq A_N$ and $x_m \in A_m \subseteq A_N$. But $\mathsf{d}(x_n, x_m) \leq \operatorname{diam} A_N < \varepsilon$, it means that $\{x_n\}$ is a Cauchy sequence. Because $\{x_n\}$ is complete, $x_n \to x$. It implies $x \in (A_n)^- = A_n$, and, hence, $|\,A_n\,| \geq 1$. $\square$

Definition 20 ([10])**.** *Let (X, d) be a distance space. Let C be the set of all convergent sequences in (X, d). The distance function d is continuous in (X, d) if*

$$\{x_n\}, \{y_n\} \in C \quad\Longrightarrow\quad \lim_{n\to\infty}\{\mathsf{d}(x_n, y_n)\} = \mathsf{d}\Big(\lim_{n\to\infty}\{x_n\}, \lim_{n\to\infty}\{y_n\}\Big).$$

A distance function is discontinuous if it is not continuous.

Remark 2. *Rather than defining continuity of a distance function in terms of the* sequential characterization of continuity *as in Definition 20, we could define continuity using an* inverse image characterization of continuity *(see Definition 16). Assuming an equivalent topological space is used for both characterizations, the two characterizations are equivalent (Theorem 15). In fact, one could construct an equivalence such as the following:*

$$\left\{\begin{array}{l}\mathsf{d} \text{ is continuous in } \mathbb{R}^{X^2}\\ \text{(Definition 28)}\\ \text{(inverse image characterization}\\ \text{of continuity)}\end{array}\right\} \iff \left\{\begin{array}{l}\{x_n\}, \{y_n\} \in C \quad\Longrightarrow\\ \lim_{n\to\infty}\{\mathsf{d}(x_n, y_n)\} = \mathsf{d}\Big(\lim_{n\to\infty}\{x_n\}, \lim_{n\to\infty}\{y_n\}\Big)\\ \text{(Definition 29)}\\ \text{(sequential characterization of continuity)}\end{array}\right\}.$$

Note that just as $\{x_n\}$ is a sequence in X, so the ordered pair $(\{x_n\}, \{y_n\})$ is a sequence in X^2. The remainder follows from Theorem 15. However, use of the inverse image characterization *is somewhat troublesome because we would need a topology on X^2, and we don't immediately have one defined and ready to use. In fact, we don't even immediately have a distance space on X^2 defined or even open balls in such a distance space. The result is, for the scope of this paper, it is arguably not worthwhile constructing the extra structure, but rather instead this paper uses the* sequential characterization *as a definition (as in Definition 20).*

3.4 Examples

Similar distance functions and several of the observations for the examples in this section can be found in [10].

In a metric space, all open balls are open, the open balls form a base for a topology, the limits of convergent sequences are unique, and the metric function is continuous. In the distance space of the next example, none of these properties hold.

Example 1. [14] *Let (x, y) be an ordered pair in $\mathbb{R}^2$. Let $(a : b)$ be an open interval and $(a : b]$ a half-open interval in $\mathbb{R}$. Let $|x|$ be the absolute value of $x \in \mathbb{R}$. The function $\mathsf{d}(x, y) \in \mathbb{R}^{\mathbb{R}\times\mathbb{R}}$ such that*

$$\mathsf{d}(x,y) := \begin{cases} y, & \forall\,(x,y) \in \{4\} \times (0:2], \\ x, & \forall\,(x,y) \in (0:2] \times \{4\}, \\ |x-y|, & \textit{otherwise}, \end{cases}$$

is a distance on $\mathbb{R}$.

Note some characteristics of the distance space $(\mathbb{R}, \mathsf{d})$.

1. *$(\mathbb{R}, \mathsf{d})$ is not a metric space because d does not satisfy the triangle inequality:*
   $\mathsf{d}(0,4) := |0-4| = 4 \not\leq 2 = |0-1| + 1 := \mathsf{d}(0,1) + \mathsf{d}(1,4)$.

2. *Not every open ball in $(\mathbb{R}, \mathsf{d})$ is open. For example, the open ball $\mathsf{B}(3,2)$ is not open because $4 \in \mathsf{B}(3,2)$ but for all $0 < \varepsilon < 1$*
   $\mathsf{B}(4,\varepsilon) = (4-\varepsilon : 4+\varepsilon) \cup (0 : \varepsilon) \not\subseteq (1:5) = \mathsf{B}(3,2)$.

3. *The open balls of $(\mathbb{R}, \mathsf{d})$ do not form a base for a topology on $\mathbb{R}$. This follows directly from previous item and Theorem 2.*

4. *In the distance space $(\mathbb{R}, \mathsf{d})$, limits are not unique. For example, the sequence $\{1/n\}_1^\infty$ converges both to the limit 0 and the limit 4 in $(\mathbb{R}, \mathsf{d})$:*
   $$\begin{array}{llllll} \lim\limits_{n\to\infty} \mathsf{d}(x_n, 0) & := \lim\limits_{n\to\infty} \mathsf{d}(1/n, 0) & := \lim\limits_{n\to\infty} |1/n - 0| & = 0 & \Longrightarrow & \{1/n\} \to 0, \\ \lim\limits_{n\to\infty} \mathsf{d}(x_n, 4) & := \lim\limits_{n\to\infty} \mathsf{d}(1/n, 4) & := \lim\limits_{n\to\infty} \{1/n\} & = 0 & \Longrightarrow & \{1/n\} \to 4. \end{array}$$

5. *The topological space (X, T) induced by $(\mathbb{R}, \mathsf{d})$ also yields limits of 0 and 4 for the sequence $\{1/n\}_1^\infty$, just as it does in previous item. This is largely due to the fact that, for small ε, the open balls $\mathsf{B}(0,\varepsilon)$ and $\mathsf{B}(4,\varepsilon)$ are open.*
   $$\begin{array}{rcll} \mathsf{B}(0,\varepsilon) \textit{ is open} & \Longrightarrow & \textit{for each } U \in T \textit{ that contains } 0, \exists N \in \mathbb{N} \textit{ such that } 1/n \in U & \forall n > N \\ & \Longleftrightarrow & \{1/n\} \to 0 \quad \textit{by definition of convergence.} & \\ \mathsf{B}(4,\varepsilon) \textit{ is open} & \Longrightarrow & \textit{for each } U \in T \textit{ that contains } 4, \exists N \in \mathbb{N} \textit{ such that } 1/n \in U & \forall n > N \\ & \Longleftrightarrow & \{1/n\} \to 4 \quad \textit{by definition of convergence.} & \end{array}$$

6. *The distance function d is discontinuous:*
   $$\begin{aligned} \lim_{n\to\infty} \{\mathsf{d}(1-1/n, 4-1/n)\} &= \lim_{n\to\infty} \{|(1-1/n) - (4-1/n)|\} = |1-4| = 3 \neq 4 = \mathsf{d}(0,4) \\ &= \mathsf{d}\Big(\lim_{n\to\infty}\{1-1/n\}, \lim_{n\to\infty}\{4-1/n\}\Big). \end{aligned}$$

[14] A similar distance function d and item 4 can in essence be found in [10].

In a metric space, all convergent sequences are also Cauchy. However, this is not the case for all distance spaces, as demonstrated next.

Example 2. [15] *The function* $\mathsf{d}(x,y) \in \mathbb{R}^{\mathbb{R}\times\mathbb{R}}$ *such that*

$$\mathsf{d}(x,y) := \begin{cases} |x-y|, & \text{for } x=0 \text{ or } y=0 \text{ or } x=y, \\ 1, & \text{otherwise,} \end{cases}$$

is a distance on $\mathbb{R}$

Note some characteristics of the distance space $(\mathbb{R},\mathsf{d})$.

1. $(\mathbb{R},\mathsf{d})$ *is not a metric space because the triangle inequality does not hold:*
$$\mathsf{d}\left(\tfrac{1}{4},\tfrac{1}{2}\right) = 1 \not\leq \tfrac{3}{4} = \left|\tfrac{1}{4}-0\right| + \left|0-\tfrac{1}{2}\right| = \mathsf{d}\left(\tfrac{1}{4},0\right) + \mathsf{d}\left(0,\tfrac{1}{2}\right).$$

2. *The open ball* $\mathsf{B}\left(\frac{1}{4},\frac{1}{2}\right)$ *is not open because for any* $\varepsilon \in \mathbb{R}^+$, *no matter how small,*
$$\mathsf{B}(0,\varepsilon) = (-\varepsilon : +\varepsilon) \not\subseteq \left\{0, \tfrac{1}{4}\right\} = \left\{x \in X \,\middle|\, \mathsf{d}\left(\tfrac{1}{4},x\right) < \tfrac{1}{2}\right\} := \mathsf{B}\left(\tfrac{1}{4},\tfrac{1}{2}\right).$$

3. *Even though not all the open balls are open, it is still possible to have an open set in* $(\mathbb{R},\mathsf{d})$. *For example, the set* $U := \{1, 2\}$ *is open:*
$$\begin{array}{lclclclcl} \mathsf{B}(1,1) & := & \{x \in X \,|\, \mathsf{d}(1,x) < 1\} & = & \{1\} & \subseteq & \{1,2\} & := & U, \\ \mathsf{B}(2,1) & := & \{x \in X \,|\, \mathsf{d}(2,x) < 1\} & = & \{2\} & \subseteq & \{1,2\} & := & U. \end{array}$$

4. *By item 2 and Theorem 2, the open balls of the distance space* $(\mathbb{R},\mathsf{d})$ *do not form a base for a topology on* $\mathbb{R}$.

5. *Even though the open balls in* $(\mathbb{R},\mathsf{d})$ *do not induce a topology on* $\mathbb{R}$, *it is still possible to find a set of open sets in* $(\mathbb{R},\mathsf{d})$ *that is a topology. For example, the set* $\{\varnothing, \{1,2\}, \mathbb{R}\}$ *is a topology on* $\mathbb{R}$.

6. *In* $(\mathbb{R},\mathsf{d})$ *limits of convergent sequences are unique. Namely,* $\{x_n\} \to x \implies$
$$\lim_{n\to\infty} \mathsf{d}(x_n,x) = \begin{cases} \lim |x_n - 0| & = & 0, & \text{for } x = 0, \\ |x-x| & = & 0, & \text{for constant } \{x_n\} \text{ for } n > N, \\ 1 & \neq & 0, & \text{otherwise,} \end{cases}$$
which says that there are only two ways for a sequence to converge: either $x = 0$ *or the sequence eventually becomes constant (or both). Any other sequence will diverge.*

7. *In* $(\mathbb{R},\mathsf{d})$ *a convergent sequence is not necessarily Cauchy. For example, the sequence* $\{{}^1/_n\}_{n\in\mathbb{N}}$ *is convergent with limit* 0
$$\lim_{n\to\infty} \mathsf{d}({}^1/_n, 0) = \lim_{n\to\infty} {}^1/_n = 0.$$
However, even though $\{{}^1/_n\}$ *is convergent, it is not Cauchy*
$$\lim_{n,m\to\infty} \mathsf{d}({}^1/_n, {}^1/_m) = 1 \neq 0.$$

8. *The distance function* d *is discontinuous in* (X,d):
$$\lim_{n\to\infty} \{\mathsf{d}({}^1/_n, 2 - {}^1/_n)\} = 1 \neq 2 = \mathsf{d}(0,2) = \mathsf{d}\left(\lim_{n\to\infty}\{{}^1/_n\}, \lim_{n\to\infty}\{2 - {}^1/_n\}\right).$$

[15] The distance function d and item 7 can in essence be found in [10].

Example 3. [16] *The function* $\mathsf{d}(x,y) \in \mathbb{R}^{\mathbb{R}\times\mathbb{R}}$ *such that*
$$\mathsf{d}(x,y) := \begin{cases} 2|x-y|, & \forall\,(x,y) \in \{(0,1)\,,\,(1,0)\}, \\ |x-y|, & \text{otherwise}, \end{cases}$$
is a distance on $\mathbb{R}$.

Note some characteristics of the distance space $(\mathbb{R},\mathsf{d})$.

1. $(\mathbb{R},\mathsf{d})$ *is not a metric space because* d *does not satisfy the triangle inequality:*
$$\mathsf{d}(0,1) := 2|0-1| = 2 \not\leq 1 = |0-1/2| + |1/2-1| := \mathsf{d}(0,1/2) + \mathsf{d}(1/2,1)\,.$$

2. *The function* d *is discontinuous:*
$$\begin{aligned}\lim_{n\to\infty}\{\mathsf{d}(1-1/n,1/n)\} &:= \lim_{n\to\infty}\{|1-1/n-1/n|\} = 1 \neq 2 \\ &= 2|0-1| := \mathsf{d}(0,1) = \mathsf{d}\Big(\lim_{n\to\infty}\{1-1/n\}\,,\lim_{n\to\infty}\{1/n\}\Big)\,.\end{aligned}$$

3. *In* $(\mathbb{R},\mathsf{d})$ *open balls are open:*
   (a) $\mathsf{p}\,(x,y) := |x-y|$ *is a metric and thus all open balls in that do not contain both* 0 *and* 1 *are open;*
   (b) *by Example 14,* $\mathsf{q}\,(x,y) := 2|x-y|$ *is also a metric and thus all open balls containing* 0 *and* 1 *only are open;*
   (c) *the only question remaining is with regards to open balls that contain* $0, 1$ *and some other element(s) in* $\mathbb{R}$. *But even in this case, open balls are still open. For example,* $\mathsf{B}(-1,2) = (-1:2) = (-1:1) \cup (1:2)$. *Note that both* $(-1:1)$ *and* $(1:2)$ *are open, and thus by Theorem 1,* $\mathsf{B}(-1,2)$ *is open as well.*

4. *By previous item and Theorem 2, the open balls of* $(\mathbb{R},\mathsf{d})$ do *form a base for a topology on* $\mathbb{R}$.

5. *In* $(\mathbb{R},\mathsf{d})$ *the limits of convergent sequences are unique. This is demonstrated in Example 7 using additional structure developed in Section 4.*

6. *In* $(\mathbb{R},\mathsf{d})$ *convergent sequences are Cauchy. This is also demonstrated in Example 7.*

The distance functions in Examples 1–3 were all discontinuous. In the absence of the triangle inequality and in light of these examples, one might try replacing the triangle inequality with the weaker requirement of continuity. However, as demonstrated by the next example, this also leads to an arguably disastrous result.

Example 4 ([10,74])**.** *The function* $\mathsf{d} \in \mathbb{R}^{\mathbb{R}\times\mathbb{R}}$ *such that* $\mathsf{d}(x,y) := (x-y)^2$ *is a distance on* $\mathbb{R}$. *Note some characteristics of the distance space* $(\mathbb{R},\mathsf{d})$.

1. $(\mathbb{R},\mathsf{d})$ *is not a metric space because the triangle inequality does not hold:*
$$\mathsf{d}(0,2) := (0-2)^2 = 4 \not\leq 2 = (0-1)^2 + (1-2)^2 := \mathsf{d}(0,1) + \mathsf{d}(1,2)\,.$$

2. *The distance function* d *is continuous in* (X,d). *This is demonstrated in the more general setting of Section 4 in Example 8.*

[16] The distance function d and item 2 can in essence be found in [10].

3. *Calculating the length of curves in* $(\mathbb{R}, \mathsf{d})$ *leads to a paradox.*[17] *Partition* $[0:1]$ *into* 2^N *consecutive line segments connected at the points* $\left\{0, \frac{1}{2^N}, \frac{2}{2^N}, \frac{3}{2^N}, \ldots, \frac{2^{N-1}1}{2^N}, 1\right\}$. *Then the distance, as measured by* d, *between any two consecutive points is equal to* $\mathsf{d}(p_n, p_{n+1}) := (p_n - p_{n+1})^2 = \left(\frac{1}{2^N}\right)^2 = \frac{1}{2^{2N}}$. *But this leads to the paradox that the total length of* $[0:1]$ *is 0:*

$$\lim_{N\to\infty} \sum_{n=0}^{2^N-1} \frac{1}{2^{2N}} = \lim_{N\to\infty} \frac{2^N}{2^{2N}} = \lim_{N\to\infty} \frac{1}{2^N} = 0.$$

4 DISTANCE SPACES WITH POWER TRIANGLE INEQUALITIES

4.1 Definitions

This paper introduces a new relation called the power triangle inequality. It is a generalization of other common relations, including the triangle inequality. The power triangle inequality is defined in terms of a function herein called the power triangle function (next definition). This function is a special case of the power mean with $N = 2$ and $\lambda_1 = \lambda_2 = \frac{1}{2}$. Power means have the attractive properties of being continuous and strictly monotone with respect to a free parameter $p \in \mathbb{R}^*$. This fact is inherited and exploited by the power triangle inequality.

Definition 21. *Let* (X, d) *be a distance space. Let* $\mathbb{R}^+$ *be the set of all positive real numbers and* $\mathbb{R}^*$ *be the set of extended real numbers. The power triangle function* τ *on* (X, d) *is defined as*

$$\tau(p, \sigma; x, y, z; \mathsf{d}) := 2\sigma\left[\frac{1}{2}\mathsf{d}^p(x,z) + \frac{1}{2}\mathsf{d}^p(z,y)\right]^{\frac{1}{p}}, \qquad \forall (p,\sigma) \in \mathbb{R}^* \times \mathbb{R}^{\vdash}, \quad x, y, z \in X.$$

Remark 3. *In the field of probabilistic metric spaces, a function called the triangle function was introduced by Sherstnev [102]. However, the power triangle function as defined in the present paper is not a special case of (is not compatible with) the triangle function of Sherstnev. Another definition of triangle function has been offered by Bessenyei [6] with special cases of* $\Phi(u,v) := c(u+v)$ *and* $\Phi(u,v) := (u^p + v^p)^{\frac{1}{p}}$, *which are similar to the definition of power triangle function offered in the present paper.*

Definition 22. *Let* (X, d) *be a distance space. Let* 2^{XXX} *be the set of all trinomial relations on* X *(see Definition 3). A relation* $\triangle(p, \sigma; \mathsf{d})$ *in* 2^{XXX} *is a power triangle inequality on* (X, d) *if*

$$\triangle(p, \sigma; \mathsf{d}) := \left\{(x,y,z) \in X^3 \mid \mathsf{d}(x,y) \le \tau(p,\sigma;x,y,z;\mathsf{d})\right\} \qquad \text{for some } (p,\sigma) \in \mathbb{R}^* \times \mathbb{R}^+.$$

The tuple $(X, \mathsf{d}, p, \sigma)$ *is a power distance space and* d *a power distance or power distance function if* (X, d) *is a distance space in which the triangle relation* $\triangle(p, \sigma; \mathsf{d})$ *holds.*

The power triangle function can be used to define some standard inequalities (next definition). See Corollary 3 for some justification of the definitions.

[17] This is the method of "inscribed polygons" for calculating the length of a curve and goes back to Archimedes [17, 117].

Definition 23 ([6,36,41,44,46,47,52,62,63,69,119]). *Let* $\triangle(p,\sigma;\mathsf{d})$ *be a power triangle inequality on a distance space* (X,d).

1. $\triangle(\infty,{}^{\sigma}/_{2};\mathsf{d})$ *is the* σ*-inframetric inequality.*
2. $\triangle(\infty,\frac{1}{2};\mathsf{d})$ *is the inframetric inequality.*
3. $\triangle(2,\sqrt{2}/2;\mathsf{d})$ *is the quadratic inequality.*
4. $\triangle(1,\sigma;\mathsf{d})$ *is the relaxed triangle inequality.*
5. $\triangle(1,1;\mathsf{d})$ *is the triangle inequality.*
6. $\triangle({}^{1}/_{2},2;\mathsf{d})$ *is the square mean root inequality.*
7. $\triangle(0,\frac{1}{2};\mathsf{d})$ *is the geometric inequality.*
8. $\triangle(-1,\frac{1}{4};\mathsf{d})$ *is the harmonic inequality.*
9. $\triangle(-\infty,\frac{1}{2};\mathsf{d})$ *is the minimal inequality.*

Definition 24. [18] *Let* (X,d) *be a distance space.*

1. (X,d) *is a metric space if the triangle inequality holds in* X.
2. (X,d) *is a near metric space if the relaxed triangle inequality holds in* X.
3. (X,d) *is an inframetric space if the inframetric inequality holds in* X.
4. (X,d) *is a* σ*-inframetric space if the* σ*-inframetric inequality holds in* X.

4.2 Properties

4.2.1 Relationships of the power triangle function

Corollary 2. *Let* $\tau(p,\sigma;x,y,z;\mathsf{d})$ *be the power triangle function in the distance space* (X,d). *Let* $(\mathbb{R},|\cdot|,\leq)$ *be the ordered metric space with the usual ordering relation* $\leq$ *and usual metric* $|\cdot|$ *on* $\mathbb{R}$. *The function* $\tau(p,\sigma;x,y,z;\mathsf{d})$ *is continuous and strictly monotone in* $(\mathbb{R},|\cdot|,\leq)$ *with respect to both the variables* p *and* σ.

Proof. The function $\tau(p,\sigma;x,y,z;\mathsf{d})$ is continuous and strictly monotone with respect to p via Theorem 18. By definition 21 of τ we have

$$\tau(p,\sigma;x,y,z;\mathsf{d}) := 2\sigma\underbrace{\left[\frac{1}{2}\mathsf{d}^p(x,z)+\frac{1}{2}\mathsf{d}^p(z,y)\right]^{\frac{1}{p}}}_{\mathsf{f}(p,x,y,z)} = 2\sigma\mathsf{f}(p,x,y,z),$$

where f is defined as above. Therefore τ is affine with respect to σ, and, hence, $\tau(p,\sigma;x,y,z;\mathsf{d})$ is continuous and strictly monotone with respect to σ. □

Corollary 3. *Let* $\tau(p,\sigma;x,y,z;\mathsf{d})$ *be the power triangle function in the distance space* (X,d).

$$\tau(p,\sigma;x,y,z;\mathsf{d}) = \begin{cases} 2\sigma\max\{\mathsf{d}(x,z),\,\mathsf{d}(z,y)\} & \text{for } p=\infty, & \text{(maximum)},^{19} \\ 2\sigma\left[{}^{1}/_{2}\mathsf{d}^2(x,z)+{}^{1}/_{2}\mathsf{d}^2(z,y)\right]^{\frac{1}{2}} & \text{for } p=2, & \text{(quadratic mean)}, \\ \sigma[\mathsf{d}(x,z)+\mathsf{d}(z,y)] & \text{for } p=1, & \text{(arithmetic mean)},^{20} \\ 2\sigma\sqrt{\mathsf{d}(x,z)}\,\sqrt{\mathsf{d}(z,y)} & \text{for } p=0, & \text{(geometric mean)}, \\ 4\sigma\left[\frac{1}{\mathsf{d}(x,z)}+\frac{1}{\mathsf{d}(z,y)}\right]^{-1} & \text{for } p=-1, & \text{(harmonic mean)}, \\ 2\sigma\min\{\mathsf{d}(x,z),\,\mathsf{d}(z,y)\} & \text{for } p=-\infty, & \text{(minimum)}. \end{cases}$$

Proof. These follow directly from Theorem 18. □

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[18] For definitions in metric space see [30,43,48,49,60]; in near metric space see [36,41,46,47,62,69,119].

[19] The maximum $\tau(\infty,\sigma;x,y,z;d)$ corresponds to the inframetric space.

[20] The arithmetic mean $\tau(1,\sigma;x,y,z;d)$ corresponds to the near metric space.

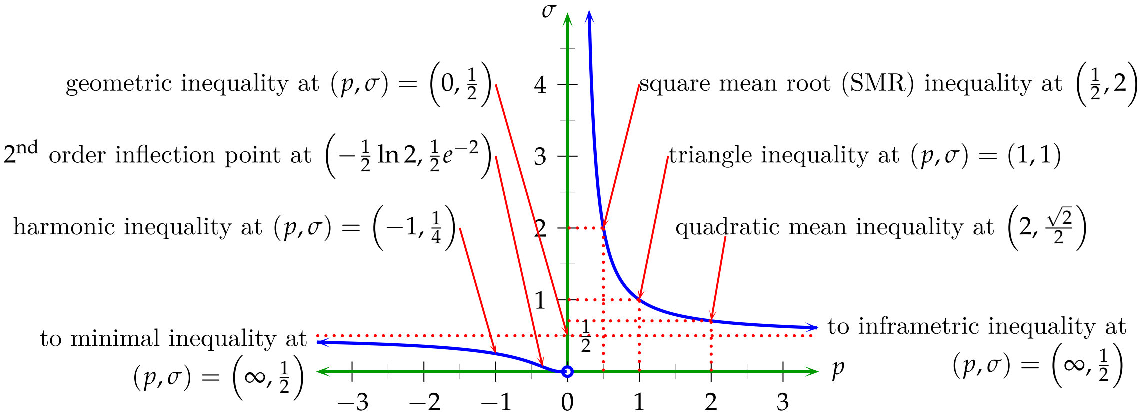


Figure 1: $\sigma = \frac{1}{2}(2^{\frac{1}{p}}) = 2^{\frac{1}{p}-1}$ or $p = \frac{\ln 2}{\ln(2\sigma)}$ (see Lemma 1, Lemma 2, Corollary 6, Corollary 7, and Theorem 9).

Corollary 4. *Let* (X, d) *be a distance space. Then*

$$\begin{array}{rcl} 2\sigma \min\{\mathsf{d}(x,z), \mathsf{d}(z,y)\} & \leq & 4\sigma\left[\frac{1}{\mathsf{d}(x,z)} + \frac{1}{\mathsf{d}(z,y)}\right]^{-1} \quad \leq \quad 2\sigma\sqrt{\mathsf{d}(x,z)}\,\sqrt{\mathsf{d}(z,y)} \\ & \leq & \sigma[\mathsf{d}(x,z)+\mathsf{d}(z,y)] \quad \leq \quad 2\sigma\max\{\mathsf{d}(x,z), \mathsf{d}(z,y)\}. \end{array}$$

Proof. These follow directly from Corollary 8. ☐

4.2.2 Properties of power distance spaces

The power triangle inequality property of a power distance space axiomatically endows a metric with an upper bound. Lemma 1 demonstrates that there is a complementary lower bound somewhat similar in form to the power triangle inequality upper bound. In the special case where $2\sigma = 2^{\frac{1}{p}}$, the lower bound helps provide a simple proof of the continuity of a large class of power distance functions (Theorem 9). The inequality $2\sigma \leq 2^{\frac{1}{p}}$ is a special relation in this paper and appears repeatedly in this paper; it appears as an inequality in Lemma 2, Corollaries 6 and 7, and as an equality in Lemma 1 and Theorem 9. It is plotted in Figure 1.

Lemma 1. [21] *Let* $(X, \mathsf{d}, p, \sigma)$ *be a power triangle triangle space. Let* $|\cdot|$ *be the absolute value function. Let* $\max\{x,y\}$ *be the maximum and* $\min\{x,y\}$ *the minimum of any* $x, y \in \mathbb{R}^*$. *Then, for all* $(p,\sigma) \in \mathbb{R}^* \times \mathbb{R}^+$,

1. $\mathsf{d}^p(x,y) \geq \max\left\{0, \frac{2}{(2\sigma)^p}\mathsf{d}^p(x,z) - \mathsf{d}^p(z,y), \frac{2}{(2\sigma)^p}\mathsf{d}^p(y,z) - \mathsf{d}^p(z,x)\right\} \quad \forall x,y,z \in X,$
2. $\mathsf{d}(x,y) \geq |\mathsf{d}(x,z) - \mathsf{d}(z,y)|$ *if* $p \neq 0$ *and* $2\sigma = 2^{\frac{1}{p}} \quad \forall x,y,z \in X.$

Proof. From power triangle inequality and symmetric property of d we obtain

$$\begin{aligned} \frac{2}{(2\sigma)^p}\mathsf{d}^p(x,z) - \mathsf{d}^p(z,y) &\leq \frac{2}{(2\sigma)^p}\left[2\sigma[{}^1\!/\!{}_2\mathsf{d}^p(x,y) + {}^1\!/\!{}_2\mathsf{d}^p(y,z)]^{\frac{1}{p}}\right]^p - \mathsf{d}^p(z,y) \\ &= \frac{2(2\sigma)^p}{(2\sigma)^p}[{}^1\!/\!{}_2\mathsf{d}^p(x,y) + {}^1\!/\!{}_2\mathsf{d}^p(y,z)] - \mathsf{d}^p(z,y) \\ &= [\mathsf{d}^p(x,y) + \mathsf{d}^p(y,z)] - \mathsf{d}^p(y,z) = \mathsf{d}^p(x,y). \end{aligned}$$

[21] For assertion in metric space, i.e. $(p,\sigma) = (1,1)$ see [5,43,83].

Using commutative and non-negative properties of d, for $(p,\sigma) \in \mathbb{R}^* \times \mathbb{R}^+$ one can derive

$$\mathsf{d}^p(x,y) \geq \frac{2}{(2\sigma)^p}\mathsf{d}^p(x,z) - \mathsf{d}^p(z,y)\,, \qquad \mathsf{d}^p(y,x) \geq \frac{2}{(2\sigma)^p}\mathsf{d}^p(y,z) - \mathsf{d}^p(z,x)\,, \qquad \mathsf{d}^p(x,y) \geq 0.$$

The rest follows because $\mathsf{g}(x) := x^{\frac{1}{p}}$ is strictly monotone in $\mathbb{R}^{\mathbb{R}}$.

In case $2\sigma = 2^{\frac{1}{p}}$ we have

$$\begin{aligned}\mathsf{d}(x,y) &\geq \max\left\{0,\, \frac{2}{(2\sigma)^p}\mathsf{d}^p(x,z) - \mathsf{d}^p(z,y)\,,\, \frac{2}{(2\sigma)^p}\mathsf{d}^p(y,z) - \mathsf{d}^p(z,x)\right\}^{\frac{1}{p}}\\ &= \max\left\{0,\, \mathsf{d}(x,z) - \mathsf{d}(z,y)\,,\, \mathsf{d}(y,z) - \mathsf{d}(z,x)\right\}\\ &= \max\left\{0,\, (\mathsf{d}(x,z) - \mathsf{d}(z,y)),\, -(\mathsf{d}(x,z) - \mathsf{d}(z,y))\right\} = |(\mathsf{d}(x,z) - \mathsf{d}(z,y))|.\end{aligned}$$

□

Theorem 5. *Let (X,d,p,σ) be a power distance space. Let B be an open ball on (X,d). Then for all $(p,\sigma) \in (\mathbb{R}^*\backslash\{0\}) \times \mathbb{R}^+$ the following implications hold:*

1. *if $2\sigma \leq 2^{\frac{1}{p}}$ and $q \in \mathsf{B}(\theta,r)$ then there exists $r_q \in \mathbb{R}^+$ such that $\mathsf{B}(q,r_q) \subseteq \mathsf{B}(\theta,r)$;*
2. *if there exists $r_q \in \mathbb{R}^+$ such that $\mathsf{B}(q,r_q) \subseteq \mathsf{B}(\theta,r)$ then $q \in \mathsf{B}(\theta,r)$.*

Proof. Using the Archimedean Property[22] we obviously obtain

$$q \in \mathsf{B}(\theta,r) \iff \mathsf{d}(\theta,q) < r \iff 0 < r - \mathsf{d}(\theta,q) \iff \exists r_q \in \mathbb{R}^+,\, 0 < r_q < r - \mathsf{d}(\theta,q)\,.$$

Therefore

$$\begin{aligned}\mathsf{B}(q,r_q) &:= \left\{x \in X \,\middle|\, \mathsf{d}(q,x) < r_q\right\} = \left\{x \in X \,\middle|\, \mathsf{d}^p(q,x) < r_q^p \in \mathbb{R}^+\right\}\\ &\subseteq \{x \in X \,|\, \mathsf{d}^p(q,x) < r^p - \mathsf{d}^p(\theta,q)\} = \{x \in X \,|\, \mathsf{d}^p(\theta,q) + \mathsf{d}^p(q,x) < r^p\}\\ &= \left\{x \in X \,\middle|\, [\mathsf{d}^p(\theta,q) + \mathsf{d}^p(q,x)]^{\frac{1}{p}} < r\right\} \subseteq \left\{x \in X \,\middle|\, 2^{1-1/p}\sigma[\mathsf{d}^p(\theta,q) + \mathsf{d}^p(q,x)]^{\frac{1}{p}} < r\right\}\\ &= \left\{x \in X \,\middle|\, 2\sigma[1/2\mathsf{d}(\theta,q)\,x + 1/2\mathsf{d}^p(q,x)]^{\frac{1}{p}} < r\right\} := \{x \in X \,|\, \tau(p,\sigma,\theta,x,q) < r\}\\ &\subseteq \{x \in X \,|\, \mathsf{d}(\theta,x) < r\} := \mathsf{B}(\theta,r)\,.\end{aligned}$$

Here we used the fact that the functions $\mathsf{f}(x) := x^p$ and $\mathsf{f}(x) := x^{\frac{1}{p}}$ are monotone. So, the first implication is proved.

The second implication follows from

$$q \in \{x \in X \,|\, \mathsf{d}(q,x) = 0\} \subseteq \left\{x \in X \,\middle|\, \mathsf{d}(q,x) < r_q\right\} := \mathsf{B}(q,r_q) \subseteq \mathsf{B}(\theta,r)\,.$$

□

The next assertion follows from Theorem 2 and Theorem 5.

Corollary 5. *Let (X,d,p,σ) be a power distance space. If the inequality $2\sigma \leq 2^{\frac{1}{p}}$ holds for all $(p,\sigma) \in (\mathbb{R}^*\backslash\{0\}) \times \mathbb{R}^+$ then every open ball in (X,d) is open.*

Corollary 6. *Let (X,d,p,σ) be a power distance space. Let B be the set of all open balls in (X,d). If the inequality $2\sigma \leq 2^{\frac{1}{p}}$ holds for all $(p,\sigma) \in (\mathbb{R}^*\backslash\{0\}) \times \mathbb{R}^+$ then B is a base for (X,T).*

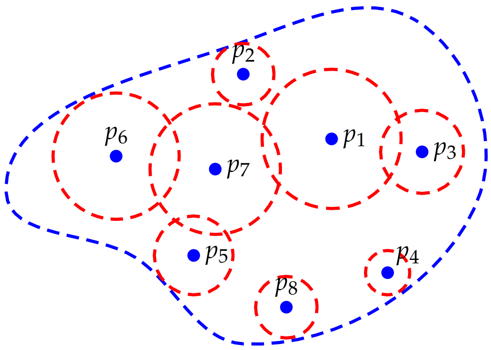


Figure 2: open set (see Lemma 2) .

Proof. The set of all open balls in (X, d) is a base for (X, T) by Corollary 5 and Theorem 11. T is a topology on X by Definition 26. $\square$

The next assertion demonstrates that every point in an open set is contained in an open ball that is contained in the original open set (see also Figure 2).

Lemma 2. *Let $(X, \mathsf{d}, p, \sigma)$ be a power distance space. Let* B *be an open ball on (X, d). Then for all $(p, \sigma) \in (\mathbb{R}^* \backslash \{0\}) \times \mathbb{R}^+$ the following implications hold:*

1. *if $2\sigma \leq 2^{\frac{1}{p}}$ and U is open in (X, d) then for all $x \in U$ there exists $r \in \mathbb{R}^+$ such that $\mathsf{B}(x, r) \subseteq U$;*
2. *if for all $x \in U$ there exists $r \in \mathbb{R}^+$ such that $\mathsf{B}(x, r) \subseteq U$ then U is open in (X, d).*

Proof. From Corollary 6 we have

$$U = \bigcup \{\mathsf{B}(x_\gamma, r_\gamma) \mid \mathsf{B}(x_\gamma, r_\gamma) \subseteq U\} \supseteq \mathsf{B}(x, r),$$

because x must be in one of those balls in U. So, the first implication is proved.

The second implication follows from

$$U = \bigcup \{x \in X \mid x \in U\} = \bigcup \{\mathsf{B}(x, r) \mid x \in U \text{ and } \mathsf{B}(x, r) \subseteq U\} \implies U \text{ is open}$$

by Corollary 6 and Corollary 1. $\square$

Corollary 7. [23] *Let $(X, \mathsf{d}, p, \sigma)$ be a power distance space. Let* B *be an open ball on (X, d). If $2\sigma \leq 2^{\frac{1}{p}}$ for all $(p, \sigma) \in (\mathbb{R}^* \backslash \{0\}) \times \mathbb{R}^+$ then every open ball $\mathsf{B}(x, r)$ in (X, d) is open.*

Proof. The union of any set of open balls is open by Corollary 6, therefore the union of a set of just one open ball is open. Hence, every open ball is open. $\square$

Theorem 6. [24] *Let $(X, \mathsf{d}, p, \sigma)$ be a power distance space. Let (X, T) be a topological space induced by (X, d). Let $\{x_n\}_{n \in \mathbb{Z}} \subset X$ be a sequence in (X, d). The sequence $\{x_n\}$ converges to a limit x iff for any $\varepsilon \in \mathbb{R}^+$ there exists $N \in \mathbb{Z}$ such that for all $n > N$, $\mathsf{d}(x_n, x) < \varepsilon$.*

[22] See [1, 121].
[23] For assertion in metric space see [1, 97].
[24] For theorem in metric space see [53, 97].

Proof. The sequence $\{x_n\}$ converges to x if and only if $x_n \in U\ \forall U \in N_x$, $n > N$. By Lemma 2 $\exists \mathsf{B}(x,\varepsilon)$ such that $x_n \in \mathsf{B}(x,\varepsilon)\ \forall n > N$. So, $\mathsf{d}(x_n, x) < \varepsilon$. $\square$

In distance spaces not all convergent sequences are Cauchy (see Example 2). However in a distance space with any power triangle inequality all convergent sequences are Cauchy.

Theorem 7. [25] *Let $(X, \mathsf{d}, p, \sigma)$ be a power distance space with any $(p,\sigma) \in \mathbb{R}^* \times \mathbb{R}^+$. Let $\{x_n\}_{n\in\mathbb{Z}} \subset X$ be a sequence in (X, d). Every convergent sequence $\{x_n\}$ is a Cauchy sequence and therefore it is bounded in (X, d).*

Proof. Let $\{x_n\}_{n\in\mathbb{Z}}$ be a convergent sequence in (X, d). Then we have

$$\mathsf{d}(x_n, x_m) \le \tau(p,\sigma; x_n, x_m, x) := 2\sigma\left[\frac{1}{2}\mathsf{d}^p(x_n, x) + \frac{1}{2}\mathsf{d}^p(x_m, x)\right]^{\frac{1}{p}} < 2\sigma\left[\frac{1}{2}\varepsilon^p + \frac{1}{2}\varepsilon^p\right]^{\frac{1}{p}} = 2\sigma\varepsilon.$$

By Corollary 3 and definitions of power triangle inequality at $p = \infty$, $p = -\infty$ and $p = 0$ we have

$$\begin{aligned}
\mathsf{d}(x_n, x_m) &\le \tau(\infty,\sigma; x_n, x_m, x) = 2\sigma \max\{\mathsf{d}(x_n, x),\ \mathsf{d}(x_m, x)\} = 2\sigma \max\{\varepsilon,\ \varepsilon\} = 2\sigma\varepsilon;\\
\mathsf{d}(x_n, x_m) &\le \tau(-\infty,\sigma; x_n, x_m, x) = 2\sigma \min\{\mathsf{d}(x_n, x),\ \mathsf{d}(x_m, x)\} = 2\sigma \min\{\varepsilon,\ \varepsilon\} = 2\sigma\varepsilon;\\
\mathsf{d}(x_n, x_m) &\le \tau(0,\sigma; x_n, x_m, x) = 2\sigma\sqrt{\mathsf{d}(x_n, x)}\,\sqrt{\mathsf{d}(x_m, x)} = 2\sigma\sqrt{\varepsilon}\,\sqrt{\varepsilon} = 2\sigma\varepsilon.
\end{aligned}$$

Therefore the sequence $\{x_n\}$ is Cauchy. By Proposition 1 every Cauchy sequence is bounded. $\square$

Theorem 8. [26] *Let $(X, \mathsf{d}, p, \sigma)$ be a power distance space with any $(p,\sigma) \in \mathbb{R}^* \times \mathbb{R}^+$. Let $\mathsf{f} \in \mathbb{Z}^{\mathbb{Z}}$ be a strictly monotone function such that $\mathsf{f}(n) < \mathsf{f}(n+1)$. If $\{x_n\}_{n\in\mathbb{Z}}$ is a Cauchy sequence and $\left\{x_{\mathsf{f}(n)}\right\}_{n\in\mathbb{Z}}$ is convergent then $\{x_n\}_{n\in\mathbb{Z}}$ is convergent.*

Proof. It is easy to see that

$$\begin{aligned}
\mathsf{d}(x_n, x) = \mathsf{d}(x, x_n) &\le \tau(p,\sigma; x, x_n, x_{\mathsf{f}(n)}) := 2\sigma\left[\frac{1}{2}\mathsf{d}^p\left(x, x_{\mathsf{f}(n)}\right) + \frac{1}{2}\mathsf{d}^p\left(x_{\mathsf{f}(n)}, x_n\right)\right]^{\frac{1}{p}}\\
&= 2\sigma\left[\frac{1}{2}\varepsilon + \frac{1}{2}\mathsf{d}^p\left(x_{\mathsf{f}(n)}, x_n\right)\right]^{\frac{1}{p}} = 2\sigma\left[\frac{1}{2}\varepsilon^p + \frac{1}{2}\varepsilon^p\right]^{\frac{1}{p}} = 2\sigma\varepsilon,
\end{aligned}$$

so, the sequence $\{x_n\}_{n\in\mathbb{Z}}$ is convergent. $\square$

Theorem 9. [27] *Let $(X, \mathsf{d}, p, \sigma)$ be a power distance space. Let $(\mathbb{R}, \mathsf{q})$ be a metric space of real numbers with the usual metric $\mathsf{q}(x,y) := |x-y|$. If $2\sigma = 2^{\frac{1}{p}}$ then d is continuous in $(\mathbb{R}, \mathsf{q})$.*

Proof. Using triangle inequality of $(\mathbb{R}, |x-y|)$ and Lemma 1 we obtain

$$\begin{aligned}
|\mathsf{d}(x,y) - \mathsf{d}(x_n, y_n)| &\le |\mathsf{d}(x,y) - \mathsf{d}(x_n, y)| + |\mathsf{d}(x_n, y) - \mathsf{d}(x_n, y_n)|\\
&= |\mathsf{d}(x,y) - \mathsf{d}(y, x_n)| + |\mathsf{d}(y, x_n) - \mathsf{d}(x_n, y_n)|\\
&\le \mathsf{d}(x, x_n) + \mathsf{d}(y, y_n) \to 0 \qquad \text{as } n \to \infty.
\end{aligned}$$

$\square$

[25] For theorem in metric space see [2,53,97].
[26] For theorem in metric space see [97].
[27] For theorem in metric space see [5].

In distance spaces and topological spaces, limits of convergent sequences are in general not unique (see Example 1, Example 12). However the next theorem demonstrates that in a power distance space limits are unique.

Theorem 10 (Uniqueness of limit). [28] *Let $(X, \mathsf{d}, p, \sigma)$ be a power distance space with any $(p, \sigma) \in \mathbb{R}^* \times \mathbb{R}^+$. Let $x, y \in X$ and let $\{x_n\} \subset X$ be an X-valued sequence.*

If $(\{x_n\}, \{y_n\}) \to (x, y)$ then $x = y$.

Proof. Let us prove that for all $(p, \sigma) \in \mathbb{R}^* \times \mathbb{R}^+$ and for any $\varepsilon \in \mathbb{R}^+$, there exists N such that $\mathsf{d}(x, y) < 2\sigma\varepsilon$. For $p \in \mathbb{R}^* \setminus \{-\infty, 0, \infty\}$ we have

$$\mathsf{d}(x,y) \leq \tau(p,\sigma;x,y,x_n) := 2\sigma\left[\frac{1}{2}\mathsf{d}^p(x,x_n) + \frac{1}{2}\mathsf{d}^p(x_n,y)\right]^{\frac{1}{p}} < 2\sigma\left[\frac{1}{2}\varepsilon^p + \frac{1}{2}\varepsilon^p\right]^{\frac{1}{p}} = 2\sigma\varepsilon.$$

By Corollary 3 and definition of power triangle inequality at $p = \infty$, $p = -\infty$, $p = 0$ we have

$$\begin{aligned}
\mathsf{d}(x,y) &\leq \tau(\infty,\sigma;x,y,x_n) = 2\sigma \max\{\mathsf{d}(x,x_n),\, \mathsf{d}(x_n,y)\} < 2\sigma\varepsilon,\\
\mathsf{d}(x,y) &\leq \tau(-\infty,\sigma;x,y,x_n) = 2\sigma \min\{\mathsf{d}(x,x_n),\, \mathsf{d}(x_n,y)\} < 2\sigma\varepsilon,\\
\mathsf{d}(x,y) &\leq \tau(0,\sigma;x,y,x_n) = 2\sigma\sqrt{\mathsf{d}(x,x_n)}\,\sqrt{\mathsf{d}(x_n,y)} = 2\sigma\sqrt{\varepsilon}\,\sqrt{\varepsilon} < 2\sigma\varepsilon
\end{aligned}$$

respectively.

Suppose that $x \neq y$. Then $\mathsf{d}(x, y) \neq 0$, and therefore $\mathsf{d}(x, y) > 0$. It implies that there exists ε such that $\mathsf{d}(x, y) > 2\sigma\varepsilon$, which contradicts the proved above inequality $\mathsf{d}(x, y) < 2\sigma\varepsilon$. ☐

4.3 Examples

It is not always possible to find a triangle relation $\triangle(p, \sigma; \mathsf{d})$ that holds in every distance space, as demonstrated by Example 5 and Example 6 (next two examples).

Example 5. *Let $\mathsf{d}(x, y) \in \mathbb{R}^{\mathbb{R}\times\mathbb{R}}$ be defined as follows*

$$\mathsf{d}(x,y) := \begin{cases} y, & \forall\,(x,y) \in \{4\} \times (0:2],\\ x, & \forall\,(x,y) \in (0:2] \times \{4\},\\ |x-y|, & \text{otherwise.} \end{cases}$$

Note the following about the pair $(\mathbb{R}, \mathsf{d})$.

1. *By Example 1, $(\mathbb{R}, \mathsf{d})$ is a distance space, but not a metric space, that is, the triangle relation $\triangle(1, 1; \mathsf{d})$ does not hold in $(\mathbb{R}, \mathsf{d})$.*

2. *Observe further that $(\mathbb{R}, \mathsf{d})$ is not a power distance space. In particular, the triangle relation $\triangle(p, \sigma; \mathsf{d})$ does not hold in $(\mathbb{R}, \mathsf{d})$ for any finite value of σ (does not hold for any $\sigma \in \mathbb{R}^+$)*

$$\begin{aligned}
\mathsf{d}(0,4) = 4 \not\leq 0 = \lim_{\varepsilon\to 0} 2\sigma\varepsilon &= \lim_{\varepsilon\to 0} 2\sigma\left[1/2|0-\varepsilon|^p + 1/2\varepsilon^p\right]^{\frac{1}{p}}\\
&:= \lim_{\varepsilon\to 0} 2\sigma[1/2\mathsf{d}^p(0,\varepsilon) + 1/2\mathsf{d}^p(\varepsilon,4)]^{\frac{1}{p}} := \lim_{\varepsilon\to 0} \triangle(p,\sigma;0,4,\varepsilon;\mathsf{d}).
\end{aligned}$$

[28] For theorem in metric space see [97, 109].

Example 6. *Let* $\mathsf{d}(x,y) \in \mathbb{R}^{\mathbb{R}\times\mathbb{R}}$ *be defined as follows*

$$\mathsf{d}(x,y) := \begin{cases} |x-y|, & \text{for } x=0 \text{ or } y=0 \text{ or } x=y, \\ 1, & \text{otherwise.} \end{cases}$$

Note the following about the pair $(\mathbb{R},\mathsf{d})$.

1. *By Example 2,* $(\mathbb{R},\mathsf{d})$ *is a distance space, but not a metric space, that is, the triangle relation* $\triangle(1,1;\mathsf{d})$ *does not hold in* $(\mathbb{R},\mathsf{d})$.

2. *Observe further that* $(\mathbb{R},\mathsf{d})$ *is not a power distance space, that is, the triangle relation* $\triangle(p,\sigma;\mathsf{d})$ *does not hold in* $(\mathbb{R},\mathsf{d})$ *for any value of* $(p,\sigma) \in \mathbb{R}^* \times \mathbb{R}^+$.

   Let us prove that $\triangle(p,\sigma;\mathsf{d})$ *does not hold for any* $(p,\sigma) \in \{\infty\} \times \mathbb{R}^+$. *Indeed, Corollary 3 and Corollary 2 imply*

$$\begin{aligned}\lim_{n,m\to\infty} \mathsf{d}(1/n,1/m) &:= 1 \nleq 0 = 2\sigma \max\{0,0\} = 2\sigma \lim_{n,m\to\infty} \max\{\mathsf{d}(1/n,0), \mathsf{d}(0,1/m)\} \\ &\geq \lim_{n,m\to\infty} 2\sigma[1/2\mathsf{d}^p(1/n,0) + 1/2\mathsf{d}^p(0,1/m)]^{\frac{1}{p}} := \lim_{n,m\to\infty} \tau(p,\sigma,1/n,1/m,0).\end{aligned}$$

   The triangle relation $\triangle(p,\sigma;\mathsf{d})$ *does not hold for any* $(p,\sigma) \in \mathbb{R}^* \times \mathbb{R}^+$ *also. The triangle function* $\tau(p,\sigma;x,y,z;\mathsf{d})$ *is continuous and strictly monotone in* $(\mathbb{R},|\cdot|,\leq)$ *with respect to the variable* p *via Corollary 2. From proved above it follows that* $\triangle(p,\sigma;\mathsf{d})$ *fails to hold at the best case of* $p=\infty$*, and so by Corollary 2, it doesn't hold for any other value of* $p \in \mathbb{R}^*$ *either.*

Example 7. *Let* d *be a function in* $\mathbb{R}^{\mathbb{R}\times\mathbb{R}}$ *such that*

$$\mathsf{d}(x,y) := \begin{cases} 2|x-y|, & \forall (x,y) \in \{(0,1),(1,0)\}, \\ |x-y|, & \text{otherwise.} \end{cases}$$

Note the following about the pair $(\mathbb{R},\mathsf{d})$.

1. *By Example 3,* $(\mathbb{R},\mathsf{d})$ *is a distance space, but not a metric space, that is, the triangle relation* $\triangle(1,1;\mathsf{d})$ *does not hold in* $(\mathbb{R},\mathsf{d})$.

2. *But observe further that* $(\mathbb{R},\mathsf{d},1,2)$ *is a power distance space. Let us prove that* $\triangle(1,2;\mathsf{d})$ *holds for all* $(x,y) \in \{(0,1),(1,0)\}$. *Indeed, for any* $z \in \mathbb{R}$ *we have*

$$\begin{aligned}\mathsf{d}(1,0) = \mathsf{d}(0,1) &:= 2|0-1| = 2 \leq 2 \leq 2(|0-z|+|z-1|) \\ &= 2\sigma\left(1/2|0-z|^p + 1/2|z-1|^p\right)^{\frac{1}{p}} := 2\sigma(1/2\mathsf{d}^p(0,z)+\mathsf{d}^p(z,1))^{\frac{1}{p}} := \tau(1,2;0,1,z).\end{aligned}$$

   Let us show that $\triangle(1,2;\mathsf{d})$ *holds for all other* $(x,y) \in \mathbb{R}^* \times \mathbb{R}^+$. *Using Corollary 2 we obtain*

$$\begin{aligned}\mathsf{d}(x,y) := 2|x-y| \leq (|x-z|+|z-y|) = 2\sigma\left(1/2|0-z|^p + 1/2|z-1|^p\right)^{\frac{1}{p}} \\ := \tau(1,1;x,y,z) \leq \tau(1,2;x,y,z).\end{aligned}$$

3. *In* (X,d)*, the limits of convergent sequences are unique. This follows directly from the fact that* $(\mathbb{R},\mathsf{d},1,2)$ *is a power distance space and by Theorem 10.*

4. *In* (X,d)*, convergent sequences are Cauchy. This follows directly from the fact that* $(\mathbb{R},\mathsf{d},1,2)$ *is a power distance space and by Theorem 7.*

Example 8. *Let* d *be a function in* $\mathbb{R}^{\mathbb{R}\times\mathbb{R}}$ *such that* $\mathsf{d}(x,y) := (x-y)^2$. *Note the following about the pair* $(\mathbb{R},\mathsf{d})$.

1. *It was demonstrated in Example 4 that* $(\mathbb{R},\mathsf{d})$ *is a distance space, but that it is not a metric space because the triangle inequality does not hold.*

2. *However, the tuple* $(\mathbb{R},\mathsf{d},p,\sigma)$ *is a power distance space for any* $(p,\sigma)\in\mathbb{R}^*\times[2:\infty)$. *In particular, for all* $x,y,z\in\mathbb{R}$, *the power triangle inequality must hold. The "worst case" for this is when a third point* z *is exactly "halfway between"* x *and* y *in* $\mathsf{d}(x,y)$; *that is, when* $z=\frac{x+y}{2}$:

$$\begin{aligned}(x-y)^2 &:= \mathsf{d}(x,y) \le \tau(p,\sigma;x,y,z;\mathsf{d}) := 2\sigma[1/2\mathsf{d}^p(x,z)+1/2\mathsf{d}^p(z,y)]^{\frac{1}{p}}\\ &:= 2\sigma\left[1/2(x-z)^{2p}+1/2(z-y)^{2p}\right]^{\frac{1}{p}} = 2\sigma\left[1/2|x-z|^{2p}+1/2|z-y|^{2p}\right]^{\frac{1}{p}}\\ &= 2\sigma\left[1/2\left|x-\frac{x+y}{2}\right|^{2p}+1/2\left|\frac{x+y}{2}-y\right|^{2p}\right]^{\frac{1}{p}}\\ &= 2\sigma\left[1/2\left|\frac{y-x}{2}\right|^{2p}+1/2\left|\frac{x-y}{2}\right|^{2p}\right]^{\frac{1}{p}} = 2\sigma\left[\left|\frac{x-y}{2}\right|^{2p}\right]^{\frac{1}{p}} = \frac{2\sigma}{4}|x-y|^2.\end{aligned}$$

   It follows $(p,\sigma)\in\mathbb{R}^*\times[2:\infty)$.

3. *The power distance function* d *is continuous in* $(\mathbb{R},\mathsf{d},p,\sigma)$ *for any* (p,σ) *such that* $\sigma\ge 2$ *and* $2\sigma = p^{\frac{1}{p}}$. *This follows directly from Theorem 9.*

APPENDIX A TOPOLOGICAL SPACES

Definition 25 ([59, 60, 89, 96, 111]). *Let* Γ *be a set with an arbitrary (possibly uncountable) number of elements. Let* 2^X *be the power set of a set* X. *A family of sets* $T\subseteq 2^X$ *is a topology on* X *if*

1. $\varnothing\in T$ *and*
2. $X\in T$ *and*
3. $U,V\in T \implies U\cap V\in T$ *and*
4. $\{U_\gamma\,|\,\gamma\in\Gamma\}\subseteq T \implies \bigcup_{\gamma\in\Gamma}U_\gamma\in T$.

The ordered pair (X,T) *is a topological space if* T *is a topology on* X. *A set* U *is open in* (X,T) *if* U *is any element of* T. *A set* D *is closed in* (X,T) *if* D^c *is open in* (X,T).

Just as the power set 2^X and the set $\{\varnothing,X\}$ are algebras of sets on a set X, so also are these sets topologies on X.

Example 9 ([42, 73, 89, 105]). *Let* $\mathcal{T}(X)$ *be the set of topologies on a set* X *and* 2^X *the power set on* X. *Then* $\{\varnothing,X\}$ *is a topology in* $\mathcal{T}(X)$, *which is called indiscrete topology or trivial topology;* 2^X *is a topology in* $\mathcal{T}(X)$, *which is called discrete topology.*

Definition 26 ([37, 66]). *Let* (X,T) *be a topological space. A set* $B\subseteq 2^X$ *is a base for* T *if* $B\subseteq T$ *and for all* $U\in T$ *there exist* $\{B_\gamma\in B\}$ *such that* $U=\bigcup_\gamma B_\gamma$.

Theorem 11 ([37,66]). *Let (X,T) be a topological space. Let B be a subset of 2^X. If B is a base for (X,T) then for every $x \in X$ and for every open set U containing x, there exists $B_x \in B$ such that $x \in B_x \subseteq U$.*

Theorem 12 ([11]). *Let (X,T) be a topological space and $B \subseteq 2^X$. If B is a base for (X,T) then*
1. $x \in X \implies \exists B_x \in B$ *such that* $x \in B_x$ *and*
2. $B_1, B_2 \in B \implies B_1 \cap B_2 \in B$.

Example 10 ([37]). *Let (X,d) be a metric space. The set $B := \{\mathsf{B}(x,r) \,|\, x \in X,\ r \in \mathbb{N}\}$ (the set of all open balls in (X,d)) is a base for a topology on (X,d).*

Example 11 (the standard topology on the real line). [29] *The set $B := \{(a:b) \,|\, a,b \in \mathbb{R},\ a < b\}$ is a base for the metric space $(\mathbb{R}, |b-a|)$ (the usual metric space on $\mathbb{R}$).*

Definition 27 ([51,67,72,81,89,110]). *Let (X,T) be a topological space. Let 2^X be the power set of X. The set A^- is the closure of $A \in 2^X$ if $A^- := \bigcap \left\{D \in 2^X \,|\, A \subseteq D \text{ and } D \text{ is closed}\right\}$. The set A° is the interior of $A \in 2^X$ if $A^\circ := \bigcup \left\{U \in 2^X \,|\, U \subseteq A \text{ and } U \text{ is open}\right\}$. A point x is a closure point of A if $x \in A^-$. A point x is an interior point of A if $x \in A^\circ$. A point x is an accumulation point of A if $x \in (A \backslash \{x\})^-$. A point x in A^- is a point of adherence in A or is adherent to A if $x \in A^-$.*

Lemma 3 ([1,81]). *Let A^- be the closure, and A° the interior of a set $A \in 2^X$ in a topological space (X,T). Then $A^\circ \subseteq A \subseteq A^-$; $A = A^\circ$ iff A is open; $A = A^-$ iff A is closed.*

Definition 28 ([37]). *Let (X,T_x) and (Y,T_y) be topological spaces. Let f be a function in Y^X. A function $\mathsf{f} \in Y^X$ is continuous if for any open set $U \in T_y$ in (Y,T_y) the set $\mathsf{f}^{-1}(U) \in T_x$ is open in (X,T_x). A function is discontinuous in $(X,T_y)^{(X,T_x)}$ if it is not continuous in $(X,T_y)^{(X,T_x)}$.*

Definition 28 defines continuity using open sets. Continuity can alternatively be defined using closed sets or closure.

Theorem 13 ([81,101]). *Let (X,T) and (Y,S) be topological spaces. Let f be a function in Y^X. The following are equivalent:*
1. f *is continuous;*
2. *if B is closed in (Y,S) then $\mathsf{f}^{-1}(B)$ is closed in (X,T);*
3. $\mathsf{f}(A^-) \subseteq \mathsf{f}(A)^-$;
4. $\mathsf{f}^{-1}(B) \subseteq \mathsf{f}^{-1}(B^-)$.

Remark 4. *A word of warning about defining continuity in terms of topological spaces — continuity is defined in terms of a pair of topological spaces, and whether function is continuous or discontinuous in general depends very heavily on the selection of these spaces. This is illustrated in Proposition 4. The ramification of this is that when declaring a function to be continuous or discontinuous, one must make clear the assumed topological spaces.*

Proposition 4 ([35,94]). *Let (X,T) and (Y,S) be topological spaces. Let f be a function in $(Y,S)^{(X,T)}$. If T is the discrete topology then f is continuous. If S is the indiscrete topology then f is continuous.*

[29] See [37,89].

Definition 29 ([66, 75]). *Let (X, T) be a topological space. A sequence $\{x_n\}_{n\in\mathbb{Z}}$ converges in (X, T) to a point x if for each open set $U \in T$ that contains x there exists $N \in \mathbb{N}$ such that $x_n \in U$ for all $n > N$. This condition can be expressed in any of the following forms:*

1. *The limit of the sequence $\{x_n\}$ is x.*
2. *The sequence $\{x_n\}$ is convergent with limit x.*
3. $\lim_{n\to\infty} \{x_n\} = x$.
4. $\{x_n\} \to x$.

A sequence that converges is convergent. A sequence that does not converge is said to diverge, or is divergent. An element $x \in A$ is a limit point of A if it is the limit of some A-valued sequence $\{x_n\} \subset A$.

Example 12 ([89]). *Let $X := \{x,y,z\}$ and $T_{31} := \{\varnothing, \{x\}, \{x,y\}, \{x,z\}, \{x,y,z\}\}$. Then (X, T_{31}) is a topological space. In this space, the sequence $\{x, x, x, \ldots\}$ converges to x. But this sequence also converges to both y and z because x is in every open set that contains y and x is in every open set that contains z. So, the limit of the sequence is not unique.*

Example 13. *In contrast to the low resolution topological space of Example 12, the limit of the sequence $\{x, x, x, \ldots\}$ is unique in a topological space with sufficiently high resolution with respect to y and z such as the following. Define a topological space (X, T_{56}) where $X := \{x,y,z\}$ and $T_{56} := \{\varnothing, \{y\}, \{z\}, \{x,y\}, \{y,z\}, \{x,y,z\}\}$. In this space, the sequence $\{x, x, x, \ldots\}$ converges to x only. The sequence does not converge to y or z because there are open sets containing y or z that do not contain x (the open sets $\{y\}$, $\{z\}$, and $\{y,z\}$).*

Theorem 14 (The Closed Set Theorem). [30] *Let (X, T) be a topological space. Let A be a subset of X. Then A is closed in (X, T) if and only if every A-valued sequence $\{x_n\}_{n\in\mathbb{Z}} \subset A$ that converges in (X, T) has its limit in A.*

Theorem 15 ([94]). *Let (X, T) and (Y, S) be topological spaces. Let f be a function in $(Y, S)^{(X,T)}$. Then inverse image characterization of continuity (see Definition 28) is equivalent to sequential characterization of continuity (see Definition 29).*

APPENDIX B FINITE SUMS

B.1 Convexity

Definition 30 ([3, 11, 64, 103]). *A function $\mathsf{f} \in \mathbb{R}^{\mathbb{R}}$ is said to be*

convex if $\mathsf{f}(\lambda x + [1-\lambda]y) \le \lambda \mathsf{f}(x) + (1-\lambda)\,\mathsf{f}(y)$, $\forall x, y \in \mathbb{R}$, $\forall \lambda \in (0:1)$;
strictly convex if $\mathsf{f}(\lambda x + [1-\lambda]y) = \lambda \mathsf{f}(x) + (1-\lambda)\,\mathsf{f}(y)$, $\forall x, y \in \mathbb{R}$, $x \ne y$, $\forall \lambda \in (0:1)$;
concave if $-\mathsf{f}$ is convex;
affine if f is convex and concave.

Theorem 16 (Jensen's Inequality). [31] *Let $\mathsf{f} \in \mathbb{R}^{\mathbb{R}}$ be a function. If f is convex and $\sum_{n=1}^{N} \lambda_n = 1$ then $\mathsf{f}\left(\sum_{n=1}^{N} \lambda_n x_n\right) \le \sum_{n=1}^{N} \lambda_n \,\mathsf{f}(x_n)$ for all $x_n \in \mathbb{R}$ and $N \in \mathbb{N}$.*

[30] See [54, 72, 97].
[31] See [11, 64, 86].

B.2 Power means

Definition 31 ([11]). *The $(\!|\lambda_n|\!)_1^N$ weighted φ-**mean** of a tuple $(\!|x_n|\!)_1^N$ is defined as*

$$\mathsf{M}_\varphi((\!|x_n|\!)) := \varphi^{-1}\left(\sum_{n=1}^{N} \lambda_n \varphi(x_n)\right)$$

where φ is a continuous and strictly monotonic function in $\mathbb{R}^{\mathbb{R}^{\vdash}}$ and $(\!|\lambda_n|\!)_{n=1}^N$ is a sequence of weights for which $\sum_{n=1}^{N} \lambda_n = 1$.

Lemma 4 ([11,58,93]). *Let $\mathsf{M}_\varphi((\!|x_n|\!))$ be the $(\!|\lambda_n|\!)_1^N$ weighted φ-mean and $\mathsf{M}_\psi((\!|x_n|\!))$ the $(\!|\lambda_n|\!)_1^N$ weighted ψ-mean of a tuple $(\!|x_n|\!)_1^N$.*

If $\varphi\psi^{-1}$ is convex and φ is increasing then $\mathsf{M}_\varphi((\!|x_n|\!)) \geq \mathsf{M}_\psi((\!|x_n|\!))$.
If $\varphi\psi^{-1}$ is convex and φ is decreasing then $\mathsf{M}_\varphi((\!|x_n|\!)) \leq \mathsf{M}_\psi((\!|x_n|\!))$.
If $\varphi\psi^{-1}$ is concave and φ is increasing then $\mathsf{M}_\varphi((\!|x_n|\!)) \leq \mathsf{M}_\psi((\!|x_n|\!))$.
If $\varphi\psi^{-1}$ is concave and φ is decreasing then $\mathsf{M}_\varphi((\!|x_n|\!)) \geq \mathsf{M}_\psi((\!|x_n|\!))$.

One of the most well known inequalities in mathematics is Minkowski's Inequality. In 1946, H.P. Mulholland submitted a result that generalizes Minkowski's Inequality to an equal weighted φ-mean. In 1979, G.V. Milovanović and I. Milovanović generalized this even further to a *weighted* φ-mean.[32]

Theorem 17 ([20,84]). *Let φ be a convex strictly monotone function in $\mathbb{R}^{\mathbb{R}}$, such that $\varphi(0) = 0$ and $\log \circ \varphi \circ \exp$ is convex. Then*

$$\varphi^{-1}\left(\sum_{n=1}^{N} \lambda_n \varphi(x_n + y_n)\right) \leq \varphi^{-1}\left(\sum_{n=1}^{N} \lambda_n \varphi(x_n)\right) + \varphi^{-1}\left(\sum_{n=1}^{N} \lambda_n \varphi(y_n)\right).$$

Definition 32 ([11,20]). *Let $\mathsf{M}_{\varphi(x;p)}((\!|x_n|\!))$ be the $(\!|\lambda_n|\!)_1^N$ weighted φ-mean of a non-negative tuple $(\!|x_n|\!)_1^N$. A mean $\mathsf{M}_{\varphi(x;p)}((\!|x_n|\!))$ is a power mean with parameter p if $\varphi(x) := x^p$. That is,*

$$\mathsf{M}_{\varphi(x;p)}((\!|x_n|\!)) = \left(\sum_{n=1}^{N} \lambda_n (x_n)^p\right)^{\frac{1}{p}}.$$

Theorem 18 ([7,8,11,14,19,20]). *Let $\mathsf{M}_{\varphi(x;p)}((\!|x_n|\!))$ be the power mean with parameter p of an N-tuple $(\!|x_n|\!)_1^N$ in which the elements are not all equal. Then $\mathsf{M}_{\varphi(x;p)}((\!|x_n|\!)) := \left(\sum_{n=1}^{N} \lambda_n (x_n)^p\right)^{\frac{1}{p}}$ is continuous and strictly monotone in $\mathbb{R}^*$ and*

$$\mathsf{M}_{\varphi(x;p)}((\!|x_n|\!)) = \begin{cases} \max\limits_{n=1,2,\ldots,N} (\!|x_n|\!), & \text{for } p = +\infty, \\ \prod\limits_{n=1}^{N} x_n^{\lambda_n}, & \text{for } p = 0, \\ \min\limits_{n=1,2,\ldots,N} (\!|x_n|\!), & \text{for } p = -\infty. \end{cases}$$

[32] See also [20,22,58,79,85,88,112].

Proof. Let p and s be such that $-\infty < p < s < \infty$. Let $\varphi_p := x^p$ and $\varphi_s := x^s$. Then $\varphi_p\varphi_s^{-1} = x^{\frac{p}{s}}$. The composite function $\varphi_p\varphi_s^{-1}$ is convex or concave depending on the values of p and s:

	$p < 0$ (φ_p decreasing)	$p > 0$ (φ_p increasing)
$s < 0$	convex	(not possible)
$s > 0$	convex	concave

Therefore by Lemma 4, we obtain $\mathsf{M}_{\varphi(x;p)}(\langle x_n \rangle) < \mathsf{M}_{\varphi(x;s)}(\langle x_n \rangle)$. So, $M_{\varphi(x;p)}$ is strictly monotone in p.

The sum of continuous functions is continuous. Therefore, $\mathsf{M}_{\varphi(x;p)}$ is continuous in p for $p \in \mathbb{R}\backslash 0$. The cases of $p \in \{-\infty, 0, \infty\}$ we consider below.

Note that using the definition of M_φ we obtain

$$\left\{\mathsf{M}_{\varphi(x;p)}\left(\langle x_n^{-1} \rangle\right)\right\}^{-1} = \left\{\left(\sum_{n=1}^{N} \lambda_n \left(x_n^{-1}\right)^p\right)^{\frac{1}{p}}\right\}^{-1} = \left(\sum_{n=1}^{N} \lambda_n (x_n)^{-p}\right)^{\frac{1}{-p}} = \mathsf{M}_{\varphi(x;-p)}(\langle x_n \rangle). \quad (1)$$

Denote $x_m := \max_{n\in\mathbb{Z}} \langle x_n \rangle$. Note that $\lim_{p\to\infty} \mathsf{M}_\varphi \le \max_{n\in\mathbb{Z}} \langle x_n \rangle$. Indeed, using the definition of M_φ, we obtain

$$\begin{aligned}\lim_{p\to\infty} \mathsf{M}_\varphi(\langle x_n \rangle) &= \lim_{p\to\infty} \left(\sum_{n=1}^{N} \lambda_n x_n^p\right)^{\frac{1}{p}} \le \lim_{p\to\infty} \left(\sum_{n=1}^{N} \lambda_n x_m^p\right)^{\frac{1}{p}} \\ &= \lim_{p\to\infty} \left(x_m^p \underbrace{\sum_{n=1}^{N} \lambda_n}_{1}\right)^{\frac{1}{p}} = \lim_{p\to\infty} \left(x_m^p \cdot 1\right)^{\frac{1}{p}} = x_m = \max_{n\in\mathbb{Z}} \langle x_n \rangle.\end{aligned}$$

But also note that $\lim_{p\to\infty} \mathsf{M}_\varphi \ge \max_{n\in\mathbb{Z}} \langle x_n \rangle$ because

$$\lim_{p\to\infty} \mathsf{M}_\varphi(\langle x_n \rangle) = \lim_{p\to\infty} \left(\sum_{n=1}^{N} \lambda_n x_n^p\right)^{\frac{1}{p}} \ge \lim_{p\to\infty} \left(w_m x_m^p\right)^{\frac{1}{p}} = \lim_{p\to\infty} w_m^{\frac{1}{p}} x_m^{\frac{p}{p}} = x_m = \max_{n\in\mathbb{Z}} \langle x_n \rangle.$$

Here we used the fact, that $\varphi(x) := x^p$ and φ^{-1} are both increasing or both decreasing. So, $\lim_{p\to\infty} \mathsf{M}_\varphi(\langle x_n \rangle) = \max_{n\in\mathbb{Z}} \langle x_n \rangle$.

Let us prove that $\lim_{p\to-\infty} \mathsf{M}_\varphi(\langle x_n \rangle) = \min_{n\in\mathbb{Z}} \langle x_n \rangle$. From the equation (1) it follows

$$\begin{aligned}\lim_{p\to-\infty} \mathsf{M}_{\varphi(x;p)}(\langle x_n \rangle) &= \lim_{p\to\infty} \mathsf{M}_{\varphi(x;-p)}(\langle x_n \rangle) = \lim_{p\to\infty} \left\{\mathsf{M}_{\varphi(x;p)}\left(\langle x_n^{-1} \rangle\right)\right\}^{-1} = \lim_{p\to\infty} \frac{1}{\mathsf{M}_{\varphi(x;p)}\left(\langle x_n^{-1} \rangle\right)} \\ &= \frac{\lim_{p\to\infty} 1}{\lim_{p\to\infty} \mathsf{M}_{\varphi(x;p)}\left(\langle x_n^{-1} \rangle\right)} = \frac{1}{\max_{n\in\mathbb{Z}} \langle x_n^{-1} \rangle} = \frac{1}{\left(\min_{n\in\mathbb{Z}} \langle x_n \rangle\right)^{-1}} = \min_{n\in\mathbb{Z}} \langle x_n \rangle.\end{aligned}$$

It remains to prove that $\lim_{p\to 0} \mathsf{M}_\varphi(\langle x_n \rangle) = \prod_{n=1}^{N} x_n^{\lambda_n}$. Using the definition of M_φ and l'Hôpital's rule[33] we obtain

$$\lim_{p\to 0} \mathsf{M}_{\varphi}(\llparenthesis x_n \rrparenthesis) = \lim_{p\to 0} \exp\left\{\ln\left\{\mathsf{M}_{\varphi}(\llparenthesis x_n \rrparenthesis)\right\}\right\} = \lim_{p\to 0} \exp\left\{\ln\left\{\left(\sum_{n=1}^{N} \lambda_n \left(x_n^p\right)\right)^{\frac{1}{p}}\right\}\right\}$$

$$= \exp\left\{\frac{\frac{\partial}{\partial p}\ln\left(\sum_{n=1}^{N}\lambda_n\left(x_n^p\right)\right)}{\frac{\partial}{\partial p}p}\right\}_{p=0} = \exp\left\{\frac{\sum_{n=1}^{N}\lambda_n\frac{\partial}{\partial p}\left(x_n^p\right)}{\sum_{n=1}^{N}\lambda_n\left(x_n^p\right)}\right\}_{p=0}$$

$$= \exp\left\{\frac{\sum_{n=1}^{N}\lambda_n\frac{\partial}{\partial p}\exp\left(\ln\left(x_n^p\right)\right)}{\sum_{n=1}^{N}\lambda_n}\right\}_{p=0} = \exp\left\{\frac{\sum_{n=1}^{N}\lambda_n\frac{\partial}{\partial p}\exp\left(r\ln\left(x_n\right)\right)}{1}\right\}_{p=0}$$

$$= \exp\left\{\sum_{n=1}^{N}\lambda_n\frac{\partial}{\partial p}\exp\left(p\ln\left(x_n\right)\right)\right\}_{p=0} = \exp\left\{\sum_{n=1}^{N}\lambda_n\exp\left\{p\ln x_n\right\}\ln\left(x_n\right)\right\}_{p=0}$$

$$= \exp\left\{\sum_{n=1}^{N}\lambda_n\ln\left(x_n\right)\right\} = \exp\left\{\sum_{n=1}^{N}\ln\left(x_n^{\lambda_n}\right)\right\} = \exp\left\{\ln\prod_{n=1}^{N}x_n^{\lambda_n}\right\} = \prod_{n=1}^{N}x_n^{\lambda_n}.$$

□

Corollary 8 ([11, 20, 23, 63, 64])**.** *Let $\llparenthesis x_n \rrparenthesis_1^N$ be a tuple. Let $\llparenthesis \lambda_n \rrparenthesis_1^N$ be a tuple of weighting values such that $\sum_{n=1}^{N}\lambda_n = 1$. Then*

$$\min\llparenthesis x_n \rrparenthesis \le \underbrace{\left(\sum_{n=1}^{N}\lambda_n\frac{1}{x_n}\right)^{-1}}_{\text{harmonic mean}} \le \underbrace{\prod_{n=1}^{N}x_n^{\lambda_n}}_{\text{geometric mean}} \le \underbrace{\sum_{n=1}^{N}\lambda_n x_n}_{\text{arithmetic mean}} \le \max\llparenthesis x_n \rrparenthesis.$$

Proof. These five means are all special cases of the power mean $\mathsf{M}_{\varphi(x:p)}$, namely

$p=\infty$: $\max\llparenthesis x_n \rrparenthesis$,
$p=1$: arithmetic mean,
$p=0$: geometric mean, So, the inequalities follow directly from Theorem 18.
$p=-1$: harmonic mean,
$p=-\infty$: $\min\llparenthesis x_n \rrparenthesis$.

If one is only concerned with the arithmetic mean and geometric mean, their relationship can be established directly using Jensen's Inequality (Theorem 16):

$$\sum_{n=1}^{N}\lambda_n x_n = b^{\log_b\left(\sum_{n=1}^{N}\lambda_n x_n\right)} \ge b^{\left(\sum_{n=1}^{N}\lambda_n\log_b x_n\right)} = \prod_{n=1}^{N}b^{\left(\lambda_n\log_b x_n\right)} = \prod_{n=1}^{N}b^{\left(\log_b x_n\right)\lambda_n} = \prod_{n=1}^{N}x_n^{\lambda_n}.$$

□

[33] See [98].

B.3 Inequalities

Lemma 5 (Young's Inequality). [34]

$$\begin{aligned} xy &< \frac{x^p}{p} + \frac{y^q}{q} \quad \text{with } \tfrac{1}{p} + \tfrac{1}{q} = 1 \quad \forall 1 < p < \infty,\ x, y \geq 0, \quad \text{but } y \neq x^{p-1}, \\ xy &= \frac{x^p}{p} + \frac{y^q}{q} \quad \text{with } \tfrac{1}{p} + \tfrac{1}{q} = 1 \quad \forall 1 < p < \infty,\ x, y \geq 0, \quad \text{and } y = x^{p-1}. \end{aligned}$$

Theorem 19 (Minkowski's Inequality for sequences). [35] *Let* $(\!(x_n)\!)_1^N \subset \mathbb{C}$ *and* $(\!(y_n)\!)_1^N \subset \mathbb{C}$ *be complex* N*-tuples. Then*

$$\left(\sum_{n=1}^{N} |x_n + y_n|^p\right)^{\frac{1}{p}} \leq \left(\sum_{n=1}^{N} |x_n|^p\right)^{\frac{1}{p}} + \left(\sum_{n=1}^{N} |y_n|^p\right)^{\frac{1}{p}} \qquad \forall\, 1 < p < \infty.$$

APPENDIX C METRIC PRESERVING FUNCTIONS

Definition 33 ([31, 40, 113]). *Let* $\mathbb{M}$ *be the set of all metric spaces on a set* X*. A function* $\varphi \in \mathbb{R}^{\vdash \mathbb{R}^{\vdash}}$ *is a metric preserving function if* $\mathsf{d}(x, y) := \varphi \circ \mathsf{p}(x, y)$ *is a metric on* X *for all* $(X, \mathsf{p}) \in \mathbb{M}$*.*

Theorem 20 (necessary conditions). [36] *Let* $\mathcal{R}\varphi$ *be the range of a function* φ*. If* φ *is a metric preserving function then* $\varphi^{-1}(0) = \{0\}$*,* $\mathcal{R}\varphi \subseteq \mathbb{R}^{\vdash}$*, and the function* φ *is subadditive, i.e.* $\varphi(x + y) \leq \varphi(x) + \varphi(y)$*.*

Theorem 21 (sufficient conditions). [37] *Let* φ *be a function in* $\mathbb{R}^{\mathbb{R}}$*. If the conditions*

1. $x \geq y \implies \varphi(x) \geq \varphi(y), \quad \forall x, y \in \mathbb{R}^{\vdash}$,
2. $\varphi(0) = 0$,
3. $\varphi(x + y) \leq \varphi(x) + \varphi(y), \quad \forall x, y \in \mathbb{R}^{\vdash}$,

hold, then φ *is a metric preserving function.*

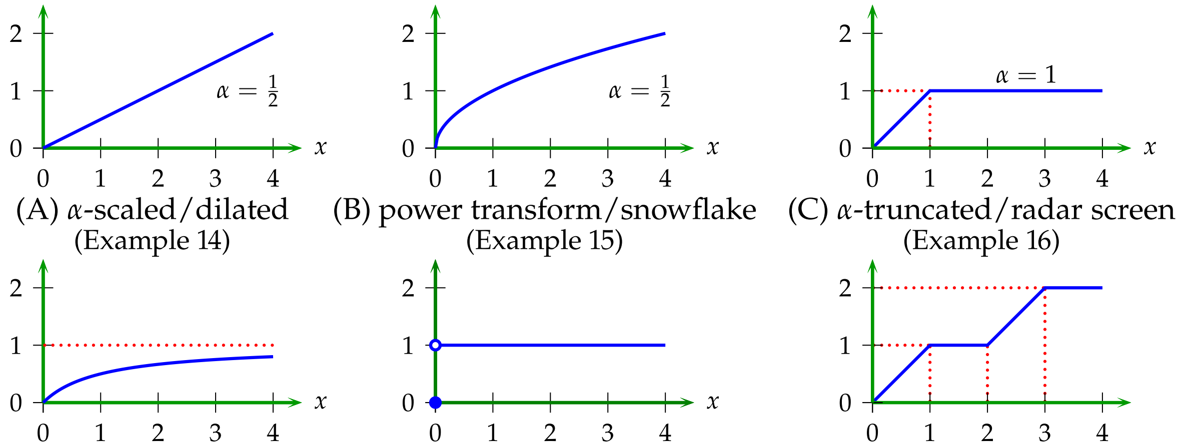
The proofs for Example 14–Example 19 follow from Theorem 21.

Example 14 (α-scaled metric/dilated metric). [38] *Let* (X, d) *be a metric space. The function* $\varphi(x) := \alpha x$*,* $\alpha \in \mathbb{R}^+$*, is a metric preserving function (see Figure 3 (A)).*

Example 15 (power transform metric/snowflake transform metric). [39] *Let* (X, d) *be a metric space. The function* $\varphi(x) := x^\alpha$*,* $\alpha \in (0 : 1]$*, is a metric preserving function (see Figure 3 (B)).*

Example 16 (α-truncated metric/radar screen metric). [40] *Let* (X, d) *be a metric space. The function* $\varphi(x) := \min\{\alpha,\ x\}$*,* $\alpha \in \mathbb{R}^+$*, is a metric preserving function (see Figure 3 (C)).*

[34] See [22, 58, 79, 112, 120].
[35] See [20, 22, 58, 79, 85, 112].
[36] See [31, 40].
[37] See [31, 40, 67].
[38] See [39].
[39] See [39, 40].
[40] See [39, 53].



(A) α-scaled/dilated (Example 14)
(B) power transform/snowflake (Example 15)
(C) α-truncated/radar screen (Example 16)
(D) bounded (Example 17)
(E) discrete (Example 18)
(F) (Example 19)

Figure 3: metric preserving functions.

Example 17 (bounded metric)**.** [41] *Let* (X, d) *be a metric space. The function* $\varphi(x) := \dfrac{x}{1+x}$ *is a metric preserving function (see Figure 3 (D)).*

Example 18 (discrete metric preserving function)**.** [42] *The function* $\varphi(x) := \begin{cases} 0, & \text{for } x \le 0, \\ 1, & \text{otherwise,} \end{cases}$ *from* $\mathbb{R}^{\mathbb{R}}$ *is a metric preserving function (see Figure 3 (E)).*

Example 19. *The function*

$$\varphi(x) := \begin{cases} x, & \text{for } 0 \le x < 1, \\ 1, & \text{for } 1 \le x \le 2, \\ x-1, & \text{for } 2 < x < 3, \\ 2, & \text{for } x \ge 3, \end{cases}$$

from $\mathbb{R}^{\mathbb{R}}$ *is a metric preserving function (see Figure 3 (F)).*

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[41] See [1, 113].
[42] See [31].

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Ґрінхое Д.Дж. *Властивості просторів з відстанню, що задовольняють степеневі нерівності трикутника* // Карпатські матем. публ. — 2016. — Т.8, №1. — С. 51–82.

Метричні простори забезпечують основу для математичного аналізу і мають ряд дуже корисних властивостей. Багато з цих властивостей випливають зокрема з нерівності трикутника. Однак є багато застосувань, в яких нерівність трикутника не справджується, але в яких ми все ще можемо здійснювати аналіз. У цій статті досліджуємо, що трапиться, якщо нерівність трикутника вилучено з переліку аксіом метрики, при цьому метричний простір стає так званим простором з відстанню. Також нас цікавить, що буде коли нерівність трикутника замінена на більш загальне двохпараметричне співвідношення, яке ми називаємо степеневою нерівністю трикутника. Таке узагальнення нерівності трикутника дає незліченно великий клас нерівностей, і включає при цьому звичайну нерівність трикутника, слабку нерівність трикутника та інфраметричну нерівність як частинні випадки. Степенева нерівність трикутника визначена в термінах функції, яку ми називаємо степеневою трикутною функцією. Ця функція є неперервною і монотонною відносно свого експоненціального параметру, є степеневим середнім, і також включає як частинні випадки максимум, мінімум, середнє квадратичне, середнє арифметичне, середнє геометричне і середнє гармонійне.

Ключові слова і фрази: метричний простір, простір з відстанню, напівметричний простір, квазі-метричний простір, нерівність трикутника, слабка нерівність трикутника, інфраметрика, середнє арифметичне, середнє квадратичне, середнє геометричне, середнє гармонійне, максимум, мінімум, середнє степеневе.

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KACHANOVSKY N.A.

OPERATORS OF STOCHASTIC DIFFERENTIATION ON SPACES OF NONREGULAR GENERALIZED FUNCTIONS OF LÉVY WHITE NOISE ANALYSIS

The operators of stochastic differentiation, which are closely related with the extended Skorohod stochastic integral and with the Hida stochastic derivative, play an important role in the classical (Gaussian) white noise analysis. In particular, these operators can be used in order to study some properties of the extended stochastic integral and of solutions of stochastic equations with Wick-type nonlinearities.

During recent years the operators of stochastic differentiation were introduced and studied, in particular, in the framework of the Meixner white noise analysis, in the same way as on spaces of regular test and generalized functions and on spaces of nonregular test functions of the Lévy white noise analysis. In the present paper we make the next natural step: introduce and study operators of stochastic differentiation on spaces of nonregular generalized functions of the Lévy white noise analysis (i.e., on spaces of generalized functions that belong to the so-called nonregular rigging of the space of square integrable with respect to the measure of a Lévy white noise functions). In so doing, we use Lytvynov's generalization of the chaotic representation property. The researches of the present paper can be considered as a contribution in a further development of the Lévy white noise analysis.

Key words and phrases: operator of stochastic differentiation, stochastic derivative, extended stochastic integral, Lévy process.

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INTRODUCTION

Let $L = (L_t)_{t\in[0,+\infty)}$ be a Lévy process (i.e., a random process on $[0,+\infty)$ with stationary independent increments and such that $L_0 = 0$, see, e.g., [5, 30, 31] for details) without Gaussian part and drift. In [23] the extended Skorohod stochastic integral with respect to L and the corresponding Hida stochastic derivative on the space of square integrable random variables (L^2) were constructed in terms of Lytvynov's generalization of the *chaotic representation property* (CRP) (see [27] and Subsection 1.2), some properties of these operators were established; and it was shown that the above-mentioned integral coincides with the well-known (constructed in terms of Itô's generalization of the CRP [14]) extended stochastic integral with respect to a Lévy process (e.g., [6, 7]). In [10, 21] the notion of stochastic integral and derivative was widened to spaces of regular and nonregular test and generalized functions that belong to so-called regular parametrized and nonregular riggings of (L^2) respectively, this gives a possibility to extend an area of possible applications of the above-mentioned operators (in particular, now it is possible to define the stochastic integral and derivative as linear *continuous* operators). Together with the stochastic integral and derivative, it is natural to introduce

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and to study so-called *operators of stochastic differentiation* in the Lévy white noise analysis, by analogy with the Gaussian analysis [1, 37], the Gamma-analysis [17, 18], and the Meixner analysis [19, 20]. These operators are closely related with the extended Skorohod stochastic integral with respect to a Lévy process and with the corresponding Hida stochastic derivative and, by analogy with the classical Gaussian case, can be used, in particular, in order to study some properties of the extended stochastic integral and of solutions of normally ordered stochastic equations (stochastic equations with Wick-type nonlinearities in another terminology). In [9, 8] the operators of stochastic differentiation on spaces that belong to a *regular parametrized* rigging of (L^2) ([21]) were introduced and studied. This rigging plays a very important role in the Lévy analysis; but, in order to solve some problems that arise in this analysis (in particular, in the theory of normally ordered stochastic equations), it is necessary to introduce into consideration another, *nonregular* rigging of (L^2) (see [21] and Subsection 1.3), and operators (e.g., the extended stochastic integral, the Hida stochastic derivative) on spaces (of nonregular test and generalized functions) that belong to this rigging. Therefore it is natural to introduce and to study operators of stochastic differentiation on the just now mentioned spaces.

In the paper [24] the operators of stochastic differentiation were introduced and studied on the spaces of nonregular test functions of the Lévy white noise analysis. In particular, it was shown that, roughly speaking, these operators are the restrictions to the above-mentioned spaces of the corresponding operators on (L^2). The next natural step is, of course, to consider operators of stochastic differentiation on the spaces of nonregular generalized functions. But here there is a problem: in contrast to the classical Gaussian case and to the "regular case", the operators of stochastic differentiation on (L^2) cannot be naturally continued to the just now mentioned spaces (to the point, actually for the same reason the Hida stochastic derivative also cannot be naturally continued from (L^2) to the spaces of nonregular generalized functions). Nevertheless, it is possible to introduce on these spaces natural analogs of the above-mentioned operators. These analogs have properties quite analogous to the properties of operators of stochastic differentiation, and can be accepted as operators of stochastic differentiation on the spaces of nonregular generalized functions. In the present paper we introduce and study in detail the just now mentioned operators. In forthcoming papers we'll consider elements of the so-called Wick calculus in the Lévy white noise analysis, this will give us the possibility to continue the study of properties and to consider some applications of the operators of stochastic differentiation.

The paper is organized in the following manner. In the first section we introduce a Lévy process L and construct a convenient for our considerations probability triplet connected with L; then, following [21, 23, 27], we describe in detail Lytvynov's generalization of the CRP, the nonregular rigging of (L^2), and stochastic derivatives and integrals on the spaces that belong to this rigging. In the second section we deal with the operators of stochastic differentiation on the spaces of nonregular generalized functions, considering separately the cases of bounded and unbounded operators. Note that some results of this paper were announced without proofs in [25].

1 PRELIMINARIES

In this paper we denote by $\|\cdot\|_H$ or $|\cdot|_H$ the norm in a space H; by $(\cdot,\cdot)_H$ the scalar product in a space H; and by $\langle\cdot,\cdot\rangle_H$ or $\langle\!\langle\cdot,\cdot\rangle\!\rangle_H$ the dual pairing generated by the scalar product in a space H. Another notation for norms, scalar products and dual pairings will be introduced when it will be necessary.

1.1 Lévy processes

Denote $\mathbb{R}_+ := [0, +\infty)$. In this paper we deal with a real-valued locally square integrable Lévy process $L = (L_t)_{t\in\mathbb{R}_+}$ (a random process on $\mathbb{R}_+$ with stationary independent increments and such that $L_0 = 0$) without Gaussian part and drift (it is comparatively simple to consider such processes from technical point of view). As is well known (e.g., [7]), the characteristic function of L is

$$\mathbb{E}[e^{i\theta L_t}] = \exp\Big[t \int_{\mathbb{R}} (e^{i\theta x} - 1 - i\theta x)\nu(dx)\Big], \tag{1}$$

where ν is the Lévy measure of L, which is a measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, here and below $\mathcal{B}$ denotes the Borel σ-algebra; $\mathbb{E}$ denotes the expectation. We assume that ν *is a Radon measure whose support contains an infinite number of points,* $\nu(\{0\}) = 0$, *there exists* $\varepsilon > 0$ *such that*

$$\int_{\mathbb{R}} x^2 e^{\varepsilon|x|}\nu(dx) < \infty,$$

and

$$\int_{\mathbb{R}} x^2\nu(dx) = 1. \tag{2}$$

Let us define a measure of the white noise of L. Let $\mathcal{D}$ denote the set of all real-valued infinite-differentiable functions on $\mathbb{R}_+$ with compact supports. As is well known, $\mathcal{D}$ can be endowed by the projective limit topology generated by a family of Sobolev spaces (e.g., [4]). Let $\mathcal{D}'$ be the set of linear continuous functionals on $\mathcal{D}$. For $\omega \in \mathcal{D}'$ and $\varphi \in \mathcal{D}$ denote $\omega(\varphi)$ by $\langle\omega, \varphi\rangle$; note that one can understand $\langle\cdot,\cdot\rangle$ as the dual pairing generated by the scalar product in the space $L^2(\mathbb{R}_+)$ of (classes of) square integrable with respect to the Lebesgue measure real-valued functions on $\mathbb{R}_+$, see Subsection 1.3 for details. The notation $\langle\cdot,\cdot\rangle$ will be preserved for dual pairings in tensor powers of spaces.

Definition. *A probability measure μ on $(\mathcal{D}', \mathcal{C}(\mathcal{D}'))$, where $\mathcal{C}$ denotes the cylindrical σ-algebra, with the Fourier transform*

$$\int_{\mathcal{D}'} e^{i\langle\omega,\varphi\rangle}\mu(d\omega) = \exp\Big[\int_{\mathbb{R}_+\times\mathbb{R}} (e^{i\varphi(u)x} - 1 - i\varphi(u)x)du\nu(dx)\Big], \quad \varphi \in \mathcal{D}, \tag{3}$$

is called the measure of a Lévy white noise.

The existence of μ follows from the Bochner–Minlos theorem (e.g., [13]), see [27]. Below we assume that the *σ-algebra $\mathcal{C}(\mathcal{D}')$ is complete with respect to μ*, i.e., $\mathcal{C}(\mathcal{D}')$ contains all subsets of all measurable sets O such that $\mu(O) = 0$.

Denote $(L^2) := L^2(\mathcal{D}', \mathcal{C}(\mathcal{D}'), \mu)$ the space of (classes of) real-valued square integrable with respect to μ functions on $\mathcal{D}'$; let also $\mathcal{H} := L^2(\mathbb{R}_+)$. Substituting in (3) $\varphi = t\psi$, $t \in \mathbb{R}$, $\psi \in \mathcal{D}$, and using the Taylor decomposition by t and (2), one can show that

$$\int_{\mathcal{D}'} \langle\omega, \psi\rangle^2\mu(d\omega) = \int_{\mathbb{R}_+} \big(\psi(u)\big)^2 du \tag{4}$$

(this statement follows also from results of [27] and [7]). Let $f \in \mathcal{H}$ and $\mathcal{D} \ni \varphi_k \to f$ in $\mathcal{H}$ as $k \to \infty$ (it is well known (e.g., [4]) that $\mathcal{D}$ is a dense set in $\mathcal{H}$). It follows from (4) that

$\{\langle\circ,\varphi_k\rangle\}_{k\geq 1}$ is a Cauchy sequence in (L^2), therefore one can define $\langle\circ,f\rangle := (L^2)-\lim\limits_{k\to\infty}\langle\circ,\varphi_k\rangle$. It is easy to show (by the method of "mixed sequences") that $\langle\circ,f\rangle$ does not depend on the choice of an approximating sequence for f and therefore is well defined in (L^2).

Let us consider $\langle\circ,1_{[0,t)}\rangle\in(L^2)$, $t\in\mathbb{R}_+$ (here and below 1_A denotes the indicator of a set A). It follows from (1) and (3) that $\big(\langle\circ,1_{[0,t)}\rangle\big)_{t\in\mathbb{R}_+}$ can be identified with a Lévy process on the probability space $(\mathcal{D}',\mathcal{C}(\mathcal{D}'),\mu)$, i.e., one can write $L_t=\langle\circ,1_{[0,t)}\rangle\in(L^2)$.

Remark. *Note that one can understand the Lévy white noise as a generalized random process (in the sense of [11]) with trajectories from $\mathcal{D}'$: formally $L'_\cdot(\omega)=\langle\omega,1_{[0,\cdot)}\rangle'=\langle\omega,\delta_\cdot\rangle=\omega(\cdot)$, where $\delta_\cdot$ is the Dirac delta-function concentrated at $\cdot$. Therefore μ is the measure of L' in the classical sense of this notion [12].*

Remark. *A Lévy process L without Gaussian part and drift is a* Poisson process *if its Lévy measure $\nu(\Delta)=\delta_1(\Delta)$, $\Delta\in\mathcal{B}(\mathbb{R})$, i.e., if ν is a point mass at 1. This measure does not satisfy the conditions accepted above (the support of δ_1 does not contain an infinite number of points); nevertheless, all results of the present paper have natural (and often strong) analogs in the Poissonian analysis. The reader can find more information about peculiarities of the Poissonian case in [23], Subsection 1.2.*

1.2 Lytvynov's generalization of the CRP

As is known, some random processes L have a so-called *chaotic representation property* (CRP) that consists, roughly speaking, in the following: any square integrable random variable can be decomposed in a series of repeated stochastic integrals from nonrandom functions with respect to L (see, e.g., [28] for a detailed presentation). The CRP plays a very important role in the stochastic analysis (in particular, for processes with the CRP this property can be used in order to construct extended stochastic integrals [16, 34, 15], stochastic derivatives and operators of stochastic differentiation, e.g., [37, 1]), but, unfortunately, the only Lévy processes with this property are Wiener and Poisson processes (e.g., [36]).

There are different approaches to a generalization of the CRP for Lévy processes: Itô's approach [14], Nualart-Schoutens' approach [29, 32], Lytvynov's approach [27], Oksendal's approach [7, 6] etc. The interconnections between these generalizations of the CRP are described in, e.g., [27, 2, 7, 35, 6, 23]. In the present paper we deal with Lytvynov's generalization of the CRP that will be described now in detail.

Denote by $\widehat{\otimes}$ a symmetric tensor product and set $\mathbb{Z}_+:=\mathbb{N}\cup\{0\}$. Let $\mathcal{P}\equiv\mathcal{P}(\mathcal{D}')$ be the set of polynomials on $\mathcal{D}'$, i.e., $\mathcal{P}$ consists of zero and elements of the form

$$f(\omega)=\sum_{n=0}^{N_f}\langle\omega^{\otimes n},f^{(n)}\rangle,\quad \omega\in\mathcal{D}',\ N_f\in\mathbb{Z}_+,\ f^{(n)}\in\mathcal{D}^{\widehat{\otimes}n},\ f^{(N_f)}\neq 0,$$

here N_f is called the *power of a polynomial* f; $\langle\omega^{\otimes 0},f^{(0)}\rangle:=f^{(0)}\in\mathcal{D}^{\widehat{\otimes}0}:=\mathbb{R}$. Since the measure μ of a Lévy white noise has a holomorphic at zero Laplace transform (this follows from (3) and properties of the measure ν, see also [27]), $\mathcal{P}$ is a dense set in (L^2) [33]. Denote by $\mathcal{P}_n$ the set of polynomials of power not greater than n, by $\overline{\mathcal{P}}_n$ the closure of $\mathcal{P}_n$ in (L^2). Let for $n\in\mathbb{N}$ $\mathbf{P}_n:=\overline{\mathcal{P}}_n\ominus\overline{\mathcal{P}}_{n-1}$ (the orthogonal difference in (L^2)), $\mathbf{P}_0:=\overline{\mathcal{P}}_0$. It is clear now that

$$(L^2)=\bigoplus_{n=0}^{\infty}\mathbf{P}_n.$$

Let $f^{(n)} \in \mathcal{D}^{\widehat{\otimes} n}$, $n \in \mathbb{Z}_+$. Denote by $:\langle \circ^{\otimes n}, f^{(n)}\rangle:$ the orthogonal projection in (L^2) of a monomial $\langle \circ^{\otimes n}, f^{(n)}\rangle$ onto $\mathbf{P}_n$. Let us define scalar products $(\cdot,\cdot)_{ext}$ on $\mathcal{D}^{\widehat{\otimes} n}$, $n \in \mathbb{Z}_+$, by setting for $f^{(n)}, g^{(n)} \in \mathcal{D}^{\widehat{\otimes} n}$

$$(f^{(n)}, g^{(n)})_{ext} := \frac{1}{n!}\int_{\mathcal{D}'} :\langle \omega^{\otimes n}, f^{(n)}\rangle::\langle \omega^{\otimes n}, g^{(n)}\rangle: \mu(d\omega),$$

and let $|\cdot|_{ext}$ be the corresponding norms, i.e., $|f^{(n)}|_{ext} = \sqrt{(f^{(n)}, f^{(n)})_{ext}}$. Denote by $\mathcal{H}_{ext}^{(n)}$, $n \in \mathbb{Z}_+$, the completions of $\mathcal{D}^{\widehat{\otimes} n}$ with respect to the norms $|\cdot|_{ext}$. For $F^{(n)} \in \mathcal{H}_{ext}^{(n)}$ define a Wick monomial $:\langle \circ^{\otimes n}, F^{(n)}\rangle: \stackrel{\text{def}}{=} (L^2) - \lim_{k\to\infty} :\langle \circ^{\otimes n}, f_k^{(n)}\rangle:$, where $\mathcal{D}^{\widehat{\otimes} n} \ni f_k^{(n)} \to F^{(n)}$ as $k \to \infty$ in $\mathcal{H}_{ext}^{(n)}$ (well-posedness of this definition can be proved by the method of "mixed sequences"). Since, as is easy to see, for each $n \in \mathbb{Z}_+$ the set $\{:\langle \circ^{\otimes n}, f^{(n)}\rangle: | f^{(n)} \in \mathcal{D}^{\widehat{\otimes} n}\}$ is a dense one in $\mathbf{P}_n$, we have the next statement (which describes Lytvynov's generalization of the CRP).

Theorem. ([27]) *A random variable $F \in (L^2)$ if and only if there exists a unique sequence of kernels $F^{(n)} \in \mathcal{H}_{ext}^{(n)}$, $n \in \mathbb{Z}_+$, such that*

$$F = \sum_{n=0}^{\infty} :\langle \circ^{\otimes n}, F^{(n)}\rangle: \tag{5}$$

(the series converges in (L^2)) and

$$\|F\|^2_{(L^2)} = \int_{\mathcal{D}'} |F(\omega)|^2 \mu(d\omega) = \mathbb{E}|F|^2 = \sum_{n=0}^{\infty} n! |F^{(n)}|^2_{ext} < \infty.$$

So, for $F, G \in (L^2)$ the scalar product has the form

$$(F,G)_{(L^2)} = \int_{\mathcal{D}'} F(\omega)G(\omega)\mu(d\omega) = \mathbb{E}[FG] = \sum_{n=0}^{\infty} n!(F^{(n)}, G^{(n)})_{ext},$$

where $F^{(n)}, G^{(n)} \in \mathcal{H}_{ext}^{(n)}$ are the kernels from decompositions (5) for F and G respectively. In particular, for $F^{(n)} \in \mathcal{H}_{ext}^{(n)}$ and $G^{(m)} \in \mathcal{H}_{ext}^{(m)}$, $n, m \in \mathbb{Z}_+$,

$$\begin{aligned}\big(:\langle \circ^{\otimes n}, F^{(n)}\rangle:, :\langle \circ^{\otimes m}, G^{(m)}\rangle:\big)_{(L^2)} &= \int_{\mathcal{D}'} :\langle \omega^{\otimes n}, F^{(n)}\rangle::\langle \omega^{\otimes m}, G^{(m)}\rangle: \mu(d\omega) \\ &= \mathbb{E}\big[:\langle \circ^{\otimes n}, F^{(n)}\rangle::\langle \circ^{\otimes m}, G^{(m)}\rangle:\big] = \delta_{n,m} n!(F^{(n)}, G^{(n)})_{ext}.\end{aligned}$$

Note that in the space (L^2) we have $:\langle \circ^{\otimes 0}, F^{(0)}\rangle: = \langle \circ^{\otimes 0}, F^{(0)}\rangle = F^{(0)}$ and $:\langle \circ, F^{(1)}\rangle: = \langle \circ, F^{(1)}\rangle$ [27].

Remark. *In order to make calculations connected with the spaces $\mathcal{H}_{ext}^{(n)}$, it is necessary to know explicit formulas for the scalar products $(\cdot,\cdot)_{ext}$. Such formulas were obtained by E.W. Lytvynov in [27]. Here, following [23], we write out it for convenience of a reader. Denote by $\|\cdot\|_\nu$ the norm in the space $L^2(\mathbb{R}, \nu)$ of (classes of) square integrable with respect to ν real-valued functions on $\mathbb{R}$. Let*

$$p_n(x) := x^n + a_{n,n-1}x^{n-1} + \cdots + a_{n,1}x, \quad a_{n,j} \in \mathbb{R},\ j \in \{1, \ldots, n-1\},\ n \in \mathbb{N}, \tag{6}$$

be orthogonal in $L^2(\mathbb{R},\nu)$ polynomials, i.e., for natural numbers n,m such that $n \neq m$, $\int_{\mathbb{R}} p_n(x)p_m(x)\nu(dx) = 0$. Then for $F^{(n)}, G^{(n)} \in \mathcal{H}_{ext}^{(n)}$, $n \in \mathbb{N}$, we have

$$
\begin{aligned}
(F^{(n)},G^{(n)})_{ext} = &\sum_{\substack{k,l_j,s_j\in\mathbb{N}:\ j=1,\dots,k,\ l_1>l_2>\dots>l_k,\\ l_1s_1+\dots+l_ks_k=n}} \frac{n!}{s_1!\cdots s_k!}\Big(\frac{\|p_{l_1}\|_\nu}{l_1!}\Big)^{2s_1}\cdots\Big(\frac{\|p_{l_k}\|_\nu}{l_k!}\Big)^{2s_k}\\
&\times \int_{\mathbb{R}_+^{s_1+\dots+s_k}} F^{(n)}(\underbrace{u_1,\dots,u_1}_{l_1},\dots,\underbrace{u_{s_1},\dots,u_{s_1}}_{l_1},\dots,\underbrace{u_{s_1+\dots+s_k},\dots,u_{s_1+\dots+s_k}}_{l_k}) \qquad (7)\\
&\times G^{(n)}(\underbrace{u_1,\dots,u_1}_{l_1},\dots,\underbrace{u_{s_1},\dots,u_{s_1}}_{l_1},\dots,\underbrace{u_{s_1+\dots+s_k},\dots,u_{s_1+\dots+s_k}}_{l_k})du_1\cdots du_{s_1+\dots+s_k}.
\end{aligned}
$$

In particular, for $n = 1$ $(F^{(1)},G^{(1)})_{ext} = \|p_1\|_\nu^2 \int_{\mathbb{R}_+} F^{(1)}(u)G^{(1)}(u)du$; if $n = 2$ then we have $(F^{(2)},G^{(2)})_{ext} = \|p_1\|_\nu^4 \int_{\mathbb{R}_+^2} F^{(2)}(u,v)G^{(2)}(u,v)dudv + \frac{\|p_2\|_\nu^2}{2}\int_{\mathbb{R}_+} F^{(2)}(u,u)G^{(2)}(u,u)du$, etc.

It follows from (7) that $\mathcal{H}_{ext}^{(1)} = \mathcal{H} \equiv L^2(\mathbb{R}_+)$: by (6) $p_1(x) = x$ and therefore by (2) $\|p_1\|_\nu = 1$; and for $n \in \mathbb{N}\backslash\{1\}$ one can identify $\mathcal{H}^{\widehat{\otimes}n}$ with the proper subspace of $\mathcal{H}_{ext}^{(n)}$ that consists of "vanishing on diagonals" elements (i.e., $F^{(n)}(u_1,\dots,u_n) = 0$ if there exist $k,j \in \{1,\dots,n\}$ such that $k \neq j$ but $u_k = u_j$). In this sense the space $\mathcal{H}_{ext}^{(n)}$ is an extension of $\mathcal{H}^{\widehat{\otimes}n}$ (this explains why we use the subscript ext in the notations $\mathcal{H}_{ext}^{(n)}$, $(\cdot,\cdot)_{ext}$ and $|\cdot|_{ext}$).

1.3 A nonregular rigging of (L^2)

Denote by T the set of indexes $\tau = (\tau_1,\tau_2)$, where $\tau_1 \in \mathbb{N}$, τ_2 is an infinite differentiable function on $\mathbb{R}_+$ such that for all $u \in \mathbb{R}_+$ $\tau_2(u) \geq 1$. Let $\mathcal{H}_\tau$ be the Sobolev space on $\mathbb{R}_+$ of order τ_1 weighted by the function τ_2, i.e., $\mathcal{H}_\tau$ is a completion of the set of infinite differentiable functions on $\mathbb{R}_+$ with compact supports with respect to the norm generated by the scalar product

$$(\varphi,\psi)_{\mathcal{H}_\tau} = \int_{\mathbb{R}_+}\Big(\varphi(u)\psi(u) + \sum_{k=1}^{\tau_1}\varphi^{[k]}(u)\psi^{[k]}(u)\Big)\tau_2(u)du,$$

here $\varphi^{[k]}$ and $\psi^{[k]}$ are derivatives of order k of functions φ and ψ respectively. It is well known (e.g., [4]) that $\mathcal{D} = \mathrm{pr\,lim}_{\tau\in T}\mathcal{H}_\tau$ (moreover, $\mathcal{D}^{\widehat{\otimes}n} = \mathrm{pr\,lim}_{\tau\in T}\mathcal{H}_\tau^{\widehat{\otimes}n}$, see, e.g., [3] for details) and for each $\tau \in T$ $\mathcal{H}_\tau$ is densely and continuously embedded into $\mathcal{H} \equiv L^2(\mathbb{R}_+)$, therefore one can consider the chain

$$\mathcal{D}' \supset \mathcal{H}_{-\tau} \supset \mathcal{H} \supset \mathcal{H}_\tau \supset \mathcal{D},$$

where $\mathcal{H}_{-\tau}$, $\tau \in T$, are the spaces dual of $\mathcal{H}_\tau$ with respect to $\mathcal{H}$. Note that by the Schwartz theorem [4] $\mathcal{D}' = \mathrm{ind\,lim}_{\tau\in T}\mathcal{H}_{-\tau}$ (it is convenient for us to consider $\mathcal{D}'$ as a topological space with the inductive limit topology). By analogy with [22] one can easily show that the measure μ of a Lévy white noise is concentrated on $\mathcal{H}_{-\widetilde{\tau}}$ with some $\widetilde{\tau} \in T$, i.e., $\mu(\mathcal{H}_{-\widetilde{\tau}}) = 1$. Excepting from T the indexes τ such that μ is not concentrated on $\mathcal{H}_{-\tau}$, we will assume, in what follows, that *for each $\tau \in T$ $\mu(\mathcal{H}_{-\tau}) = 1$.*

Denote the norms in $\mathcal{H}_\tau$ and its tensor powers by $|\cdot|_\tau$, i.e., for $f^{(n)} \in \mathcal{H}_\tau^{\widehat{\otimes}n}$, $n \in \mathbb{N}$, $|f^{(n)}|_\tau = \sqrt{(f^{(n)},f^{(n)})_{\mathcal{H}_\tau^{\widehat{\otimes}n}}}$.

Lemma. ([21]) *There exists $\tau' \in T$ such that for each $n \in \mathbb{N}$ the space $\mathcal{H}_{\tau'}^{\widehat{\otimes} n}$ is densely and continuously embedded into the space $\mathcal{H}_{ext}^{(n)}$. Moreover, for all $f^{(n)} \in \mathcal{H}_{\tau'}^{\widehat{\otimes} n}$*

$$|f^{(n)}|_{ext}^2 \leq n! c^n |f^{(n)}|_{\tau'}^2,$$

where $c > 0$ is some constant.

It follows from this lemma that if for some $\tau \in T$ the space $\mathcal{H}_\tau$ is continuously embedded into the space $\mathcal{H}_{\tau'}$ then for each $n \in \mathbb{N}$ the space $\mathcal{H}_\tau^{\widehat{\otimes} n}$ is densely and continuously embedded into the space $\mathcal{H}_{ext}^{(n)}$, and there exists $c(\tau) > 0$ such that for all $f^{(n)} \in \mathcal{H}_\tau^{\widehat{\otimes} n}$

$$|f^{(n)}|_{ext}^2 \leq n! c(\tau)^n |f^{(n)}|_\tau^2. \tag{8}$$

In what follows, it will be convenient to assume that the *indexes τ such that $\mathcal{H}_\tau$ is not continuously embedded into $\mathcal{H}_{\tau'}$, are removed from T.*

Denote $\mathcal{P}_W := \{f = \sum_{n=0}^{N_f} :\langle \circ^{\otimes n}, f^{(n)}\rangle :, f^{(n)} \in \mathcal{D}^{\widehat{\otimes} n}, N_f \in \mathbb{Z}_+\} \subset (L^2)$. Accept on default $q \in \mathbb{Z}_+$, $\tau \in T$; set $\mathcal{H}_\tau^{\widehat{\otimes} 0} := \mathbb{R}$; and define scalar products $(\cdot,\cdot)_{\tau,q}$ on $\mathcal{P}_W$ by setting for

$$f = \sum_{n=0}^{N_f} :\langle \circ^{\otimes n}, f^{(n)}\rangle :, \ g = \sum_{n=0}^{N_g} :\langle \circ^{\otimes n}, g^{(n)}\rangle : \in \mathcal{P}_W$$

$$(f,g)_{\tau,q} := \sum_{n=0}^{\min(N_f,N_g)} (n!)^2 2^{qn} (f^{(n)}, g^{(n)})_{\mathcal{H}_\tau^{\widehat{\otimes} n}}. \tag{9}$$

Let $\|\cdot\|_{\tau,q}$ be the corresponding norms, i.e., $\|f\|_{\tau,q} = \sqrt{(f,f)_{\tau,q}}$. In order to verify the well-posedness of this definition, i.e., that formula (9) defines *scalar*, and not just quasiscalar products, we note that if for $f \in \mathcal{P}_W$ $\|f\|_{\tau,q} = 0$ then by (9) for each coefficient $f^{(n)}$ of f $|f^{(n)}|_\tau = 0$ and therefore by (8) $|f^{(n)}|_{ext} = 0$. So, in this case $f = 0$ in (L^2).

Definition. *We define Kondratiev spaces of nonregular test functions $(\mathcal{H}_\tau)_q$ as completions of $\mathcal{P}_W$ with respect to the norms $\|\cdot\|_{\tau,q}$, and set*

$$(\mathcal{H}_\tau) := \operatorname*{pr\,lim}_{q\in\mathbb{Z}_+} (\mathcal{H}_\tau)_q, \quad (\mathcal{D}) := \operatorname*{pr\,lim}_{q\in\mathbb{Z}_+,\tau\in T} (\mathcal{H}_\tau)_q.$$

As is easy to see, $f \in (\mathcal{H}_\tau)_q$ if and only if f can be presented in the form

$$f = \sum_{n=0}^{\infty} :\langle \circ^{\otimes n}, f^{(n)}\rangle :, \ f^{(n)} \in \mathcal{H}_\tau^{\widehat{\otimes} n} \tag{10}$$

(the series converges in $(\mathcal{H}_\tau)_q$), with

$$\|f\|_{\tau,q}^2 := \|f\|_{(\mathcal{H}_\tau)_q}^2 = \sum_{n=0}^{\infty} (n!)^2 2^{qn} |f^{(n)}|_\tau^2 < \infty; \tag{11}$$

and for $f, g \in (\mathcal{H}_\tau)_q$

$$(f,g)_{(\mathcal{H}_\tau)_q} = \sum_{n=0}^{\infty} (n!)^2 2^{qn} (f^{(n)}, g^{(n)})_{\mathcal{H}_\tau^{\widehat{\otimes} n}},$$

where $f^{(n)}, g^{(n)} \in \mathcal{H}_\tau^{\widehat{\otimes} n}$ are the kernels from decompositions (10) for f and g respectively (since for each $n \in \mathbb{Z}_+$ $\mathcal{H}_\tau^{\widehat{\otimes} n} \subseteq \mathcal{H}_{ext}^{(n)}$, for $f^{(n)} \in \mathcal{H}_\tau^{\widehat{\otimes} n}$ $:\langle \circ^{\otimes n}, f^{(n)}\rangle :$ is a well defined Wick monomial, see Subsection 1.2). Further, $f \in (\mathcal{H}_\tau)$ ($f \in (\mathcal{D})$) if and only if f can be presented in form (10) and norm (11) is finite for each $q \in \mathbb{Z}_+$ (for each $q \in \mathbb{Z}_+$ and each $\tau \in T$).

Proposition. ([21]) *For each $\tau \in T$ there exists $q_0 = q_0(\tau) \in \mathbb{Z}_+$ such that for each $q \in \mathbb{N}_{q_0} := \{q_0, q_0 + 1, \cdots\}$ the space $(\mathcal{H}_\tau)_q$ is densely and continuously embedded into (L^2).*

In view of this proposition for $\tau \in T$ and $q \geq q_0(\tau)$ one can consider a chain

$$(\mathcal{D}') \supset (\mathcal{H}_{-\tau}) \supset (\mathcal{H}_{-\tau})_{-q} \supset (L^2) \supset (\mathcal{H}_\tau)_q \supset (\mathcal{H}_\tau) \supset (\mathcal{D}), \tag{12}$$

where $(\mathcal{H}_{-\tau})_{-q}$, $(\mathcal{H}_{-\tau}) = \operatorname{ind\,lim}_{q\to\infty}(\mathcal{H}_{-\tau})_{-q}$ and $(\mathcal{D}') = \operatorname{ind\,lim}_{q\to\infty,\tau\in T}(\mathcal{H}_{-\tau})_{-q}$ are the spaces dual of $(\mathcal{H}_\tau)_q$, $(\mathcal{H}_\tau)$ and $(\mathcal{D})$ with respect to (L^2).

Definition. *Chain* (12) *is called a nonregular rigging of the space (L^2). The negative spaces of this chain $(\mathcal{H}_{-\tau})_{-q}$, $(\mathcal{H}_{-\tau})$ and $(\mathcal{D}')$ are called Kondratiev spaces of nonregular generalized functions.*

Finally, we describe natural orthogonal bases in the spaces $(\mathcal{H}_{-\tau})_{-q}$. Let us consider chains

$$\mathcal{D}'^{(m)} \supset \mathcal{H}_{-\tau}^{(m)} \supset \mathcal{H}_{ext}^{(m)} \supset \mathcal{H}_\tau^{\widehat{\otimes} m} \supset \mathcal{D}^{\widehat{\otimes} m}, \tag{13}$$

$m \in \mathbb{Z}_+$ (for $m = 0$ $\mathcal{D}^{\widehat{\otimes} 0} = \mathcal{H}_\tau^{\widehat{\otimes} 0} = \mathcal{H}_{ext}^{(0)} = \mathcal{H}_{-\tau}^{(0)} = \mathcal{D}'^{(0)} = \mathbb{R}$), where $\mathcal{H}_{-\tau}^{(m)}$ and $\mathcal{D}'^{(m)} = \operatorname{ind\,lim}_{\tau\in T}\mathcal{H}_{-\tau}^{(m)}$ are the spaces dual of $\mathcal{H}_\tau^{\widehat{\otimes} m}$ and $\mathcal{D}^{\widehat{\otimes} m}$ with respect to $\mathcal{H}_{ext}^{(m)}$. The next statement follows from the definition of the spaces $(\mathcal{H}_{-\tau})_{-q}$ and the general duality theory (cf. [22]).

Proposition. ([21]) *There exists a system of generalized functions*

$$\{:\langle \circ^{\otimes m}, F_{ext}^{(m)}\rangle: \in (\mathcal{H}_{-\tau})_{-q} \mid F_{ext}^{(m)} \in \mathcal{H}_{-\tau}^{(m)},\ m \in \mathbb{Z}_+\}$$

such that

1) for $F_{ext}^{(m)} \in \mathcal{H}_{ext}^{(m)} \subset \mathcal{H}_{-\tau}^{(m)}$ $:\langle \circ^{\otimes m}, F_{ext}^{(m)}\rangle:$ is a Wick monomial that was defined in Subsection 1.2;

2) any generalized function $F \in (\mathcal{H}_{-\tau})_{-q}$ can be presented as a series

$$F = \sum_{m=0}^{\infty} :\langle \circ^{\otimes m}, F_{ext}^{(m)}\rangle:,\ F_{ext}^{(m)} \in \mathcal{H}_{-\tau}^{(m)}, \tag{14}$$

that converges in $(\mathcal{H}_{-\tau})_{-q}$, i.e.,

$$\|F\|^2_{-\tau,-q} := \|F\|^2_{(\mathcal{H}_{-\tau})_{-q}} = \sum_{m=0}^{\infty} 2^{-qm} |F_{ext}^{(m)}|^2_{\mathcal{H}_{-\tau}^{(m)}} < \infty; \tag{15}$$

and, vice versa, any series (14) *with finite norm* (15) *is a generalized function from $(\mathcal{H}_{-\tau})_{-q}$ (i.e., such a series converges in $(\mathcal{H}_{-\tau})_{-q}$);*

3) for $F, G \in (\mathcal{H}_{-\tau})_{-q}$ the scalar product has a form

$$(F, G)_{(\mathcal{H}_{-\tau})_{-q}} = \sum_{m=0}^{\infty} 2^{-qm} (F_{ext}^{(m)}, G_{ext}^{(m)})_{\mathcal{H}_{-\tau}^{(m)}},$$

where $F_{ext}^{(m)}, G_{ext}^{(m)} \in \mathcal{H}_{-\tau}^{(m)}$ are the kernels from decompositions (14) *for F and G respectively;*

4) the dual pairing between $F \in (\mathcal{H}_{-\tau})_{-q}$ and $f \in (\mathcal{H}_\tau)_q$ that is generated by the scalar product in (L^2), has the form

$$\langle\!\langle F, f\rangle\!\rangle_{(L^2)} = \sum_{m=0}^{\infty} m! \langle F_{ext}^{(m)}, f^{(m)}\rangle_{ext}, \tag{16}$$

where $F_{ext}^{(m)} \in \mathcal{H}_{-\tau}^{(m)}$ and $f^{(m)} \in \mathcal{H}_\tau^{\widehat{\otimes} m}$ are the kernels from decompositions (14) *and* (10) *for F and f respectively, $\langle\cdot,\cdot\rangle_{ext}$ denotes the dual pairings between elements of negative and positive spaces from chains* (13)*, these pairings are generated by the scalar products in $\mathcal{H}_{ext}^{(m)}$.*

It is clear that $F \in (\mathcal{H}_{-\tau})$ ($F \in (\mathcal{D}')$) if and only if F can be presented in form (14) and norm (15) is finite for some $q \in \mathbb{N}_{q_0(\tau)}$ (for some $\tau \in T$ and some $q \in \mathbb{N}_{q_0(\tau)}$).

1.4 Stochastic derivatives and integrals

First, following [24], we recall the notion of the Hida stochastic derivative on the spaces of nonregular test functions, and of the extended stochastic integral on the spaces of nonregular generalized functions. Decomposition (5) for elements of (L^2) defines an isometric isomorphism (a generalized Wiener-Itô-Sigal isomorphism) $\mathbf{I} : (L^2) \to \bigoplus_{n=0}^{\infty} n!\mathcal{H}_{ext}^{(n)}$, where $\bigoplus_{n=0}^{\infty} n!\mathcal{H}_{ext}^{(n)}$ is a weighted extended Fock space (cf. [26]): for $F \in (L^2)$ of form (5) $\mathbf{I}F = (F^{(0)}, F^{(1)}, \dots, F^{(n)}, \dots) \in \bigoplus_{n=0}^{\infty} n!\mathcal{H}_{ext}^{(n)}$. Let $\mathbf{1} : \mathcal{H} \to \mathcal{H}$ be the identity operator. Then the operator $\mathbf{I} \otimes \mathbf{1} : (L^2) \otimes \mathcal{H} \to \big(\bigoplus_{n=0}^{\infty} n!\mathcal{H}_{ext}^{(n)}\big) \otimes \mathcal{H} \cong \bigoplus_{n=0}^{\infty} n!(\mathcal{H}_{ext}^{(n)} \otimes \mathcal{H})$ is an isometric isomorphism between the spaces $(L^2) \otimes \mathcal{H}$ and $\bigoplus_{n=0}^{\infty} n!(\mathcal{H}_{ext}^{(n)} \otimes \mathcal{H})$. It is clear that for arbitrary $n \in \mathbb{Z}_+$ and $F_\cdot^{(n)} \in \mathcal{H}_{ext}^{(n)} \otimes \mathcal{H}$ a vector $(\underbrace{0, \dots, 0}_{n}, F_\cdot^{(n)}, 0, \dots)$ belongs to $\bigoplus_{n=0}^{\infty} n!(\mathcal{H}_{ext}^{(n)} \otimes \mathcal{H})$. Set

$$:\langle \circ^{\otimes n}, F_\cdot^{(n)} \rangle: \overset{def}{=} (\mathbf{I} \otimes \mathbf{1})^{-1}(\underbrace{0, \dots, 0}_{n}, F_\cdot^{(n)}, 0, \dots) \in (L^2) \otimes \mathcal{H}. \tag{17}$$

By the construction elements $:\langle \circ^{\otimes n}, F_\cdot^{(n)} \rangle:$, $n \in \mathbb{Z}_+$, form an orthogonal basis in the space $(L^2) \otimes \mathcal{H}$ in the sense that any $F \in (L^2) \otimes \mathcal{H}$ can be presented as

$$F(\cdot) = \sum_{n=0}^{\infty} :\langle \circ^{\otimes n}, F_\cdot^{(n)} \rangle:, \quad F_\cdot^{(n)} \in \mathcal{H}_{ext}^{(n)} \otimes \mathcal{H}$$

(the series converges in $(L^2) \otimes \mathcal{H}$), with $\|F\|^2_{(L^2)\otimes\mathcal{H}} = \sum_{n=0}^{\infty} n!|F_\cdot^{(n)}|^2_{\mathcal{H}_{ext}^{(n)}\otimes\mathcal{H}} < \infty$. Since, as is easily seen, the restriction of the generalized Wiener-Itô-Sigal isomorphism $\mathbf{I}$ to the space $(\mathcal{H}_\tau)_q$ is an isometric isomorphism between $(\mathcal{H}_\tau)_q$ and a weighted Fock space $\bigoplus_{n=0}^{\infty} (n!)^2 2^{qn} \mathcal{H}_\tau^{\widehat{\otimes} n}$ (cf. [26]), and, of course, the restriction of the identity operator on $\mathcal{H}$ to the space $\mathcal{H}_\tau$ is the identity operator on $\mathcal{H}_\tau$, for arbitrary $n \in \mathbb{Z}_+$ and $f_\cdot^{(n)} \in \mathcal{H}_\tau^{\widehat{\otimes} n} \otimes \mathcal{H}_\tau \subset \mathcal{H}_{ext}^{(n)} \otimes \mathcal{H}$ we have $:\langle \circ^{\otimes n}, f_\cdot^{(n)} \rangle: \in (\mathcal{H}_\tau)_q \otimes \mathcal{H}_\tau$. Moreover, elements $:\langle \circ^{\otimes n}, f_\cdot^{(n)} \rangle:$, $f_\cdot^{(n)} \in \mathcal{H}_\tau^{\widehat{\otimes} n} \otimes \mathcal{H}_\tau$, $n \in \mathbb{Z}_+$, form orthogonal bases (in the above-described sense) in the spaces $(\mathcal{H}_\tau)_q \otimes \mathcal{H}_\tau$.

Definition. *For $g \in (\mathcal{H}_\tau)_q$ we define a Hida stochastic derivative $\partial_\cdot g \in (\mathcal{H}_\tau)_q \otimes \mathcal{H}_\tau$ by the formula*

$$\partial_\cdot g := \sum_{n=0}^{\infty} (n+1) :\langle \circ^{\otimes n}, g^{(n+1)}(\cdot) \rangle:, \tag{18}$$

where $g^{(n+1)} \in \mathcal{H}_\tau^{\widehat{\otimes} n+1}$, $n \in \mathbb{Z}_+$, are the kernels from decomposition (10) *for g considered as elements of $\mathcal{H}_\tau^{\widehat{\otimes} n} \otimes \mathcal{H}_\tau$.*

Since (see (11))

$$\begin{aligned} \|\partial_\cdot g\|^2_{(\mathcal{H}_\tau)_q\otimes\mathcal{H}_\tau} &= \sum_{n=0}^{\infty} ((n+1)!)^2 2^{qn} |g^{(n+1)}(\cdot)|^2_{\mathcal{H}_\tau^{\widehat{\otimes} n}\otimes\mathcal{H}_\tau} \\ &= 2^{-q} \sum_{n=0}^{\infty} ((n+1)!)^2 2^{q(n+1)} |g^{(n+1)}|^2_\tau \le 2^{-q}\|g\|^2_{\tau,q}, \end{aligned} \tag{19}$$

this definition is well posed and, moreover, the Hida stochastic derivative

$$\partial_\cdot : (\mathcal{H}_\tau)_q \to (\mathcal{H}_\tau)_q \otimes \mathcal{H}_\tau \tag{20}$$

is a linear *continuous* operator. It is shown in [24] that this derivative is (generated by) the restriction to $(\mathcal{H}_\tau)_q$ of the Hida stochastic derivative on (L^2). We note also that the restrictions of derivative (20) to $(\mathcal{H}_\tau)$ and to $(\mathcal{D})$ generate linear continuous operators $\partial_\cdot : (\mathcal{H}_\tau) \to (\mathcal{H}_\tau) \otimes \mathcal{H}_\tau := \mathrm{pr}\lim_{q\in\mathbb{Z}_+}(\mathcal{H}_\tau)_q \otimes \mathcal{H}_\tau$ and $\partial_\cdot : (\mathcal{D}) \to (\mathcal{D}) \otimes \mathcal{D} := \mathrm{pr}\lim_{q\in\mathbb{Z}_+,\tau\in T}(\mathcal{H}_\tau)_q \otimes \mathcal{H}_\tau$ respectively.

Definition. *We define an extended stochastic integral*

$$\int \circ(u)\widehat{d}L_u : (\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}_{-\tau} \to (\mathcal{H}_{-\tau})_{-q} \tag{21}$$

as a linear continuous operator adjoint to Hida stochastic derivative (20)*: for* $F \in (\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}_{-\tau}$

$$\int F(u)\widehat{d}L_u := \partial_\cdot^* F \in (\mathcal{H}_{-\tau})_{-q}, \tag{22}$$

i.e., for arbitrary $g \in (\mathcal{H}_\tau)_q$ $\langle\langle \int F(u)\widehat{d}L_u, g\rangle\rangle_{(L^2)} = \langle\langle F(\cdot), \partial_\cdot g\rangle\rangle_{(L^2)\otimes\mathcal{H}}$.

It is shown in [24] that integral (21) is an extension of the extended Skorohod stochastic integral on $(L^2) \otimes \mathcal{H}$.

By analogy one can define linear continuous operators $\int \circ(u)\widehat{d}L_u : (\mathcal{H}_{-\tau}) \otimes \mathcal{H}_{-\tau} \to (\mathcal{H}_{-\tau})$ and $\int \circ(u)\widehat{d}L_u : (\mathcal{D}') \otimes \mathcal{D}' \to (\mathcal{D}')$, where $(\mathcal{H}_{-\tau}) \otimes \mathcal{H}_{-\tau} := \mathrm{ind}\lim_{q\to\infty}(\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}_{-\tau}$, $(\mathcal{D}') \otimes \mathcal{D}' := \mathrm{ind}\lim_{q\to\infty,\tau\in T}(\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}_{-\tau}$.

In contrast to formula (18) for the Hida stochastic derivative, formula (22) for integral (21) is inconvenient for calculations. Therefore let us write out a representation for this integral in terms of orthogonal bases in the spaces of nonregular generalized functions.

First we note that, as in the case of the spaces $(\mathcal{H}_{-\tau})_{-q}$, it follows from the general duality theory that there exists a system of orthogonal in $(\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}_{-\tau}$ generalized functions $\{:\langle\circ^{\otimes m}, F_{ext,\cdot}^{(m)}\rangle: \in (\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}_{-\tau} \mid F_{ext,\cdot}^{(m)} \in \mathcal{H}_{-\tau}^{(m)} \otimes \mathcal{H}_{-\tau},\ m \in \mathbb{Z}_+\}$ such that for $F_{ext,\cdot}^{(m)} \in \mathcal{H}_{ext}^{(m)} \otimes \mathcal{H} \subset \mathcal{H}_{-\tau}^{(m)} \otimes \mathcal{H}_{-\tau}$ $:\langle\circ^{\otimes m}, F_{ext,\cdot}^{(m)}\rangle:$ is given by (17); and any generalized function $F \in (\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}_{-\tau}$ can be presented as a convergent in $(\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}_{-\tau}$ series

$$F(\cdot) = \sum_{m=0}^{\infty} :\langle\circ^{\otimes m}, F_{ext,\cdot}^{(m)}\rangle:,\ F_{ext,\cdot}^{(m)} \in \mathcal{H}_{-\tau}^{(m)} \otimes \mathcal{H}_{-\tau}, \tag{23}$$

now

$$\|F\|^2_{(\mathcal{H}_{-\tau})_{-q}\otimes\mathcal{H}_{-\tau}} = \sum_{m=0}^{\infty} 2^{-qm}|F_{ext,\cdot}^{(m)}|^2_{\mathcal{H}_{-\tau}^{(m)}\otimes\mathcal{H}_{-\tau}} < \infty. \tag{24}$$

Consider a family of chains

$$\mathcal{D}'^{\widehat{\otimes}m} \supset \mathcal{H}_{-\tau}^{\widehat{\otimes}m} \supset \mathcal{H}^{\widehat{\otimes}m} \supset \mathcal{H}_{\tau}^{\widehat{\otimes}m} \supset \mathcal{D}^{\widehat{\otimes}m},\ m \in \mathbb{Z}_+ \tag{25}$$

(as is well known (e.g., [4]), $\mathcal{H}_{-\tau}^{\widehat{\otimes}m}$ and $\mathcal{D}'^{\widehat{\otimes}m} = \mathrm{ind}\lim_{\tau\in T}\mathcal{H}_{-\tau}^{\widehat{\otimes}m}$ are the spaces dual of $\mathcal{H}_{\tau}^{\widehat{\otimes}m}$ and $\mathcal{D}^{\widehat{\otimes}m}$ respectively; in the case $m = 0$ all spaces from chain (25) are equal to $\mathbb{R}$). Since the spaces of test functions in chains (25) and (13) coincide, there exists a family of natural isomorphisms

$$U_m : \mathcal{D}'^{(m)} \to \mathcal{D}'^{\widehat{\otimes}m},\ m \in \mathbb{Z}_+,$$

such that for all $F_{ext}^{(m)} \in \mathcal{D}'^{(m)}$ and $f^{(m)} \in \mathcal{D}^{\widehat{\otimes}m}$

$$\langle F_{ext}^{(m)}, f^{(m)} \rangle_{ext} = \langle U_m F_{ext}^{(m)}, f^{(m)} \rangle. \tag{26}$$

It is easy to see that the restrictions of U_m to $\mathcal{H}_{-\tau}^{(m)}$ are isometric isomorphisms between the spaces $\mathcal{H}_{-\tau}^{(m)}$ and $\mathcal{H}_{-\tau}^{\widehat{\otimes}m}$.

Remark. *As we saw above, $\mathcal{H}_{ext}^{(1)} = \mathcal{H}$, and therefore in the case $m = 1$ chains* (25) *and* (13) *coincide. Thus $U_1 = \mathbf{1}$ is the identity operator on $\mathcal{D}'^{(1)} = \mathcal{D}'$. In the case $m = 0$ U_0 is, obviously, the identity operator on $\mathbb{R}$.*

Proposition. ([24]) *Let $F \in (\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}_{-\tau}$. The extended stochastic integral can be presented in the form*

$$\int F(u)\widehat{d}L_u = \sum_{m=0}^{\infty} :\langle \circ^{\otimes m+1}, \widehat{F}_{ext}^{(m)} \rangle:, \tag{27}$$

where

$$\widehat{F}_{ext}^{(m)} := U_{m+1}^{-1}\{Pr[(U_m \otimes \mathbf{1})F_{ext,\cdot}^{(m)}]\} \in \mathcal{H}_{-\tau}^{(m+1)}, \tag{28}$$

Pr is the symmetrization operator (more exactly, the orthoprojector acting for each $m \in \mathbb{Z}_+$ from $\mathcal{H}_{-\tau}^{\widehat{\otimes}m} \otimes \mathcal{H}_{-\tau}$ to $\mathcal{H}_{-\tau}^{\widehat{\otimes}m+1}$), $F_{ext,\cdot}^{(m)} \in \mathcal{H}_{-\tau}^{(m)} \otimes \mathcal{H}_{-\tau}$, $m \in \mathbb{Z}_+$, are the kernels from decomposition (23) *for F.*

Remark. *Sometimes it can be convenient to introduce the Hida stochastic derivative and the extended stochastic integral as linear continuous operators acting from $(\mathcal{H}_\tau)_q$ to $(\mathcal{H}_\tau)_q \otimes \mathcal{H}$ and from $(\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}$ to $(\mathcal{H}_{-\tau})_{-q}$ respectively, this case is described in detail in [21].*

Unfortunately, in contrast to the Hida stochastic derivative, the extended stochastic integral with respect to a Lévy process cannot be naturally restricted to the spaces of nonregular test functions. More precisely, for $f \in (\mathcal{H}_\tau)_q \otimes \mathcal{H}_\tau$ $\int f(u)\widehat{d}L_u$ not necessary a nonregular test function (one can show that for $\tau \in T$ and $q \in \mathbb{Z}_+$ such that $q > \log_2 c(\tau)$, where $c(\tau) > 0$ from estimate (8), if $f \in (\mathcal{H}_\tau)_q \otimes \mathcal{H}_\tau$ then $\int f(u)\widehat{d}L_u \in (L^2)$; and for q sufficiently large this integral is a *regular* test function [21]). Nevertheless, one can introduce on each space of nonregular test functions a linear operator that has properties quite analogous to the properties of the extended stochastic integral. Now we'll introduce such operators (which will be called *generalized stochastic integrals*) and consider them in detail.

Let $f \in (\mathcal{H}_\tau)_q \otimes \mathcal{H}_\tau$. Using the above-described orthogonal basis in this space, we can write

$$f(\cdot) = \sum_{n=0}^{\infty} :\langle \circ^{\otimes n}, f_\cdot^{(n)} \rangle:, \ f_\cdot^{(n)} \in \mathcal{H}_\tau^{\widehat{\otimes}n} \otimes \mathcal{H}_\tau \tag{29}$$

(the series converges in $(\mathcal{H}_\tau)_q \otimes \mathcal{H}_\tau$), in this case

$$\|f\|_{(\mathcal{H}_\tau)_q \otimes \mathcal{H}_\tau}^2 = \sum_{n=0}^{\infty} (n!)^2 2^{qn} |f_\cdot^{(n)}|_{\mathcal{H}_\tau^{\widehat{\otimes}n} \otimes \mathcal{H}_\tau}^2 < \infty. \tag{30}$$

Definition. *We define a generalized stochastic integral*

$$\mathbb{I} : (\mathcal{H}_\tau)_{q+1} \otimes \mathcal{H}_\tau \to (\mathcal{H}_\tau)_q \tag{31}$$

as a linear continuous operator given for $f \in (\mathcal{H}_\tau)_{q+1} \otimes \mathcal{H}_\tau$ by the formula

$$\mathbb{I}(f) := \sum_{n=0}^{\infty} :\langle \circ^{\otimes n+1}, \hat{f}^{(n)}\rangle: \tag{32}$$

(cf. (27)), where $\hat{f}^{(n)} := Pr f_\cdot^{(n)} \in \mathcal{H}_\tau^{\widehat{\otimes} n+1}$ are the orthoprojections onto $\mathcal{H}_\tau^{\widehat{\otimes} n+1}$ (the symmetrizations by all variables) of the kernels $f_\cdot^{(n)} \in \mathcal{H}_\tau^{\widehat{\otimes} n} \otimes \mathcal{H}_\tau$ from decomposition (29) for f.

Since (see (11), (32) and (30))

$$\begin{aligned}\|\mathbb{I}(f)\|^2_{\tau,q} &= \sum_{n=0}^{\infty} ((n+1)!)^2 2^{q(n+1)} |\hat{f}^{(n)}|^2_\tau \le 2^q \sum_{n=0}^{\infty} (n!)^2 2^{(q+1)n} [(n+1)^2 2^{-n}] |f_\cdot^{(n)}|^2_{\mathcal{H}_\tau^{\widehat{\otimes} n} \otimes \mathcal{H}_\tau} \\ &\le 9 \cdot 2^{q-2} \|f\|^2_{(\mathcal{H}_\tau)_{q+1} \otimes \mathcal{H}_\tau},\end{aligned}$$

this definition is well posed. It is clear that the restriction of the operator $\mathbb{I}$ to the space $(\mathcal{H}_\tau) \otimes \mathcal{H}_\tau$ (respectively to the space $(\mathcal{D}) \otimes \mathcal{D}$) is a linear continuous operator acting from $(\mathcal{H}_\tau) \otimes \mathcal{H}_\tau$ to $(\mathcal{H}_\tau)$ (respectively from $(\mathcal{D}) \otimes \mathcal{D}$ to $(\mathcal{D})$).

The Hida stochastic derivative, in turn, has no a natural extension to the spaces of nonregular generalized functions (the kernels from decompositions (14) for elements of $(\mathcal{H}_{-\tau})_{-q}$ belong to the spaces $\mathcal{H}_{-\tau}^{(m)}$, $m \in \mathbb{Z}_+$, and for elements of these spaces it is impossible "to separate a variable"). Nevertheless, one can define a natural analog of this derivative (a *generalized Hida derivative*) on each of the above-mentioned spaces as an operator adjoint to $\mathbb{I}$.

Definition. *We define a generalized Hida derivative*

$$\widetilde{\partial}_\cdot : (\mathcal{H}_{-\tau})_{-q} \to (\mathcal{H}_{-\tau})_{-q-1} \otimes \mathcal{H}_{-\tau} \tag{33}$$

as a linear continuous operator adjoint to generalized stochastic integral (31) ($\widetilde{\partial}_\cdot := \mathbb{I}^*$), i.e., for all $F \in (\mathcal{H}_{-\tau})_{-q}$ and $f \in (\mathcal{H}_\tau)_{q+1} \otimes \mathcal{H}_\tau$

$$\langle\langle \widetilde{\partial}_\cdot F, f(\cdot)\rangle\rangle_{(L^2)\otimes\mathcal{H}} = \langle\langle F, \mathbb{I}(f)\rangle\rangle_{(L^2)}. \tag{34}$$

By analogy one can define linear continuous operators $\widetilde{\partial}_\cdot : (\mathcal{H}_{-\tau}) \to (\mathcal{H}_{-\tau}) \otimes \mathcal{H}_{-\tau}$ and $\widetilde{\partial}_\cdot : (\mathcal{D}') \to (\mathcal{D}') \otimes \mathcal{D}'$. We note also that since operators (33) and (31) are continuous, $\widetilde{\partial}_\cdot^* = \mathbb{I}^{**} = \mathbb{I}$ and $\widetilde{\partial}_\cdot^{**} = \mathbb{I}^* = \widetilde{\partial}_\cdot$.

In order to make calculations with derivative (33), let us obtain a representation for this operator in terms of orthogonal bases in the spaces of nonregular generalized functions.

Proposition. *Let $F \in (\mathcal{H}_{-\tau})_{-q}$. Then*

$$\widetilde{\partial}_\cdot F = \sum_{m=0}^{\infty} (m+1) :\langle \circ^{\otimes m}, F_{ext}^{(m+1)}(\cdot)\rangle: \in (\mathcal{H}_{-\tau})_{-q-1} \otimes \mathcal{H}_{-\tau} \tag{35}$$

(cf. (18)), where

$$F_{ext}^{(m+1)}(\cdot) := (U_m^{-1} \otimes \mathbf{1})(U_{m+1} F_{ext}^{(m+1)})(\cdot) \in \mathcal{H}_{-\tau}^{(m)} \otimes \mathcal{H}_{-\tau}, \tag{36}$$

here $F_{ext}^{(m+1)} \in \mathcal{H}_{-\tau}^{(m+1)}$ are the kernels from decomposition (14) for F.

Proof. Using (34), (14), (32), (16), (26), (36) and (29), for $F \in (\mathcal{H}_{-\tau})_{-q}$ and $f \in (\mathcal{H}_\tau)_{q+1} \otimes \mathcal{H}_\tau$ we obtain

$$\begin{aligned}
\langle\langle \widetilde{\partial}_{\cdot} F, f\rangle\rangle_{(L^2)\otimes\mathcal{H}} &= \langle\langle F, \mathbb{I}(f)\rangle\rangle_{(L^2)} = \langle\langle \sum_{m=0}^{\infty} :\langle \circ^{\otimes m}, F_{ext}^{(m)}\rangle :, \sum_{n=0}^{\infty} :\langle \circ^{\otimes n+1}, \hat{f}^{(n)}\rangle :\rangle\rangle_{(L^2)} \\
&= \sum_{m=0}^{\infty} (m+1)!\langle F_{ext}^{(m+1)}, \hat{f}^{(m)}\rangle_{\mathcal{H}_{ext}^{(m+1)}} = \sum_{m=0}^{\infty} (m+1)!\langle U_{m+1} F_{ext}^{(m+1)}, Pr f_{\cdot}^{(m)}\rangle_{\mathcal{H}^{\widehat{\otimes} m+1}} \\
&= \sum_{m=0}^{\infty} (m+1)!\langle (U_{m+1} F_{ext}^{(m+1)})(\cdot), f_{\cdot}^{(m)}\rangle_{\mathcal{H}^{\widehat{\otimes} m}\otimes\mathcal{H}} \\
&= \sum_{m=0}^{\infty} m!(m+1)\langle (U_m^{-1}\otimes \mathbf{1})(U_{m+1} F_{ext}^{(m+1)})(\cdot), f_{\cdot}^{(m)}\rangle_{\mathcal{H}_{ext}^{(m)}\otimes\mathcal{H}} \\
&= \langle\langle \sum_{m=0}^{\infty} (m+1) :\langle \circ^{\otimes m}, (U_m^{-1}\otimes \mathbf{1})(U_{m+1} F_{ext}^{(m+1)})(\cdot)\rangle :, \sum_{n=0}^{\infty} :\langle \circ^{\otimes n}, f_{\cdot}^{(n)}\rangle :\rangle\rangle_{(L^2)\otimes\mathcal{H}} \\
&= \langle\langle \sum_{m=0}^{\infty} (m+1) :\langle \circ^{\otimes m}, F_{ext}^{(m+1)}(\cdot)\rangle :, f\rangle\rangle_{(L^2)\otimes\mathcal{H}},
\end{aligned} \tag{37}$$

whence the result follows. ☐

Sometimes it can be necessary to define a generalized stochastic integral by formula (32) as a linear unbounded operator

$$\mathbb{I} : (\mathcal{H}_\tau)_q \otimes \mathcal{H}_\tau \to (\mathcal{H}_\tau)_q \tag{38}$$

with the domain

$$\mathrm{dom}(\mathbb{I}) := \{ f \in (\mathcal{H}_\tau)_q \otimes \mathcal{H}_\tau : \|\mathbb{I}(f)\|_{\tau,q}^2 = \sum_{n=0}^{\infty} ((n+1)!)^2 2^{q(n+1)} |\hat{f}^{(n)}|_\tau^2 < \infty \}. \tag{39}$$

Since set (39) is dense in $(\mathcal{H}_\tau)_q \otimes \mathcal{H}_\tau$, one can define now a corresponding generalized Hida derivative as an unbounded operator adjoint to operator (38):

$$\widetilde{\partial}_{\cdot} := \mathbb{I}^* : (\mathcal{H}_{-\tau})_{-q} \to (\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}_{-\tau}. \tag{40}$$

The domain of operator (40) by definition consists of $F \in (\mathcal{H}_{-\tau})_{-q}$ such that $(\mathcal{H}_\tau)_q \otimes \mathcal{H}_\tau \supset \mathrm{dom}(\mathbb{I}) \ni f \mapsto \langle\langle F, \mathbb{I}(f)\rangle\rangle_{(L^2)}$ is a linear continuous functional. By properties of Hilbert equipments the last is possible if and only if there exists $H \in (\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}_{-\tau}$ such that $\langle\langle F, \mathbb{I}(f)\rangle\rangle_{(L^2)} = \langle\langle H, f\rangle\rangle_{(L^2)\otimes\mathcal{H}}$. But by definition of $\widetilde{\partial}_{\cdot}$ we have $H = \widetilde{\partial}_{\cdot} F$ and therefore the domain of operator (40) can be described by the condition $\widetilde{\partial}_{\cdot} F \in (\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}_{-\tau}$. Since for $f \in \mathrm{dom}(\mathbb{I})$ and $F \in \mathrm{dom}(\widetilde{\partial}_{\cdot})$ calculation (37) is, obviously, valid, $\widetilde{\partial}_{\cdot} F$ has form (35). So, the domain of operator (40) can be described as follows:

$$\begin{aligned}
\mathrm{dom}(\widetilde{\partial}_{\cdot}) = \{ F \in (\mathcal{H}_{-\tau})_{-q} : \|\widetilde{\partial}_{\cdot} F\|_{(\mathcal{H}_{-\tau})_{-q}\otimes\mathcal{H}_{-\tau}}^2 &= \sum_{m=0}^{\infty} 2^{-qm}(m+1)^2 |F_{ext}^{(m+1)}(\cdot)|_{\mathcal{H}_{-\tau}^{(m)}\otimes\mathcal{H}_{-\tau}}^2 \\
&= \sum_{m=0}^{\infty} 2^{-qm}(m+1)^2 |F_{ext}^{(m+1)}|_{\mathcal{H}_{-\tau}^{(m+1)}}^2 < \infty \}
\end{aligned} \tag{41}$$

(see (36)).

Proposition. *Generalized stochastic integral* (38) *and generalized Hida derivative* (40) *are mutually adjoint and, in particular, closed operators.*

Proof. Since set (41) is dense in $(\mathcal{H}_{-\tau})_{-q}$, the operator $\widetilde{\partial}^*_\cdot = \mathbb{I}^{**} : (\mathcal{H}_\tau)_q \otimes \mathcal{H}_\tau \to (\mathcal{H}_\tau)_q$ is well defined as a linear unbounded operator with the domain that consists of $f \in (\mathcal{H}_\tau)_q \otimes \mathcal{H}_\tau$ such that $(\mathcal{H}_{-\tau})_{-q} \supset \mathrm{dom}(\widetilde{\partial}_\cdot) \ni F \mapsto \langle\langle \widetilde{\partial}_\cdot F, f\rangle\rangle_{(L^2)\otimes\mathcal{H}}$ is a linear continuous functional. By properties of Hilbert equipments the last is possible if and only if there exists $h \in (\mathcal{H}_\tau)_q$ such that $\langle\langle \widetilde{\partial}_\cdot F, f\rangle\rangle_{(L^2)\otimes\mathcal{H}} = \langle\langle F, h\rangle\rangle_{(L^2)}$. But by (40) $h = \mathbb{I}(f)$ and therefore the domain of $\widetilde{\partial}^*_\cdot$ can be described by the condition $\mathbb{I}(f) \in (\mathcal{H}_\tau)_q$. Compareing this condition with (39) one can conclude that $\mathrm{dom}(\widetilde{\partial}^*_\cdot) = \mathrm{dom}(\mathbb{I})$, therefore $\widetilde{\partial}^*_\cdot = \mathbb{I}^{**} = \mathbb{I}$. The equality $\mathbb{I}^* = \widetilde{\partial}_\cdot$ is a definition of $\widetilde{\partial}_\cdot$. □

2 OPERATORS OF STOCHASTIC DIFFERENTIATION

2.1 The case of bounded operators

As we said above, just as the Hida stochastic derivative, the operators of stochastic differentiation on (L^2) [8, 9] cannot be naturally continued to the spaces of nonregular generalized functions (because the kernels from decompositions (14) for elements of $(\mathcal{H}_{-\tau})_{-q}$ belong to too wide spaces). Nevertheless, one can introduce on these spaces natural analogs of the above-mentioned operators. These analogs have properties similar to the properties of operators of stochastic differentiation, and can be accepted as operators of stochastic differentiation on the spaces of nonregular generalized functions. In order to give an exact definition of the just now mentioned operators, we need a preparation.

Let $F_{ext}^{(m)} \in \mathcal{H}_{-\tau}^{(m)}$, $f^{(n)} \in \mathcal{H}_\tau^{\widehat{\otimes} n}$, $n, m \in \mathbb{N}$, $m > n$. We define a generalized partial pairing $\langle F_{ext}^{(m)}, f^{(n)}\rangle_{ext} \in \mathcal{H}_{-\tau}^{(m-n)}$ by setting for any $g^{(m-n)} \in \mathcal{H}_\tau^{\widehat{\otimes} m-n}$

$$\langle\langle F_{ext}^{(m)}, f^{(n)}\rangle_{ext}, g^{(m-n)}\rangle_{ext} = \langle F_{ext}^{(m)}, f^{(n)}\widehat{\otimes} g^{(m-n)}\rangle_{ext}. \tag{42}$$

Since by the generalized Cauchy-Bunyakovsky inequality

$$|\langle F_{ext}^{(m)}, f^{(n)}\widehat{\otimes} g^{(m-n)}\rangle_{ext}| \le |F_{ext}^{(m)}|_{\mathcal{H}_{-\tau}^{(m)}} |f^{(n)}\widehat{\otimes} g^{(m-n)}|_\tau \le |F_{ext}^{(m)}|_{\mathcal{H}_{-\tau}^{(m)}} |f^{(n)}|_\tau |g^{(m-n)}|_\tau,$$

this definition is well posed and

$$|\langle F_{ext}^{(m)}, f^{(n)}\rangle_{ext}|_{\mathcal{H}_{-\tau}^{(m-n)}} \le |F_{ext}^{(m)}|_{\mathcal{H}_{-\tau}^{(m)}} |f^{(n)}|_\tau. \tag{43}$$

Definition. *Let $n \in \mathbb{N}$, $f^{(n)} \in \mathcal{H}_\tau^{\widehat{\otimes} n}$. We define (the analog of) the operator of stochastic differentiation*

$$(\widetilde{D}^n \circ)(f^{(n)}) : (\mathcal{H}_{-\tau})_{-q} \to (\mathcal{H}_{-\tau})_{-q-1}$$

as a linear continuous operator that is given by the formula

$$\begin{aligned}(\widetilde{D}^n F)(f^{(n)}) :&= \sum_{m=n}^{\infty} \frac{m!}{(m-n)!} :\langle \circ^{\otimes m-n}, \langle F_{ext}^{(m)}, f^{(n)}\rangle_{ext}\rangle: \\ &\equiv \sum_{m=0}^{\infty} \frac{(m+n)!}{m!} :\langle \circ^{\otimes m}, \langle F_{ext}^{(m+n)}, f^{(n)}\rangle_{ext}\rangle:,\end{aligned} \tag{44}$$

where $F_{ext}^{(m)} \in \mathcal{H}_{-\tau}^{(m)}$ are the kernels from decomposition (14) *for $F \in (\mathcal{H}_{-\tau})_{-q}$.*

Since (see (15), (44) and (43))

$$\|(\widetilde{D}^n F)(f^{(n)})\|^2_{-\tau,-q-1} = \sum_{m=0}^{\infty} 2^{(-q-1)m} \frac{((m+n)!)^2}{(m!)^2} |\langle F_{ext}^{(m+n)}, f^{(n)}\rangle_{ext}|^2_{\mathcal{H}_{-\tau}^{(m)}}$$
$$\leq |f^{(n)}|^2_\tau 2^{qn} \sum_{m=0}^{\infty} 2^{-q(m+n)} |F_{ext}^{(m+n)}|^2_{\mathcal{H}_{-\tau}^{(m+n)}} \Big[2^{-m}\frac{((m+n)!)^2}{(m!)^2}\Big] \leq |f^{(n)}|^2_\tau 2^{qn} C(n) \|F\|^2_{-\tau,-q},$$

where $C(n) := \max_{m\in\mathbb{Z}_+}[2^{-m}\frac{((m+n)!)^2}{(m!)^2}] \leq \max_{m\in\mathbb{Z}_+}[2^{-m}(m+n)^{2n}] < \infty$, this definition is well posed. It is clear that the operator $(\widetilde{D}^n\circ)(f^{(n)})$ can be naturally continued to a linear continuous operator on the space $(\mathcal{H}_{-\tau})$ (or $(\mathcal{D}')$).

Let us consider main properties of the operator $\widetilde{D}^n$.

Theorem. *1) For $k_1,\dots,k_m \in \mathbb{N}$, $f_j^{(k_j)} \in \mathcal{H}_\tau^{\widehat{\otimes} k_j}$, $j \in \{1,\dots,m\}$,*

$$(\widetilde{D}^{k_m}(\cdots(\widetilde{D}^{k_2}((\widetilde{D}^{k_1}\circ)(f_1^{(k_1)})))(f_2^{(k_2)})\cdots))(f_m^{(k_m)}) = (\widetilde{D}^{k_1+\cdots+k_m}\circ)(f_1^{(k_1)}\widehat{\otimes}\cdots\widehat{\otimes} f_m^{(k_m)}).$$

2) For each $F \in (\mathcal{H}_{-\tau})_{-q}$ the kernels $F_{ext}^{(n)} \in \mathcal{H}_{-\tau}^{(n)}$, $n \in \mathbb{N}$, from decomposition (14) *can be presented in the form*

$$F_{ext}^{(n)} = \frac{1}{n!}\mathbb{E}(\widetilde{D}^n F),$$

i.e., for each $f^{(n)} \in \mathcal{H}_\tau^{\widehat{\otimes} n}$ $\langle F_{ext}^{(n)}, f^{(n)}\rangle_{ext} = \frac{1}{n!}\mathbb{E}((\widetilde{D}^n F)(f^{(n)}))$, here $\mathbb{E}\circ := \langle\!\langle \circ, 1\rangle\!\rangle_{(L^2)}$ is a generalized expectation.

3) The adjoint to $\widetilde{D}^n$ operator has the form

$$(\widetilde{D}^n g)(f^{(n)})^* = \sum_{m=0}^{\infty} :\langle \circ^{m+n}, f^{(n)}\widehat{\otimes} g^{(m)}\rangle: \in (\mathcal{H}_\tau)_q, \tag{45}$$

where $g \in (\mathcal{H}_\tau)_{q+1}$, $f^{(n)} \in \mathcal{H}_\tau^{\widehat{\otimes} n}$, and $g^{(m)} \in \mathcal{H}_\tau^{\widehat{\otimes} m}$ are the kernels from decomposition (10) *for g.*

Proof. 1) The proof consists in the application of the mathematical induction method.

2) Using (44) and (16) we obtain

$$\mathbb{E}((\widetilde{D}^n F)(f^{(n)})) = \langle\!\langle (\widetilde{D}^n F)(f^{(n)}), 1\rangle\!\rangle_{(L^2)} = n!\langle F_{ext}^{(n)}, f^{(n)}\rangle_{ext}.$$

3) Since (see (11), (10))

$$\|\sum_{m=0}^{\infty} :\langle \circ^{m+n}, f^{(n)}\widehat{\otimes} g^{(m)}\rangle: \|^2_{\tau,q} = \sum_{m=0}^{\infty} ((m+n)!)^2 2^{q(m+n)} |f^{(n)}\widehat{\otimes} g^{(m)}|^2_\tau$$
$$\leq |f^{(n)}|^2_\tau 2^{qn} \sum_{m=0}^{\infty} (m!)^2 2^{(q+1)m} |g^{(m)}|^2_\tau \Big[2^{-m}\frac{((m+n)!)^2}{(m!)^2}\Big] \leq |f^{(n)}|^2_\tau 2^{qn} C(n)\|g\|^2_{\tau,q+1} < \infty$$

(here $C(n) = \max_{m\in\mathbb{Z}_+}[2^{-m}\frac{((m+n)!)^2}{(m!)^2}]$ as above), the right hand side of (45) is well defined as an element of $(\mathcal{H}_\tau)_q$. Further, using (44), (10), (16) and (42), for $F \in (\mathcal{H}_{-\tau})_{-q}$ of form (14) we obtain

$$\begin{aligned}
&\langle\langle F,(\widetilde{D}^n g)(f^{(n)})^*\rangle\rangle_{(L^2)} = \langle\langle(\widetilde{D}^n F)(f^{(n)}),g\rangle\rangle_{(L^2)}\\
&= \langle\langle\sum_{m=0}^{\infty}\frac{(m+n)!}{m!}:\langle\circ^{\otimes m},\langle F_{ext}^{(m+n)},f^{(n)}\rangle_{ext}\rangle:,\sum_{k=0}^{\infty}:\langle\circ^{\otimes k},g^{(k)}\rangle:\rangle\rangle_{(L^2)}\\
&= \sum_{m=0}^{\infty}(m+n)!\langle\langle F_{ext}^{(m+n)},f^{(n)}\rangle_{ext},g^{(m)}\rangle_{ext} = \sum_{m=0}^{\infty}(m+n)!\langle F_{ext}^{(m+n)},f^{(n)}\widehat{\otimes}g^{(m)}\rangle_{ext}\\
&= \langle\langle\sum_{k=0}^{\infty}:\langle\circ^{\otimes k},F_{ext}^{(k)}\rangle:,\sum_{m=0}^{\infty}:\langle\circ^{\otimes m+n},f^{(n)}\widehat{\otimes}g^{(m)}\rangle:\rangle\rangle_{(L^2)}\\
&= \langle\langle F,\sum_{m=0}^{\infty}:\langle\circ^{\otimes m+n},f^{(n)}\widehat{\otimes}g^{(m)}\rangle:\rangle\rangle_{(L^2)},
\end{aligned} \tag{46}$$

whence the result follows. □

Now we consider in more detail the case $n = 1$. Denote $\widetilde{D} := \widetilde{D}^1$.

Theorem. *1) For all $g \in (\mathcal{H}_\tau)_{q+1}$ and $f^{(1)} \in \mathcal{H}_\tau$*

$$(\widetilde{D}g)(f^{(1)})^* = \mathbb{I}(g\otimes f^{(1)}) \in (\mathcal{H}_\tau)_q. \tag{47}$$

2) For all $F \in (\mathcal{H}_{-\tau})_{-q}$ and $f^{(1)} \in \mathcal{H}_\tau$

$$(\widetilde{D}F)(f^{(1)}) = \langle\widetilde{\partial}.F, f^{(1)}(\cdot)\rangle \in (\mathcal{H}_{-\tau})_{-q-1}, \tag{48}$$

where $\langle\widetilde{\partial}.F, f^{(1)}(\cdot)\rangle$ is a partial pairing, i.e., the unique element of $(\mathcal{H}_{-\tau})_{-q-1}$ such that for arbitrary $g \in (\mathcal{H}_\tau)_{q+1}$ $\langle\langle\langle\widetilde{\partial}.F, f^{(1)}(\cdot)\rangle, g\rangle\rangle_{(L^2)} = \langle\langle\widetilde{\partial}.F, g\otimes f^{(1)}(\cdot)\rangle\rangle_{(L^2)\otimes\mathcal{H}}$.

Remark. *Similarly to the proof of the fact that the generalized partial pairing $\langle\cdot,\cdot\rangle_{ext}$ is well posed and satisfies estimate (43), one can easily show that a partial pairing is well posed and satisfies a generalized Cauchy-Bunyakovsky inequality (in our case this inequality has the form $\|\langle\widetilde{\partial}.F, f^{(1)}(\cdot)\rangle\|_{-\tau,-q-1} \le \|\widetilde{\partial}.F\|_{(\mathcal{H}_{-\tau})_{-q-1}\otimes\mathcal{H}_{-\tau}}|f^{(1)}|_\tau$).*

Proof. 1) The result follows from representation (45) with $n = 1$ and the definition of an operator $\mathbb{I}$ (see (32)).

2) Taking into account (47) and (34), for all $g \in (\mathcal{H}_\tau)_{q+1}$ we obtain

$$\begin{aligned}
\langle\langle(\widetilde{D}F)(f^{(1)}),g\rangle\rangle_{(L^2)} &= \langle\langle F,(\widetilde{D}g)(f^{(1)})^*\rangle\rangle_{(L^2)} = \langle\langle F,\mathbb{I}(g\otimes f^{(1)})\rangle\rangle_{(L^2)}\\
&= \langle\langle\widetilde{\partial}.F, g\otimes f^{(1)}(\cdot)\rangle\rangle_{(L^2)\otimes\mathcal{H}} = \langle\langle\langle\widetilde{\partial}.F, f^{(1)}(\cdot)\rangle, g\rangle\rangle_{(L^2)},
\end{aligned}$$

whence the result follows. □

Remark. *Formally substituting in* (48) $f^{(1)} = \delta_t$, $t \in \mathbb{R}_+$ *(here δ_t is the Dirac delta-function concentrated at t; the substitution is formal because $\delta_t \notin \mathcal{H}_\tau$), we obtain a formal equality $\widetilde{\partial}_t\circ = (\widetilde{D}\circ)(\delta_t)$ (whence $\widetilde{\partial}.\circ = (\widetilde{D}\circ)(\delta.)$). In this connection we note that for the Hida stochastic derivative $\partial.$ and the operator of stochastic differentiation D on the spaces of nonregular test functions, for each $t \in \mathbb{R}_+$ $\partial_t\circ = (D\circ)(\delta_t)$ [24]; the formal analog of the last equality is valid on spaces that belong to the regular rigging of (L^2) [8].*

In some applications of the Gaussian analysis (in particular, in the Malliavin calculus) an important role belongs to the commutator between the extended stochastic integral and the operator of stochastic differentiation (see, e.g., [1]). Analogs of this commutator are calculated in the Meixner analysis [19, 20] and on the spaces of regular test and generalized functions of the Lévy analysis [8, 9]. Unfortunately, it is impossible to calculate a direct analog of the above-mentioned commutator on the spaces of nonregular test functions of the Lévy analysis: as we saw above, the extended stochastic integral cannot be naturally restricted to these spaces. Nevertheless, there exists an analog of this integral on the just now mentioned spaces — the generalized stochastic integral $\mathbb{I}$. So, now it is natural to calculate the commutator between $\mathbb{I}$ and the operator of stochastic differentiation, this commutator is calculated in [24]. On the spaces of nonregular generalized functions of the Lévy analysis the extended stochastic integral with respect to a Lévy process is well defined, and the role of the operator of stochastic differentiation belongs to the operator $\widetilde{D}$. So, it is natural to calculate the commutator between the above-mentioned integral and $\widetilde{D}$. In order to do this, let us introduce operators of stochastic differentiation on the spaces $(\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}_{-\tau}$ (this notion is intuitively clear and can be used without an additional explanation, but we prefer to give an exact definition).

As above, we begin with a preparation. Let $f^{(n)} \in \mathcal{H}_\tau^{\widehat{\otimes} n}$, $g_\cdot^{(m)} \in \mathcal{H}_\tau^{\widehat{\otimes} m} \otimes \mathcal{H}_\tau$. We define

$$f^{(n)} \overline{\otimes} g_\cdot^{(m)} := (Pr \otimes \mathbf{1})(f^{(n)} \otimes g_\cdot^{(m)}) \in \mathcal{H}_\tau^{\widehat{\otimes} n+m} \otimes \mathcal{H}_\tau, \tag{49}$$

where $Pr \otimes \mathbf{1}$ is the operator of symmetrization "by $n+m$ variables, except the variable $\cdot$" or, which is the same, the orthoprojector acting from $\mathcal{H}_\tau^{\widehat{\otimes} n} \otimes \mathcal{H}_\tau^{\widehat{\otimes} m} \otimes \mathcal{H}_\tau$ to $\mathcal{H}_\tau^{\widehat{\otimes} n+m} \otimes \mathcal{H}_\tau$ (of course, this operator depends on n and m, but we simplify the nonation). It is clear that

$$|f^{(n)} \overline{\otimes} g_\cdot^{(m)}|_{\mathcal{H}_\tau^{\widehat{\otimes} n+m} \otimes \mathcal{H}_\tau} \leq |f^{(n)}|_{\mathcal{H}_\tau^{\widehat{\otimes} n}} |g_\cdot^{(m)}|_{\mathcal{H}_\tau^{\widehat{\otimes} m} \otimes \mathcal{H}_\tau}, \tag{50}$$

and for $f^{(n)} \in \mathcal{H}_\tau^{\widehat{\otimes} n}$, $g^{(m)} \in \mathcal{H}_\tau^{\widehat{\otimes} m}$, $h^{(1)} \in \mathcal{H}_\tau$

$$f^{(n)} \overline{\otimes} (g^{(m)} \otimes h^{(1)}) = (f^{(n)} \widehat{\otimes} g^{(m)}) \otimes h^{(1)}. \tag{51}$$

Let $f^{(n)} \in \mathcal{H}_\tau^{\widehat{\otimes} n}$, $F_{ext,\cdot}^{(m)} \in \mathcal{H}_{-\tau}^{(m)} \otimes \mathcal{H}_{-\tau}$, $n, m \in \mathbb{N}$, $m \geq n$. We define a generalized partial pairing $\langle F_{ext,\cdot}^{(m)}, f^{(n)} \rangle_{EXT} \in \mathcal{H}_{-\tau}^{(m-n)} \otimes \mathcal{H}_{-\tau}$ by setting for arbitrary $g_\cdot^{(m-n)} \in \mathcal{H}_\tau^{\widehat{\otimes} m-n} \otimes \mathcal{H}_\tau$

$$\langle \langle F_{ext,\cdot}^{(m)}, f^{(n)} \rangle_{EXT}, g_\cdot^{(m-n)} \rangle_{\mathcal{H}_{ext}^{(m-n)} \otimes \mathcal{H}} = \langle F_{ext,\cdot}^{(m)}, f^{(n)} \overline{\otimes} g_\cdot^{(m-n)} \rangle_{\mathcal{H}_{ext}^{(m)} \otimes \mathcal{H}}. \tag{52}$$

Since by the generalized Cauchy-Bunyakovsky inequality and (50)

$$\begin{aligned} |\langle F_{ext,\cdot}^{(m)}, f^{(n)} \overline{\otimes} g_\cdot^{(m-n)} \rangle_{\mathcal{H}_{ext}^{(m)} \otimes \mathcal{H}}| &\leq |F_{ext,\cdot}^{(m)}|_{\mathcal{H}_{-\tau}^{(m)} \otimes \mathcal{H}_{-\tau}} |f^{(n)} \overline{\otimes} g_\cdot^{(m-n)}|_{\mathcal{H}_\tau^{\widehat{\otimes} m} \otimes \mathcal{H}_\tau} \\ &\leq |F_{ext,\cdot}^{(m)}|_{\mathcal{H}_{-\tau}^{(m)} \otimes \mathcal{H}_{-\tau}} |f^{(n)}|_{\mathcal{H}_\tau^{\widehat{\otimes} n}} |g_\cdot^{(m-n)}|_{\mathcal{H}_\tau^{\widehat{\otimes} m-n} \otimes \mathcal{H}_\tau}, \end{aligned}$$

this definition is well posed and

$$|\langle F_{ext,\cdot}^{(m)}, f^{(n)} \rangle_{EXT}|_{\mathcal{H}_{-\tau}^{(m-n)} \otimes \mathcal{H}_{-\tau}} \leq |F_{ext,\cdot}^{(m)}|_{\mathcal{H}_{-\tau}^{(m)} \otimes \mathcal{H}_{-\tau}} |f^{(n)}|_{\mathcal{H}_\tau^{\widehat{\otimes} n}}. \tag{53}$$

Remark. *Let $F_{ext}^{(m)} \in \mathcal{H}_{-\tau}^{(m)}$, $H^{(1)} \in \mathcal{H}_{-\tau}$; $g^{(m-n)} \in \mathcal{H}_{\tau}^{\widehat{\otimes} m-n}$, $h^{(1)} \in \mathcal{H}_{\tau}$. For $f^{(n)} \in \mathcal{H}_{\tau}^{\widehat{\otimes} n}$ by* (52), (51) *and* (42) *we obtain*

$$
\begin{aligned}
&\langle\langle F_{ext}^{(m)} \otimes H^{(1)}(\cdot), f^{(n)}\rangle_{EXT}, g^{(m-n)} \otimes h^{(1)}(\cdot)\rangle_{\mathcal{H}_{ext}^{(m-n)}\otimes\mathcal{H}} \\
&= \langle F_{ext}^{(m)} \otimes H^{(1)}(\cdot), f^{(n)}\overline{\otimes}(g^{(m-n)} \otimes h^{(1)}(\cdot))\rangle_{\mathcal{H}_{ext}^{(m)}\otimes\mathcal{H}} \\
&= \langle F_{ext}^{(m)} \otimes H^{(1)}(\cdot), (f^{(n)}\widehat{\otimes} g^{(m-n)}) \otimes h^{(1)}(\cdot)\rangle_{\mathcal{H}_{ext}^{(m)}\otimes\mathcal{H}} = \langle F_{ext}^{(m)}, f^{(n)}\widehat{\otimes} g^{(m-n)}\rangle_{\mathcal{H}_{ext}^{(m)}}\langle H^{(1)}, h^{(1)}\rangle_{\mathcal{H}} \\
&= \langle\langle F_{ext}^{(m)}, f^{(n)}\rangle_{ext}, g^{(m-n)}\rangle_{\mathcal{H}_{ext}^{(m-n)}}\langle H^{(1)}, h^{(1)}\rangle_{\mathcal{H}} \\
&= \langle\langle F_{ext}^{(m)}, f^{(n)}\rangle_{ext} \otimes H^{(1)}(\cdot), g^{(m-n)} \otimes h^{(1)}(\cdot)\rangle_{\mathcal{H}_{ext}^{(m-n)}\otimes\mathcal{H}}.
\end{aligned}
$$

Since the set $\{g^{(m-n)} \otimes h^{(1)} : g^{(m-n)} \in \mathcal{H}_{\tau}^{\widehat{\otimes} m-n},\ h^{(1)} \in \mathcal{H}_{\tau}\}$ is total in the space $\mathcal{H}_{\tau}^{\widehat{\otimes} m-n} \otimes \mathcal{H}_{\tau}$, we can conclude that

$$
\langle F_{ext}^{(m)} \otimes H^{(1)}, f^{(n)}\rangle_{EXT} = \langle F_{ext}^{(m)}, f^{(n)}\rangle_{ext} \otimes H^{(1)} \tag{54}
$$

in the space $\mathcal{H}_{-\tau}^{(m-n)} \otimes \mathcal{H}_{-\tau}$.

Definition. *Let $n \in \mathbb{N}$, $f^{(n)} \in \mathcal{H}_{\tau}^{\widehat{\otimes} n}$. We define a linear continuous operator*

$$
(\widetilde{\mathbf{D}}^n\circ)(f^{(n)}) : (\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}_{-\tau} \to (\mathcal{H}_{-\tau})_{-q-1} \otimes \mathcal{H}_{-\tau}
$$

by setting for $F \in (\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}_{-\tau}$

$$
\begin{aligned}
(\widetilde{\mathbf{D}}^n F(\cdot))(f^{(n)}) :&= \sum_{m=n}^{\infty} \frac{m!}{(m-n)!} :\langle \circ^{\otimes m-n}, \langle F_{ext,\cdot}^{(m)}, f^{(n)}\rangle_{EXT}\rangle : \\
&\equiv \sum_{m=0}^{\infty} \frac{(m+n)!}{m!} :\langle \circ^{\otimes m}, \langle F_{ext,\cdot}^{(m+n)}, f^{(n)}\rangle_{EXT}\rangle :,
\end{aligned} \tag{55}
$$

where $F_{ext,\cdot}^{(m)} \in \mathcal{H}_{-\tau}^{(m)} \otimes \mathcal{H}_{-\tau}$ are the kernels from decomposition (23) *for F.*

Since (see (24), (55) and (53))

$$
\begin{aligned}
\|(\widetilde{\mathbf{D}}^n F(\cdot))(f^{(n)})\|^2_{(\mathcal{H}_{-\tau})_{-q-1}\otimes\mathcal{H}_{-\tau}} &= \sum_{m=0}^{\infty} 2^{(-q-1)m}\frac{((m+n)!)^2}{(m!)^2}|\langle F_{ext,\cdot}^{(m+n)}, f^{(n)}\rangle_{EXT}|^2_{\mathcal{H}_{-\tau}^{(m)}\otimes\mathcal{H}_{-\tau}} \\
&\leq |f^{(n)}|_{\tau}^2 2^{qn} \sum_{m=0}^{\infty} 2^{-q(m+n)}|F_{ext,\cdot}^{(m+n)}|^2_{\mathcal{H}_{-\tau}^{(m+n)}\otimes\mathcal{H}_{-\tau}}\Big[2^{-m}\frac{((m+n)!)^2}{(m!)^2}\Big] \\
&\leq |f^{(n)}|_{\tau}^2 2^{qn} C(n)\|F\|^2_{(\mathcal{H}_{-\tau})_{-q}\otimes\mathcal{H}_{-\tau}},
\end{aligned}
$$

where, as above, $C(n) = \max_{m\in\mathbb{Z}_+}[2^{-m}\frac{((m+n)!)^2}{(m!)^2}]$, this definition is well posed.

Remark. *Let $F \in (\mathcal{H}_{-\tau})_{-q}$, $H^{(1)} \in \mathcal{H}_{-\tau}$. Using* (55), (54) *and* (44) *one can easily show that for each $n \in \mathbb{N}$ and $f^{(n)} \in \mathcal{H}_{\tau}^{\widehat{\otimes} n}$*

$$
(\widetilde{\mathbf{D}}^n F \otimes H^{(1)})(f^{(n)}) = (\widetilde{D}^n F)(f^{(n)}) \otimes H^{(1)} \in (\mathcal{H}_{-\tau})_{-q-1} \otimes \mathcal{H}_{-\tau}.
$$

Denote $\widetilde{\mathbf{D}} := \widetilde{\mathbf{D}}^1$.

Theorem. *For all $F \in (\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}_{-\tau}$ and $f^{(1)} \in \mathcal{H}_\tau$*

$$(\widetilde{D} \int F(u)\widehat{d}L_u)(f^{(1)}) = \int (\widetilde{\mathbf{D}}F(u))(f^{(1)})\widehat{d}L_u + \int F(u)f^{(1)}(u)du \in (\mathcal{H}_{-\tau})_{-q-1}, \qquad (56)$$

here $\int (\widetilde{\mathbf{D}}F(u))(f^{(1)})\widehat{d}L_u := \int J(u)\widehat{d}L_u$, where $J(\cdot) := (\widetilde{\mathbf{D}}F(\cdot))(f^{(1)}) \in (\mathcal{H}_{-\tau})_{-q-1} \otimes \mathcal{H}_{-\tau}$; $\int F(u)f^{(1)}(u)du$ is a generalized Pettis integral, i.e.,

$$\int F(u)f^{(1)}(u)du \equiv \langle F(\cdot), f^{(1)}(\cdot)\rangle \in (\mathcal{H}_{-\tau})_{-q} \subset (\mathcal{H}_{-\tau})_{-q-1}$$

($\langle F(\cdot), f^{(1)}(\cdot)\rangle$ is a partial pairing).

Proof. Using (27) and (44) we obtain

$$(\widetilde{D} \int F(u)\widehat{d}L_u)(f^{(1)}) = \sum_{m=0}^{\infty} (m+1) \colon \langle \circ^{\otimes m}, \langle \widehat{F}_{ext}^{(m)}, f^{(1)}\rangle_{ext}\rangle \colon,$$

where $\widehat{F}_{ext}^{(m)} \in \mathcal{H}_{-\tau}^{(m+1)}$ are the kernels from decomposition (27) (which is decomposition (14) for $\int F(u)\widehat{d}L_u$), i.e., $\widehat{F}_{ext}^{(m)}$ are given by formula (28) ($F_{ext,\cdot}^{(m)} \in \mathcal{H}_{-\tau}^{(m)} \otimes \mathcal{H}_{-\tau}$ in (28) are the kernels from decomposition (23) for F). On the other hand, by (55), (27) and (28)

$$\int (\widetilde{\mathbf{D}}F(u))(f^{(1)})\widehat{d}L_u = \sum_{m=0}^{\infty} m \colon \langle \circ^{\otimes m}, U_m^{-1}\{Pr[(U_{m-1} \otimes \mathbf{1})\langle F_{ext,\cdot}^{(m)}, f^{(1)}\rangle_{EXT}]\}\rangle \colon .$$

Let $g = \sum_{k=0}^{\infty} \colon \langle \circ^{\otimes k}, g^{(k)}\rangle \colon \in (\mathcal{H}_\tau)_{q+1}$, $g^{(k)} \in \mathcal{H}_\tau^{\widehat{\otimes} k}$ (see (10)). By (16) we have

$$\langle\langle (\widetilde{D} \int F(u)\widehat{d}L_u)(f^{(1)}), g\rangle\rangle_{(L^2)} = \sum_{m=0}^{\infty} m!(m+1)\langle\langle \widehat{F}_{ext}^{(m)}, f^{(1)}\rangle_{ext}, g^{(m)}\rangle_{\mathcal{H}_{ext}^{(m)}},$$

$$\langle\langle \int (\widetilde{\mathbf{D}}F(u))(f^{(1)})\widehat{d}L_u, g\rangle\rangle_{(L^2)} = \sum_{m=0}^{\infty} m!m\langle U_m^{-1}\{Pr[(U_{m-1} \otimes \mathbf{1})\langle F_{ext,\cdot}^{(m)}, f^{(1)}\rangle_{EXT}]\}, g^{(m)}\rangle_{\mathcal{H}_{ext}^{(m)}}.$$

Further, since for each m $g^{(m)}$ belongs to the *symmetric* tensor power of $\mathcal{H}_\tau$, by (26), (52) and (49)

$$\begin{aligned}
&m\langle U_m^{-1}\{Pr[(U_{m-1} \otimes \mathbf{1})\langle F_{ext,\cdot}^{(m)}, f^{(1)}\rangle_{EXT}]\}, g^{(m)}\rangle_{\mathcal{H}_{ext}^{(m)}} \\
&\quad = m\langle (U_{m-1} \otimes \mathbf{1})\langle F_{ext,\cdot}^{(m)}, f^{(1)}\rangle_{EXT}, g^{(m)}\rangle_{\mathcal{H}^{\otimes m}} \\
&\quad = m\langle (U_{m-1} \otimes \mathbf{1})\langle F_{ext,\cdot}^{(m)}, f^{(1)}\rangle_{EXT}, g^{(m)}(\cdot)\rangle_{\mathcal{H}^{\widehat{\otimes} m-1} \otimes \mathcal{H}} = m\langle\langle F_{ext,\cdot}^{(m)}, f^{(1)}\rangle_{EXT}, g^{(m)}(\cdot)\rangle_{\mathcal{H}_{ext}^{(m-1)} \otimes \mathcal{H}} \\
&\quad = m\langle F_{ext,\cdot}^{(m)}, f^{(1)} \overline{\otimes} g^{(m)}(\cdot)\rangle_{\mathcal{H}_{ext}^{(m)} \otimes \mathcal{H}} = \langle F_{ext,\cdot}^{(m)}(\cdot_1, \dots, \cdot_m), f^{(1)}(\cdot_1) \otimes g^{(m)}(\cdot_2, \dots, \cdot_m, \cdot) \\
&\quad + f^{(1)}(\cdot_2) \otimes g^{(m)}(\cdot_3, \dots, \cdot_m, \cdot_1, \cdot) + \dots + f^{(1)}(\cdot_m) \otimes g^{(m)}(\cdot_1, \dots, \cdot_{m-1}, \cdot)\rangle_{\mathcal{H}_{ext}^{(m)} \otimes \mathcal{H}};
\end{aligned}$$

and by (42), (28), (26), the symmetry of $f^{(1)} \widehat{\otimes} g^{(m)}$ and $g^{(m)}$, and the last calculation

$$\begin{aligned}
&(m+1)\langle\langle \widehat{F}_{ext}^{(m)}, f^{(1)}\rangle_{ext}, g^{(m)}\rangle_{\mathcal{H}_{ext}^{(m)}} = (m+1)\langle \widehat{F}_{ext}^{(m)}, f^{(1)} \widehat{\otimes} g^{(m)}\rangle_{\mathcal{H}_{ext}^{(m+1)}} \\
&= (m+1)\langle (U_m \otimes \mathbf{1})F_{ext,\cdot}^{(m)}, f^{(1)} \widehat{\otimes} g^{(m)}\rangle_{\mathcal{H}^{\otimes m+1}} = (m+1)\langle (U_m \otimes \mathbf{1})F_{ext,\cdot}^{(m)}, (f^{(1)} \widehat{\otimes} g^{(m)})(\cdot)\rangle_{\mathcal{H}^{\widehat{\otimes} m} \otimes \mathcal{H}} \\
&= (m+1)\langle F_{ext,\cdot}^{(m)}, (f^{(1)} \widehat{\otimes} g^{(m)})(\cdot)\rangle_{\mathcal{H}_{ext}^{(m)} \otimes \mathcal{H}} = \langle F_{ext,\cdot}^{(m)}(\cdot_1, \dots, \cdot_m), g^{(m)}(\cdot_1, \dots, \cdot_m) \otimes f^{(1)}(\cdot) \\
&+ f^{(1)}(\cdot_1) \otimes g^{(m)}(\cdot_2, \dots, \cdot_m, \cdot) + f^{(1)}(\cdot_2) \otimes g^{(m)}(\cdot_3, \dots, \cdot_m, \cdot_1, \cdot) \\
&+ \dots + f^{(1)}(\cdot_m) \otimes g^{(m)}(\cdot_1, \dots, \cdot_{m-1}, \cdot)\rangle_{\mathcal{H}_{ext}^{(m)} \otimes \mathcal{H}} = \langle F_{ext,\cdot}^{(m)}, g^{(m)} \otimes f^{(1)}(\cdot)\rangle_{\mathcal{H}_{ext}^{(m)} \otimes \mathcal{H}} \\
&+ m\langle U_m^{-1}\{Pr[(U_{m-1} \otimes \mathbf{1})\langle F_{ext,\cdot}^{(m)}, f^{(1)}\rangle_{EXT}]\}, g^{(m)}\rangle_{\mathcal{H}_{ext}^{(m)}}.
\end{aligned}$$

Later, by (23), the construction of a pairing in a tensor product of chains (e.g., [4]), (29) and the definition of a partial pairing

$$\sum_{m=0}^{\infty} m!\langle F^{(m)}_{ext,\cdot}, g^{(m)}\otimes f^{(1)}(\cdot)\rangle_{\mathcal{H}^{(m)}_{ext}\otimes\mathcal{H}} = \langle\!\langle F(\cdot), \sum_{m=0}^{\infty} :\langle\circ^{\otimes m}, g^{(m)}\otimes f^{(1)}(\cdot)\rangle:\rangle\!\rangle_{(L^2)\otimes\mathcal{H}} \\ = \langle\!\langle F(\cdot), g\otimes f^{(1)}(\cdot)\rangle\!\rangle_{(L^2)\otimes\mathcal{H}} = \langle\!\langle\langle F(\cdot), f^{(1)}(\cdot)\rangle_{\mathcal{H}}, g\rangle\!\rangle_{(L^2)}, \tag{57}$$

where $\langle F(\cdot), f^{(1)}(\cdot)\rangle_{\mathcal{H}} \equiv \langle F(\cdot), f^{(1)}(\cdot)\rangle \in (\mathcal{H}_{-\tau})_{-q} \subset (\mathcal{H}_{-\tau})_{-q-1}$ is a partial pairing.

So, for arbitrary $g \in (\mathcal{H}_\tau)_{q+1}$

$$\langle\!\langle(\widetilde{D}\int F(u)\widehat{d}L_u)(f^{(1)}), g\rangle\!\rangle_{(L^2)} = \langle\!\langle\int(\widetilde{\mathbf{D}}F(u))(f^{(1)})\widehat{d}L_u, g\rangle\!\rangle_{(L^2)} + \langle\!\langle\langle F(\cdot), f^{(1)}(\cdot)\rangle, g\rangle\!\rangle_{(L^2)},$$

from where (56) follows. □

Remark. *As follows from* (57)*, the definition of a partial pairing, and* (16)*, for* $g = \sum_{k=0}^{\infty} :\langle\circ^{\otimes k}, g^{(k)}\rangle: \in (\mathcal{H}_\tau)_q$

$$\langle\!\langle\langle F(\cdot), f^{(1)}(\cdot)\rangle_{\mathcal{H}}, g\rangle\!\rangle_{(L^2)} = \sum_{m=0}^{\infty} m!\langle\langle F^{(m)}_{ext,\cdot}, f^{(1)}(\cdot)\rangle_{\mathcal{H}}, g^{(m)}\rangle_{\mathcal{H}^{(m)}_{ext}} \\ = \langle\!\langle\sum_{m=0}^{\infty} :\langle\circ^{\otimes m}, \langle F^{(m)}_{ext,\cdot}, f^{(1)}(\cdot)\rangle_{\mathcal{H}}\rangle:, g\rangle\!\rangle_{(L^2)},$$

from where $\langle F(\cdot), f^{(1)}(\cdot)\rangle_{\mathcal{H}} = \sum_{m=0}^{\infty} :\langle\circ^{\otimes m}, \langle F^{(m)}_{ext,\cdot}, f^{(1)}(\cdot)\rangle_{\mathcal{H}}\rangle:$ *in* $(\mathcal{H}_{-\tau})_{-q}$.

Remark. *One can easily show that the restriction of an operator* $(\widetilde{\mathbf{D}}^n\circ)(f^{(n)})$*,* $n \in \mathbb{N}$*,* $f^{(n)} \in \mathcal{H}_\tau^{\widehat{\otimes}n}$*, to the space* $(\mathcal{H}_{-\tau})_{-q}\otimes\mathcal{H}$ *can be interpreted as a linear continuous operator acting from* $(\mathcal{H}_{-\tau})_{-q}\otimes\mathcal{H}$ *to* $(\mathcal{H}_{-\tau})_{-q-1}\otimes\mathcal{H}$*. Let us consider the extended stochastic integral* $\int_\Delta \circ(u)\widehat{d}L_u := \int\circ(u)1_\Delta(u)\widehat{d}L_u : (\mathcal{H}_{-\tau})_{-q}\otimes\mathcal{H} \to (\mathcal{H}_{-\tau})_{-q}$*,* $\Delta \in \mathcal{B}(\mathbb{R}_+)$ *— the Borel* σ*-algebra (this integral satisfies* (27) *with kernels* (28)*, see [21] for a detailed presentation). By analogy with the proof of the last theorem one can show that for all* $F \in (\mathcal{H}_{-\tau})_{-q}\otimes\mathcal{H}$ *and* $f^{(1)} \in \mathcal{H}_\tau$

$$(\widetilde{D}\int_\Delta F(u)\widehat{d}L_u)(f^{(1)}) = \int_\Delta(\widetilde{\mathbf{D}}F(u))(f^{(1)})\widehat{d}L_u + \int_\Delta F(u)f^{(1)}(u)du \in (\mathcal{H}_{-\tau})_{-q-1},$$

where $\int_\Delta(\widetilde{\mathbf{D}}F(u))(f^{(1)})\widehat{d}L_u := \int_\Delta J(u)\widehat{d}L_u$*,* $J(\cdot) := (\widetilde{\mathbf{D}}F(\cdot))(f^{(1)}) \in (\mathcal{H}_{-\tau})_{-q-1}\otimes\mathcal{H}$*;*

$$\int_\Delta F(u)f^{(1)}(u)du := \int F(u)f^{(1)}(u)1_\Delta(u)du \equiv \langle F(\cdot), f^{(1)}(\cdot)1_\Delta(\cdot)\rangle \in (\mathcal{H}_{-\tau})_{-q} \subset (\mathcal{H}_{-\tau})_{-q-1}$$

is a partial pairing.

As is easily seen, the results of this subsection hold true (up to obvious modifications) if we consider the operators of stochastic differentiation on the space $(\mathcal{H}_{-\tau})$ or $(\mathcal{D}')$.

Remark. *As is known [1], in the classical Gaussian white noise analysis the operator of stochastic differentiation is a differentiation with respect to a so-called Wick product. This result holds true in the so-called Gamma-analysis [17] and in a more general Meixner analysis. In forthcoming papers we'll obtain similar results on spaces of test and generalized functions of the Lévy white noise analysis.*

2.2 The case of unbounded operators

Similarly to the analysis on spaces of regular test and generalized functions [9, 8], sometimes it can be necessary to consider $(\widetilde{D}^n\circ)(f^{(n)})$, $f^{(n)} \in \mathcal{H}_\tau^{\widehat{\otimes} n}$, as a linear operator acting in $(\mathcal{H}_{-\tau})_{-q}$. Let us accept a corresponding definition.

Definition. *Let $n \in \mathbb{N}$, $f^{(n)} \in \mathcal{H}_\tau^{\widehat{\otimes} n}$. We define the operator of stochastic differentiation*

$$(\widetilde{D}^n\circ)(f^{(n)}) : (\mathcal{H}_{-\tau})_{-q} \to (\mathcal{H}_{-\tau})_{-q} \tag{58}$$

with the domain

$$\begin{aligned} &\mathrm{dom}((\widetilde{D}^n\circ)(f^{(n)})) := \{F \in (\mathcal{H}_{-\tau})_{-q} : \\ &\quad \|(\widetilde{D}^n F)(f^{(n)})\|^2_{-\tau,-q} = \sum_{m=0}^{\infty} 2^{-qm} \frac{((m+n)!)^2}{(m!)^2} |\langle F_{ext}^{(m+n)}, f^{(n)}\rangle_{ext}|^2_{\mathcal{H}_{-\tau}^{(m)}} < \infty\} \end{aligned} \tag{59}$$

(here $F_{ext}^{(m+n)} \in \mathcal{H}_{-\tau}^{(m+n)}$ are the kernels from decomposition (14) *for F) by formula* (44)*.*

Proposition. *Operator of stochastic differentiation* (58) *with domain* (59) *is closed.*

Proof. Let us show that there exists a second adjoint to $(\widetilde{D}^n\circ)(f^{(n)})$ operator $(\widetilde{D}^n\circ)(f^{(n)})^{**} = (\widetilde{D}^n\circ)(f^{(n)})$ (it is well known that an adjoint operator is closed). Since, obviously, the domain of operator (58) is a dense set in $(\mathcal{H}_{-\tau})_{-q}$, the adjoint operator $(\widetilde{D}^n\circ)(f^{(n)})^* : (\mathcal{H}_\tau)_q \to (\mathcal{H}_\tau)_q$ is well defined. By definition, $g \in \mathrm{dom}((\widetilde{D}^n\circ)(f^{(n)})^*)$ if and only if $(\mathcal{H}_{-\tau})_{-q} \supset \mathrm{dom}((\widetilde{D}^n\circ)(f^{(n)})) \ni F \mapsto \langle\langle(\widetilde{D}^n F)(f^{(n)}), g\rangle\rangle_{(L^2)}$ is a linear continuous functional. By properties of Hilbert equipments the last is possible if and only if there exists $h \in (\mathcal{H}_\tau)_q$ such that $\langle\langle(\widetilde{D}^n F)(f^{(n)}), g\rangle\rangle_{(L^2)} = \langle\langle F, h\rangle\rangle_{(L^2)}$. But by calculation (46) h has form (45), therefore

$$\begin{aligned} &\mathrm{dom}((\widetilde{D}^n\circ)(f^{(n)})^*) := \{g \in (\mathcal{H}_\tau)_q : \\ &\quad \|(\widetilde{D}^n F)(f^{(n)})^*\|^2_{\tau,q} = \sum_{m=0}^{\infty} ((m+n)!)^2 2^{q(m+n)} |f^{(n)} \widehat{\otimes} g^{(m)}|^2_\tau < \infty\} \end{aligned}$$

(see (11)), this set is dense in $(\mathcal{H}_\tau)_q$, hence the operator $(\widetilde{D}^n\circ)(f^{(n)})^{**} : (\mathcal{H}_{-\tau})_{-q} \to (\mathcal{H}_{-\tau})_{-q}$ is well defined. Now it remains to show that

$$\mathrm{dom}((\widetilde{D}^n\circ)(f^{(n)})^{**}) = \mathrm{dom}((\widetilde{D}^n\circ)(f^{(n)})). \tag{60}$$

By definition, $F \in \mathrm{dom}((\widetilde{D}^n\circ)(f^{(n)})^{**})$ if and only if $(\mathcal{H}_\tau)_q \supset \mathrm{dom}((\widetilde{D}^n\circ)(f^{(n)})^*) \ni g \mapsto \langle\langle F, (\widetilde{D}^n g)(f^{(n)})^*\rangle\rangle_{(L^2)}$ is a linear continuous functional. By properties of Hilbert equipments the last is possible if and only if there exists $H \in (\mathcal{H}_{-\tau})_{-q}$ such that $\langle\langle F, (\widetilde{D}^n g)(f^{(n)})^*\rangle\rangle_{(L^2)} = \langle\langle H, g\rangle\rangle_{(L^2)}$. It is clear that H has form (44), therefore equality (60) follows from (59). $\square$

Remark. *Let*

$$A_n := \{F \in (\mathcal{H}_{-\tau})_{-q} : \sum_{m=0}^{\infty} 2^{-qm} \frac{((m+n)!)^2}{(m!)^2} |F_{ext}^{(m+n)}|^2_{\mathcal{H}_{-\tau}^{(m+n)}} < \infty\},\ n \in \mathbb{N},$$

here $F_{ext}^{(m+n)} \in \mathcal{H}_{-\tau}^{(m+n)}$ are the kernels from decomposition (14) *for F. For each $f^{(n)} \in \mathcal{H}_\tau^{\widehat{\otimes} n}$ we define the operator of stochastic differentiation*

$$(\widehat{D}^n\circ)(f^{(n)}) : (\mathcal{H}_{-\tau})_{-q} \to (\mathcal{H}_{-\tau})_{-q} \tag{61}$$

with the domain A_n by formula (44) *with $\widehat{D}^n$ instead of $\widetilde{D}^n$. It follows from the just proved proposition that this operator is closable (its closure is operator* (58)*). Moreover, for each $F \in A_n$ the operator $(\widehat{D}^n F)(\circ) : \mathcal{H}_\tau^{\widehat{\otimes} n} \to (\mathcal{H}_{-\tau})_{-q}$ is linear bounded (and, therefore, continuous): by* (44)*,* (15) *and* (43) *for each $f^{(n)} \in \mathcal{H}_\tau^{\widehat{\otimes} n}$*

$$\begin{aligned}\|(\widehat{D}^n F)(f^{(n)})\|^2_{-\tau,-q} &= \sum_{m=0}^{\infty} 2^{-qm} \frac{((m+n)!)^2}{(m!)^2} |\langle F_{ext}^{(m+n)}, f^{(n)}\rangle_{ext}|^2_{\mathcal{H}_{-\tau}^{(m)}} \\ &\leq |f^{(n)}|^2_\tau \sum_{m=0}^{\infty} 2^{-qm} \frac{((m+n)!)^2}{(m!)^2} |F_{ext}^{(m+n)}|^2_{\mathcal{H}_{-\tau}^{(m+n)}}.\end{aligned}$$

It is clear that the results of Subsection 2.1 hold true (up to obvious modifications) for operators (58) and (61).

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Качановський М.О. *Оператори стохастичного диференціювання на просторах нерегулярних узагальнених функцій аналізу білого шуму Леві* // Карпатські матем. публ. — 2016. — Т.8, №1. — С. 83–106.

Оператори стохастичного диференціювання, які тісно пов'язані з розширеним стохастичним інтегралом Скорохода та зі стохастичною похідною Хіди, грають важливу роль у класичному (гауссівському) аналізі білого шуму. Зокрема, ці оператори можна використовувати для вивчення деяких властивостей розширеного стохастичного інтеграла та розв'язків стохастичних рівнянь з нелінійностями віківського типу.

Протягом останніх років оператори стохастичного диференціювання були уведені та вивчені, зокрема, у межах майкснерівського аналізу білого шуму, так само як і на просторах регулярних основних і узагальнених функцій та на просторах нерегулярних основних функцій аналізу білого шуму Леві. У цій статті ми робимо наступний природний крок: уводимо та вивчаємо оператори стохастичного диференціювання на просторах нерегулярних узагальнених функцій аналізу білого шуму Леві (тобто на просторах узагальнених функцій, які належать так званому нерегулярному оснащенню простору квадратично інтегровних за мірою білого шуму Леві функцій). При цьому використовується литвинівське узагальнення властивості хаотичного розкладу. Дослідження цієї статті можна розглядати як внесок у подальший розвиток аналізу білого шуму Леві.

Ключові слова і фрази: оператор стохастичного диференціювання, стохастична похідна, розширений стохастичний інтеграл, процес Леві.

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AN INVERSE PROBLEM FOR A 2D PARABOLIC EQUATION WITH NONLOCAL OVERDETERMINATION CONDITION

We consider an inverse problem of identifying the time-dependent coefficient $a(t)$ in a two-dimensional parabolic equation:

$$u_t = a(t)\Delta u + b_1(x,y,t)u_x + b_2(x,y,t)u_y + c(x,y,t)u + f(x,y,t),\ (x,y,t) \in Q_T,$$

with the initial condition, Neumann boundary data and the nonlocal overdetermination condition

$$\nu_1(t)u(0,y_0,t) + \nu_2(t)u(h,y_0,t) = \mu_3(t), \quad t \in [0,T],$$

where y_0 is a fixed number from $[0,l]$.

The conditions of existence and uniqueness of the classical solution to this problem are established. For this purpose the Green function method, Schauder fixed point theorem and the theory of Volterra intergral equations are utilized.

Key words and phrases: inverse problem, determining coefficients, parabolic equation, nonlocal overdetermination condition, rectangular domain.

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INTRODUCTION

This paper discusses the problem of identifying an unknown pair of functions $(a(t), u(x,y,t))$ for the equation

$$\begin{aligned} &u_t = a(t)\Delta u + b_1(x,y,t)u_x + b_2(x,y,t)u_y + c(x,y,t)u + f(x,y,t), \\ &(x,y,t) \in Q_T := \{(x,y,t) : 0 < x < h, 0 < y < l, 0 < t < T\} \end{aligned} \tag{1}$$

with the initial condition

$$u(x,y,0) = \varphi(x,y), \quad (x,y) \in [0,h] \times [0,l], \tag{2}$$

boundary conditions

$$u_x(0,y,t) = \mu_{11}(y,t), \quad u_x(h,y,t) = \mu_{12}(y,t), \quad (y,t) \in [0,l] \times [0,T], \tag{3}$$

$$u_y(x,0,t) = \mu_{21}(x,t), \quad u_y(x,l,t) = \mu_{22}(x,t), \quad (x,t) \in [0,h] \times [0,T]. \tag{4}$$

With the only above data this problem is underdetermined and we are forced to impose an additional condition to determine $a(t)$. In particular, we shall take a nonlocal overdetermination condition, that arises in practical applications [15]:

$$\nu_1(t)u(0,y_0,t) + \nu_2(t)u(h,y_0,t) = \mu_3(t), \quad t \in [0,T], \tag{5}$$

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where y_0 is a fixed number from $[0, l]$.

In the past few decades a great deal of interest has been directed towards the coefficient inverse problems. In 1993 Ivanchov M. considered nonlocal inverse problems of determining a leading time-dependent coefficient in a 1D heat equation [8, 9, 10]. For parabolic equations in one space variable, Bereznytska I. [1] considered the problem of determining conductivity $a(t)$ in a general parabolic equation subject to the Neumann boundary data and nonlocal overdetermination condition. Analogous problem with the Dirichlet boundary data was investigated in [12]. Later Huzyk N. investigated the problem of identifying time-dependent coefficients in a degenerate parabolic equation also subjected to Neumann boundary data and nonlocal overdetermination condition [5], [6]. All these papers are united by the approach utilized to proof the existence of solution: the inverse problem is reformulated as a fixed point problem for a certain nonlinear map, so that the Schauder theorem can be applied to it.

The other approaches to this problem addressing the question of existence and uniqueness are the Fourier method utilized by Ismailov M.I., Kanca F. [11], Oussaeif T.-E., Bouziani A. [16] and the theory of reproducing kernels used by Mohammadi M., Mokhtari R. and Isfahani F.T. [14].

The numerical results to nonlocal inverse problems have been obtained in works of Lesnic D. *et al* [13] with the help of Ritz-Galerkin method. A numerical marching scheme based on the discrete mollification for the recovery of the diffusivity coefficient in the two-dimensional inverse heat conduction problem has been presented by Coles C., Murio D.A. [2, 3].

Since the satisfactory results to the nonlocal coefficient inverse problems were successfully obtained in one-dimensional case, this paper represents an attempt to extend these results to multidimensional case, which is more interesting for its applications.

1 Notations and assumptions

Let $G_k(x,t,\xi,\tau)$ be the Green function of a 1D problem for the equation $u_t = a(t)u_{xx}$ with a Dirichlet boundary condition, when $k=1$, Neumann bondary condition, when $k=2$. These functions are defined by the equality

$$\begin{aligned} G_k(x,t,\xi,\tau) = \frac{1}{2\sqrt{\pi(\theta(t)-\theta(\tau))}} \sum_{n=-\infty}^{+\infty} \Bigg(\exp\left(-\frac{(x-\xi+2nh)^2}{4(\theta(t)-\theta(\tau))} \right) \\ +(-1)^k \exp\left(-\frac{(x+\xi+2nh)^2}{4(\theta(t)-\theta(\tau))} \right) \Bigg), \quad k=1,2, \quad \theta(t) = \int_0^t a(\tau)d\tau. \end{aligned} \tag{6}$$

At the same time we define the function $G_m(y,t,\eta,\tau)$ analogously to $G_k(x,t,\xi,\tau)$.

Now, let us introduce the 2D heat equation

$$u_t = a(t)\Delta u + f(x,y,t), \quad (x,y,t) \in Q_T. \tag{7}$$

Green functions for (7) are determined as follows

$$G_{km}(x,y,t,\xi,\eta,\tau) = G_k(x,t,\xi,\tau)G_m(y,t,\eta,\tau), \quad k,m=1,2. \tag{8}$$

The Green function of the problem (7), (2)-(4) is defined by (8), when $k=m=2$.

For $\alpha \in (0,1)$ we denote

$$C^{\alpha,0}(\overline{Q}_T) := \{f \in C(\overline{Q}_T) \,|\, |f(x_2,y_2,t) - f(x_1,y_1,t)| \leqslant C(|x_2-x_1|^{\alpha} + |y_2-y_1|^{\alpha}), \\ (x_i,y_i,t) \in \overline{Q}_T,\ i=1,2\}.$$

Throughout this paper, we assume that:

(A1) $f \in C^{\alpha,0}(\overline{Q}_T)$, $\quad b_1, b_2, c \in C^{1,0}(\overline{Q}_T)$, $\quad \varphi \in C^2([0,h]\times[0,l])$, $\quad \mu_3, \nu_1, \nu_2 \in C^1([0,T])$, $\mu_{11}, \mu_{12} \in C^{2,1}([0,l]\times[0,T])$, $\quad \mu_{21}, \mu_{22} \in C^{2,1}([0,h]\times[0,T])$;

(A2) $\mu_3'(t) - \nu_1(t)b_1(0,y_0,t)\mu_{11}(y_0,t) - \nu_2(t)b_1(h,y_0,t)\mu_{12}(y_0,t) - \nu_1(t)f(0,y_0,t) - \nu_2(t) \times f(h,y_0,t) > 0$, $\quad \nu_1'(t) + \nu_1(t)c(0,y_0,t) \leqslant 0$, $\quad \nu_2'(t) + \nu_2(t)c(h,y_0,t) \leqslant 0$, $\nu_k(t) \geqslant 0$, $k=1,2$, $\quad b_2(0,y_0,t) \leqslant 0$, $b_2(h,y_0,t) \leqslant 0$, $\quad t \in [0,T]$, $\quad \varphi(x,y) \geqslant 0$, $\varphi_y(x,y) \geqslant 0$, $(x,y) \in [0,h]\times[0,l]$, $\quad \mu_{21}(x,t) \geqslant 0$, $\mu_{22}(x,t) \geqslant 0$, $\quad (x,t) \in [0,h]\times[0,T]$;

(A3) $\nu_1(t) + \nu_2(t) > 0$, $\quad t \in [0,T]$, $\quad \Delta\varphi(x,y) > 0$, $(x,y) \in [0,h]\times[0,l]$;

(A4) $\varphi_x(0,y) = \mu_{11}(y,0)$, $\varphi_x(h,y) = \mu_{12}(y,0)$, $y \in [0,l]$, $\quad \varphi_y(x,0) = \mu_{21}(x,0)$, $\varphi_y(x,h) = \mu_{22}(x,0)$, $x \in [0,h]$, $\quad \nu_1(0)\varphi(0,y_0) + \nu_2(0)\varphi(h,y_0) = \mu_3(0)$.

2 EXISTENCE OF A SOLUTION

Theorem 1. *Provided that* **(A1)–(A4)** *hold, the problem* (1)–(5) *has at least one solution* $(a,u) \in C([0,t^*]) \times C^{2,1}(\overline{Q}_{t^*})$, $a(t) > 0, t \in [0,t^*]$, *where* $t^* \in (0,T]$ *is determined from the input data.*

Proof. To proof the existence of the solution to (1)-(5) we are first going to reduce it to an equivalent in a certain sense operator equation with respect to a and afterwards to proof the existence of the operator equation solution by the Schauder fixed point theorem.

In order to obtain an equation with respect to $a(t)$, (1) is applied to the overdetermination condition (5) previously differentiated:

$$a(t) = [\mu_3'(t) - \nu_1(t)b_1(0,y_0,t)\mu_{11}(y_0,t) - \nu_2(t)b_1(h,y_0,t)\mu_{12}(y_0,t) - \nu_1(t) \\ \times f(0,y_0,t) - \nu_2(t)f(h,y_0,t) - (\nu_1'(t) + \nu_1(t)c(0,y_0,t))u(0,y_0,t) - (\nu_2'(t) \\ + \nu_2(t)c(h,y_0,t))u(h,y_0,t) - \nu_1(t)b_2(0,y_0,t)u_y(0,y_0,t) - \nu_2(t)b_2(h,y_0,t) \\ \times u_y(h,y_0,t)][\nu_1(t)\Delta u(0,y_0,t) + \nu_2(t)\Delta u(h,y_0,t)]^{-1}, \quad t \in [0,T].$$

To continue the investigation of the equation (9), it is necessary to get some representation of the terms $u(0,y_0,t)$, $u(h,y_0,t)$, $u_y(0,y_0,t)$, $u_y(h,y_0,t)$, $\Delta u(0,y_0,t)$, $\Delta u(h,y_0,t)$.

The solution to the problem (7), (2)–(4) is denoted as $u_0(x,y,t)$ under the temporary assumption that $a \in C([0,T])$, $a(t) > 0$, $t \in [0,T]$ is a known function. Therefore, taking advantage of (8) we represent u_0 as the solution to (7), (2)–(4)

$$
\begin{aligned}
u_0(x,y,t)= &\int_0^l\int_0^h G_{22}(x,y,t,\xi,\eta,0)\varphi(\xi,\eta)d\xi d\eta - \int_0^t\int_0^h G_{22}(x,y,t,\xi,0,\tau)a(\tau) \\
&\times \mu_{21}(\xi,\tau)d\xi d\tau + \int_0^t\int_0^h G_{22}(x,y,t,\xi,l,\tau)a(\tau)\mu_{22}(\xi,\tau)d\xi d\tau \\
&- \int_0^t\int_0^l G_{22}(x,y,t,0,\eta,\tau)a(\tau)\mu_{11}(\eta,\tau)d\eta d\tau + \int_0^t\int_0^l G_{22}(x,y,t,h,\eta,\tau)a(\tau) \\
&\times \mu_{12}(\eta,\tau)d\eta d\tau + \int_0^t\int_0^l\int_0^h G_{22}(x,y,t,\xi,\eta,\tau)f(\xi,\eta,\tau)d\xi d\eta d\tau, \quad (x,y,t)\in\overline{Q}_T.
\end{aligned} \tag{9}
$$

Denote by

$$
\begin{aligned}
&v(x,y,t) := (b_1u_x + b_2u_y + cu)(x,y,t),\\
&w_1(x,y,t) := v_x(x,y,t) = (b_1u_{xx} + b_2u_{xy} + b_{2x}u_y + (b_{1x}+c)u_x + c_xu)(x,y,t),\\
&w_2(x,y,t) := v_y(x,y,t) = (b_1u_{xy} + b_2u_{yy} + (b_{2y}+c)u_y + b_{1y}u_x + c_yu)(x,y,t),\\
&(x,y,t)\in\overline{Q}_T.
\end{aligned} \tag{10}
$$

Problem (1)-(4) is reduced to the equation

$$
u(x,y,t) = u_0(x,y,t) + \int_0^t\int_0^l\int_0^h G_{22}(x,y,t,\xi,\eta,\tau)v(\xi,\eta,\tau)d\xi d\eta d\tau, \quad (x,y,t)\in\overline{Q}_T. \tag{11}
$$

Thus, from (11) we obtain

$$
\begin{aligned}
v(x,y,t) = &(b_1u_{0x} + b_2u_{0y} + cu_0)(x,y,t) + \int_0^t\int_0^l\int_0^h (b_1(x,y,t)G_{22x}(x,y,t,\xi,\eta,\tau) \\
&+ b_2(x,y,t)G_{22y}(x,y,t,\xi,\eta,\tau) + c(x,y,t)G_{22}(x,y,t,\xi,\eta,\tau))v(\xi,\eta,\tau)d\xi d\eta d\tau,\\
&(x,y,t)\in\overline{Q}_T.
\end{aligned} \tag{12}
$$

By differentiating (12) with respect to x, applying the Green function properties and integration by parts we obtain the equation

$$
\begin{aligned}
w_1(x,y,t) = &(b_1u_{0xx} + b_2u_{0xy} + b_{2x}u_{0y} + (b_{1x}+c)u_{0x} + c_xu_0)(x,y,t)\\
&+ \int_0^t\int_0^l\int_0^h (b_{1x}(x,y,t)G_{22x}(x,y,t,\xi,\eta,\tau) + b_{2x}(x,y,t)G_{22y}(x,y,t,\xi,\eta,\tau)\\
&+ c_x(x,y,t)G_{22}(x,y,t,\xi,\eta,\tau))v(\xi,\eta,\tau)d\xi d\eta d\tau + \int_0^t\int_0^l\int_0^h (b_1(x,y,t)\\
&\times G_{12x}(x,y,t,\xi,\eta,\tau) + b_2(x,y,t)G_{12y}(x,y,t,\xi,\eta,\tau) + c(x,y,t)\\
&\times G_{12}(x,y,t,\xi,\eta,\tau))w_1(\xi,\eta,\tau)d\xi d\eta d\tau, \quad (x,y,t)\in\overline{Q}_T.
\end{aligned} \tag{13}
$$

Analogously to (13), by differentiating (12) with respect to y, we obtain

$$\begin{aligned}
w_2(x,y,t) &= (b_1 u_{0xy} + b_2 u_{0yy} + (b_{2y}+c)u_{0y} + b_{1y}u_{0x} + c_y u_0)(x,y,t) \\
&\quad + \int_0^t \int_0^l \int_0^h (b_{1y}(x,y,t)G_{22x}(x,y,t,\xi,\eta,\tau) + b_{2y}(x,y,t)G_{22y}(x,y,t,\xi,\eta,\tau) \\
&\quad + c_y(x,y,t)G_{22}(x,y,t,\xi,\eta,\tau))v(\xi,\eta,\tau)d\xi d\eta d\tau + \int_0^t \int_0^l \int_0^h (b_1(x,y,t) \\
&\quad \times G_{21x}(x,y,t,\xi,\eta,\tau) + b_2(x,y,t)G_{21y}(x,y,t,\xi,\eta,\tau) + c(x,y,t) \\
&\quad \times G_{21}(x,y,t,\xi,\eta,\tau))w_2(\xi,\eta,\tau)d\xi d\eta d\tau, \quad (x,y,t) \in \overline{Q}_T.
\end{aligned} \tag{14}$$

We find from (11)

$$u_y(x,y,t) = u_{0y}(x,y,t) + \int_0^t \int_0^l \int_0^h G_{22y}(x,y,t,\xi,\eta,\tau)v(\xi,\eta,\tau)d\xi d\eta d\tau, \tag{15}$$

$$\begin{aligned}
\Delta u(x,y,t) &= \Delta u_0(x,y,t) + \int_0^t \int_0^l \int_0^h G_{12x}(x,y,t,\xi,\eta,\tau)w_1(\xi,\eta,\tau)d\xi d\eta d\tau \\
&\quad + \int_0^t \int_0^l \int_0^h G_{21y}(x,y,t,\xi,\eta,\tau)w_2(\xi,\eta,\tau)d\xi d\eta d\tau, \quad (x,y,t) \in \overline{Q}_T,
\end{aligned} \tag{16}$$

where $u_{0y}, \Delta u_0$ are calculated from (9):

$$\begin{aligned}
u_{0y}(x,y,t) &= \int_0^l \int_0^h G_{21}(x,y,t,\xi,\eta,0)\varphi_\eta(\xi,\eta)d\xi d\eta + \int_0^t \int_0^h G_{21\eta}(x,y,t,\xi,0,\tau)a(\tau) \\
&\quad \times \mu_{21}(\xi,\tau)d\xi d\tau - \int_0^t \int_0^h G_{21\eta}(x,y,t,\xi,l,\tau)a(\tau)\mu_{22}(\xi,\tau)d\xi d\tau \\
&\quad - \int_0^t \int_0^l G_{21}(x,y,t,0,\eta,\tau)a(\tau)\mu_{11\eta}(\eta,\tau)d\eta d\tau + \int_0^t \int_0^l G_{21}(x,y,t,h,\eta,\tau)a(\tau) \\
&\quad \times \mu_{12\eta}(\eta,\tau)d\eta d\tau + \int_0^t \int_0^l \int_0^h G_{22y}(x,y,t,\xi,\eta,\tau)f(\xi,\eta,\tau)d\xi d\eta d\tau,
\end{aligned} \tag{17}$$

$$\begin{aligned}\Delta u_0(x,y,t)=&\int_0^l\int_0^h G_{22}(x,y,t,\xi,\eta,0)\Delta\varphi(\xi,\eta)d\xi d\eta-\int_0^t\int_0^h G_{22}(x,y,t,\xi,0,\tau)\\&\times\mu_{21\tau}(\xi,\tau)d\xi d\tau+\int_0^t\int_0^h G_{22}(x,y,t,\xi,l,\tau)\mu_{22\tau}(\xi,\tau)d\xi d\tau\\&-\int_0^t\int_0^l G_{22}(x,y,t,0,\eta,\tau)\mu_{11\tau}(\eta,\tau)d\eta d\tau+\int_0^t\int_0^l G_{22}(x,y,t,h,\eta,\tau)\\&\times\mu_{12\tau}(\eta,\tau)d\eta d\tau+\int_0^t d\tau\int_0^l\int_0^h\Delta G_{22}(x,y,t,\xi,\eta,\tau)f(\xi,\eta,\tau)d\xi d\eta,\quad(x,y,t)\in\overline{Q}_T.\end{aligned}\tag{18}$$

By substituting (11), (16), (15) into (9) we obtain:

$$a(t)=\frac{Q_1(a,v)(t)}{Q_2(a,w_1,w_2)(t)},\tag{19}$$

where

$$\begin{aligned}Q_1(a,v)(t)=&\mu_3'(t)-\nu_1(t)b_1(0,y_0,t)\mu_{11}(y_0,t)-\nu_2(t)b_1(h,y_0,t)\mu_{12}(y_0,t)-\nu_1(t)\\&\times f(0,y_0,t)-\nu_2(t)f(h,y_0,t)-(\nu_1'(t)+\nu_1(t)c(0,y_0,t))u_0(0,y_0,t)-(\nu_2'(t)\\&+\nu_2(t)c(h,y_0,t))u_0(h,y_0,t)-\nu_1(t)b_2(0,y_0,t)u_{0y}(0,y_0,t)-\nu_2(t)b_2(h,y_0,t)\\&\times u_{0y}(h,y_0,t)+\int_0^t\int_0^l\int_0^h v(\xi,\eta,\tau)(-(\nu_1'(t)+\nu_1(t)c(0,y_0,t))G_{22}(0,y_0,t,\xi,\eta,\tau)\\&-(\nu_2'(t)+\nu_2(t)c(h,y_0,t))G_{22}(h,y_0,t,\xi,\eta,\tau)-\nu_1(t)b_2(0,y_0,t)\\&\times G_{22y}(0,y_0,t,\xi,\eta,\tau)-\nu_2(t)b_2(h,y_0,t)G_{22y}(h,y_0,t,\xi,\eta,\tau))d\xi d\eta d\tau,\end{aligned}\tag{20}$$

$$\begin{aligned}Q_2(a,w_1,w_2)(t)=&\nu_1(t)\Delta u_0(0,y_0,t)+\nu_2(t)\Delta u_0(h,y_0,t)\\&+\int_0^t\int_0^l\int_0^h(\nu_1(t)G_{12x}(0,y_0,t,\xi,\eta,\tau)+\nu_2(t)G_{12x}(h,y_0,t,\xi,\eta,\tau))w_1(\xi,\eta,\tau)d\xi d\eta d\tau\\&+\int_0^t\int_0^l\int_0^h(\nu_1(t)G_{21y}(0,y_0,t,\xi,\eta,\tau)+\nu_2(t)G_{21y}(h,y_0,t,\xi,\eta,\tau))w_2(\xi,\eta,\tau)d\xi d\eta d\tau,\end{aligned}\tag{21}$$

where v, w_1, w_2 are solutions to the system of integral equations (12)–(14).

Denote

- $\mathcal{N} := \{a \in C([0,t^*]) : A_0 \leqslant a(t) \leqslant A_1\}$, where the constants $A_0, A_1 \in \mathbb{R}_+$, $t^* \in (0,T]$ are to be established below;
- $\widehat{P} : \mathcal{N} \times \left(C(\overline{Q}_T)\right)^3 \to \mathcal{N}$, such that $\widehat{P}(a,v,w_1,w_2) = \dfrac{Q_1(a,v)}{Q_2(a,w_1,w_2)}$;
- $\widetilde{P} : \mathcal{N} \to \left(C(\overline{Q}_T)\right)^3$ an operator that maps each element $a \in \mathcal{N}$ into the solution of the system of integral equations (12)–(14).

Since the functions v, w_1, w_2 in (19) are now defined by $\widetilde{P}$, the equation (19) can be rewritten as the following operator equation:

$$a = Pa, \text{ where } Pa := \widehat{P}(a, \widetilde{P}(a)), \quad a \in \mathcal{N}. \tag{22}$$

The problem (1)–(5) is equivalent to the equation (22) in the following sense: if (a, u) is a solution to problem (1)–(5), then a is a solution of (22) and, on the other hand, if $a \in C([0, T])$ is a solution of (22), then (a, u) is a solution to the problem (1)–(5), where u is determined by the equations (11).

From the way the equation (22) has been obtained it follows, that if (a, u) is the solution to (1)–(5), then a satisfies (22).

Reciprocally, for any $a \in \mathcal{N}$ functions u, v are uniquely determined from (11), (12) and such a system of integral equations is equivalent to the direct problem (1)–(4). Thus, it is left to be shown that (5) follows from (22). By implementing all the substitutions in the reverse order we move from (22) to (9). After (9) is multiplied by its denominator and integrated with respect to time, regarding **(A4)**, the overdetermination condition (5) is obtained.

Consequently, the existence of solution to (1)—(5) is equivalent to the existence of solution to the operator equation (22).

In order to apply the Schauder fixed point theorem we show that P is compact and that it maps $\mathcal{N}$ into itself.

Since for each $a \in \mathcal{N}$ u_{0x}, u_{0y}, u_{0xx}, u_{0xy}, u_{0yy} are continuous functions according to **(A1)**, it follows from the properties of the systems of Volterra integral equations that $\widetilde{P}$ is a bounded operator. The compactness of the operator $\widehat{P}$ follows from [7]. Therefore P is compact as the composition of bounded operator $\widetilde{P}$ and compact operator $\widehat{P}$.

Thus, the next goal is to establish A_0, $A_1 \in \mathbb{R}_+$, such that $A_0 \leqslant (Pa)(t) \leqslant A_1$, $t \in [0, t^*]$, $a \in \mathcal{N}$.

From the explicit representation of u_0 and its derivative u_{0y} (9), (17), the Green function properties and **(A2)** it follows that

$$\begin{aligned}
&\lim_{t\to 0} u_0(x,y,t) = \varphi(x,y),\\
&\lim_{t\to 0} u_{0y}(x,y,t) = \lim_{t\to 0}\Big(\int_0^l G_1(y,t,\eta,0)\varphi_\eta(x,\eta)d\eta + \int_0^t G_{1\eta}(y,t,0,\tau)a(\tau)\mu_{21}(x,\tau)d\tau\\
&\qquad\qquad - \int_0^t G_{1\eta}(y,t,l,\tau)a(\tau)\mu_{22}(x,\tau)d\tau\Big).
\end{aligned}$$

Then for any $(x, y) \in [0, h] \times [0, l]$

$$\begin{aligned}
&0 \leqslant \min_{[0,h]\times[0,l]} \varphi(x,y) \leqslant \lim_{t\to 0} u_0(x,y,t) \leqslant \max_{[0,h]\times[0,l]} \varphi(x,y),\\
&0 \leqslant \min\{\min_{[0,h]\times[0,l]} \varphi_y(x,y), \min_{[0,h]\times[0,T]} \mu_{21}(x,t), \min_{[0,h]\times[0,T]} \mu_{22}(x,t)\} \leqslant \lim_{t\to 0} u_{0y}(x,y,t)\\
&\quad \leqslant \max\{\max_{[0,h]\times[0,l]} \varphi_y(x,y), \max_{[0,h]\times[0,T]} \mu_{21}(x,t), \max_{[0,h]\times[0,T]} \mu_{22}(x,t)\}.
\end{aligned}$$

The last term in (20) vanishes, when $t \to 0$, according to the properties of Newtonian potentials.

Therefore, thanks to **(A2)** there are such constants m_1, M_1 that

$$0 < m_1 \leqslant \lim_{t\to 0} Q_1(t) \leqslant M_1.$$

Namely,

$$\begin{aligned} m_1 := \min_{[0,T]} (\mu_3'(t) - \nu_1(t) b_1(0, y_0, t)\mu_{11}(y_0, t) - \nu_2(t) b_1(h, y_0, t)\mu_{12}(y_0, t) - \nu_1(t) \\ \times f(0, y_0, t) - \nu_2(t) f(h, y_0, t)), \end{aligned} \tag{23}$$

$$\begin{aligned} M_1 := \max_{[0,T]} (\mu_3'(t) - \nu_1(t) b_1(0, y_0, t)\mu_{11}(y_0, t) - \nu_2(t) b_1(h, y_0, t)\mu_{12}(y_0, t) - \nu_1(t) \\ \times f(0, y_0, t) - \nu_2(t) f(h, y_0, t)) + \max_{[0,T]}(-(\nu_1'(t) + \nu_1(t) c(0, y_0, t)) - (\nu_2'(t) + \nu_2(t) \\ \times c(h, y_0, t))) \max_{[0,h]\times[0,l]} \varphi(x, y) + \max_{[0,T]}(-\nu_1(t) b_2(0, y_0, t) - \nu_2(t) b_2(h, y_0, t)) \\ \times \max\{ \max_{[0,h]\times[0,l]} \varphi_y(x, y), \max_{[0,h]\times[0,T]} \mu_{21}(x, t), \max_{[0,h]\times[0,T]} \mu_{22}(x, t)\}. \end{aligned} \tag{24}$$

Thus from the definition of limit it derives that for $\varepsilon = \frac{1}{2} m_1$ there is such a value $t_1 \in (0, T]$, that

$$\frac{1}{2} m_1 \leqslant Q_1(t) \leqslant M_1 + \frac{1}{2} m_1, \quad t \in [0, t_1]. \tag{25}$$

Similarly, from the explicit representation (18) of Δu_0

$$\lim_{t\to 0} \Delta u_0(x, y, t) = \Delta\varphi(x, y).$$

Denote

$$m_2 := \min_{[0,T]} (\nu_1(t) + \nu_2(t)) \min_{[0,h]\times[0,l]} \Delta\varphi(x, y), \tag{26}$$

$$M_2 := \max_{[0,T]} (\nu_1(t) + \nu_2(t)) \max_{[0,h]\times[0,l]} \Delta\varphi(x, y). \tag{27}$$

Then $0 < m_2 \leqslant \lim_{t\to 0} Q_2(t) \leqslant M_2$. Analogously, there is such a value $t_2 \in (0, T]$, that

$$\frac{1}{2} m_2 \leqslant Q_2(t) \leqslant M_2 + \frac{1}{2} m_2, \quad t \in [0, t_2]. \tag{28}$$

Define

$$A_0 := \frac{\frac{1}{2} m_1}{M_2 + \frac{1}{2} m_2}, \quad A_1 := \frac{M_1 + \frac{1}{2} m_1}{\frac{1}{2} m_2}, \quad t^* := \min\{t_1, t_2\}.$$

and make sure that: if $a \in \mathcal{N}$, then $A_0 \leqslant (Pa)(t) \leqslant A_1$, $t \in [0, t^*]$.

From the Schauder fixed point theorem follows the existence of the solution to (22), and, hence, for the problem (1)–(5). $\square$

3 UNIQUENESS OF A SOLUTION

Theorem 2. *Under the condition* ***(A2)*** *the problem* (1)–(5) *cannot have more than one solution* (a, u) *in the space* $C([0, t_1]) \times C^{2,1}(\overline{Q}_{t_1})$*, such that* $\Delta u \in C^{\alpha,0}(\overline{Q}_{t_1})$ *and* $a(t) > 0, t \in [0, t_1]$*, where* $t_1 \in (0, T]$ *is determined from the input data.*

Proof. Suppose that there exist two solutions $(a_1(t), u_1(x, y, t))$ and $(a_2(t), u_2(x, y, t))$ of the problem (1)–(5). Denote

$$a_3(t) := a_1(t) - a_2(t), \quad t \in [0, T], \tag{29}$$

$$u_3(x, y, t) := u_1(x, y, t) - u_2(x, y, t), \quad (x, y, t) \in \overline{Q}_T. \tag{30}$$

Then $(a_3(t), u_3(x, y, t))$ is solution of the problem

$$u_{3t} = a_1(t)\Delta u_3 + b_1(x, y, t)u_{3x} + b_2(x, y, t)u_{3y} + c(x, y, t)u_3 + a_3(t)\Delta u_2, \ (x, y, t) \in Q_T, \tag{31}$$

$$u_3(x, y, 0) = 0, \quad (x, y) \in [0, h] \times [0, l], \tag{32}$$

$$u_{3x}(0, y, t) = 0, \quad u_{3x}(h, y, t) = 0, \quad (y, t) \in [0, l] \times [0, T], \tag{33}$$

$$u_{3y}(x, 0, t) = 0, \quad u_{3y}(x, l, t) = 0, \quad (x, t) \in [0, h] \times [0, T], \tag{34}$$

$$\nu_1(t)u_3(0, y_0, t) + \nu_2(t)u_3(h, y_0, t) = 0, \quad t \in [0, T]. \tag{35}$$

By calculating the derivative of (35) and applying (31) to it, we obtain for $t \in [0, T]$

$$\begin{aligned}(\nu_1(t)\Delta u_2(0, y_0, t) + \nu_2(t)\Delta u_2(h, y_0, t))a_3(t) = -(\nu_1'(t) + \nu_1(t)c(0, y_0, t))\\ \times u_3(0, y_0, t) - (\nu_2'(t) + \nu_2(t)c(h, y_0, t))u_3(h, y_0, t) - \nu_1(t)b_2(0, y_0, t)u_{3y}(0, y_0, t)\\ - \nu_2(t)b_2(h, y_0, t)u_{3y}(h, y_0, t) - \nu_1(t)a_1(t)\Delta u_3(0, y_0, t) - \nu_2(t)a_1(t)\Delta u_3(h, y_0, t).\end{aligned} \tag{36}$$

Denote by $\hat{G}_{22}(x, y, t, \xi, \eta, \tau)$ a Green function of the problem (31)–(34). Since $a_1(t)$ is a known function, the solution to the problem (31)–(34) is unique and can be calculated by the formula:

$$u_3(x, y, t) = \int_0^t \int_0^l \int_0^h \hat{G}_{22}(x, y, t, \xi, \eta, \tau)a_3(\tau)\Delta u_2(\xi, \eta, \tau)d\xi d\eta d\tau. \tag{37}$$

By differentiating (37) with respect to y and applying to (37) the Laplacian , we obtain

$$u_{3y}(x, y, t) = \int_0^t \int_0^l \int_0^h \hat{G}_{22y}(x, y, t, \xi, \eta, \tau)a_3(\tau)\Delta u_2(\xi, \eta, \tau)d\xi d\eta d\tau, \tag{38}$$

$$\Delta u_3(x, y, t) = \int_0^t d\tau \int_0^l \int_0^h \Delta\hat{G}_{22}(x, y, t, \xi, \eta, \tau)a_3(\tau)\Delta u_2(\xi, \eta, \tau)d\xi d\eta. \tag{39}$$

Therefore, by applying (37)–(39) to (36), we obtain an equation with respect to $a_3(t)$

$$\begin{aligned}a_3(t) = \frac{-1}{\nu_1(t)\Delta u_2(0, y_0, t) + \nu_2(t)\Delta u_2(h, y_0, t)} \int_0^t d\tau \int_0^l \int_0^h \Big((\nu_1'(t) + \nu_1(t)c(0, y_0, t))\\ \times \hat{G}_{22}(0, y_0, t, \xi, \eta, \tau) + (\nu_2'(t) + \nu_2(t)c(h, y_0, t))\hat{G}_{22}(h, y_0, t, \xi, \eta, \tau)\\ + \nu_1(t)b_2(0, y_0, t)\hat{G}_{22y}(0, y_0, t, \xi, \eta, \tau) + \nu_2(t)b_2(h, y_0, t)\hat{G}_{22y}(h, y_0, t, \xi, \eta, \tau)\\ + \nu_1(t)a_1(t)\Delta\hat{G}_{22}(0, y_0, t, \xi, \eta, \tau) + \nu_2(t)a_1(t)\Delta\hat{G}_{22}(h, y_0, t, \xi, \eta, \tau)\Big)\\ \times a_3(\tau)\Delta u_2(\xi, \eta, \tau)d\xi d\eta.\end{aligned} \tag{40}$$

It is still necessary to ensure that for

$$\nu_1(t)\Delta u_2(0,y_0,t)+\nu_2(t)\Delta u_2(h,y_0,t)>0. \tag{41}$$

Since (a_2,u_2) is a solution of (1)–(5) it follows from (9) that $t\in[0,T]$

$$\begin{aligned}\nu_1(t)\Delta u_2(0,y_0,t)+\nu_2(t)\Delta u_2(h,y_0,t)&=\frac{1}{a_2(t)}(\mu_3'(t)-\nu_1(t)f(0,y_0,t)-\nu_2(t)\\ &\times f(h,y_0,t)-(\nu_1'(t)+\nu_1(t)c(0,y_0,t))u_2(0,y_0,t)-(\nu_2'(t)+\nu_2(t)c(h,y_0,t))\\ &\times u_2(h,y_0,t)-\nu_1(t)b_2(0,y_0,t)u_{2y}(0,y_0,t)-\nu_2(t)b_2(h,y_0,t)u_{2y}(h,y_0,t)).\end{aligned} \tag{42}$$

Thus, it follows from (42), (20) and (25), ensured by **(A2)**, that

$$\nu_1(t)\Delta u_2(0,y_0,t)+\nu_2(t)\Delta u_2(h,y_0,t)\geqslant\frac{m_1}{2a_2(t)}>0,\quad t\in[0,t_1]. \tag{43}$$

Hence, (40) is a homogeneous Volterra integral equation of the second kind on $[0,t_1]$. Since $\Delta u_2\in C^{\alpha,0}(\overline{Q}_{t_1})$, according to [4] the kernel of (40) is integrable. Therefore, (40) has a unique solution $a_3(t)=0$, $t\in[0,t_1]$, and from the equality (37) it follows that $u_3(x,y,t)=0$, $(x,y,t)\in\overline{Q}_{t_1}$. The proof of the theorem is complete. □

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Кінаш Н.Є. *Обернена задача для двовимірного параболічного рівняння із нелокальними умовами перевизначення* // Карпатські матем. публ. — 2016. — Т.8, №1. — С. 107–117.

Розглядаємо обернену задачу визначення залежного від часу коефіцієнта $a(t)$ у двовимірному параболічному рівнянні:

$$u_t = a(t)\Delta u + b_1(x,y,t)u_x + b_2(x,y,t)u_y + c(x,y,t)u + f(x,y,t),\ (x,y,t) \in Q_T,$$

із початковою умовою, крайовими умовами Неймана та нелокальною умовою перевизначення

$$\nu_1(t)u(0,y_0,t) + \nu_2(t)u(h,y_0,t) = \mu_3(t), \quad t \in [0,T],$$

де y_0 фіксоване значення із $[0,l]$.

Встановлено умови існування та єдиності класичного розв'язку задачі. З цією метою застосовано метод функції Гріна, теорему Шаудера про нерухому точку та теорію інтегральних рівнянь Вольтерра.

Ключові слова і фрази: обернена задача, визначення коефіцієнтів, параболічне рівняння, нелокальна умова перевизначення, прямокутна область.

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LOPUSHANSKA H., RAPITA V.

INVERSE CAUCHY PROBLEM FOR FRACTIONAL TELEGRAPH EQUATION WITH DISTRIBUTIONS

The inverse Cauchy problem for the fractional telegraph equation

$$u_t^{(\alpha)} - r(t)u_t^{(\beta)} + a^2(-\Delta)^{\gamma/2}u = F_0(x)g(t), \quad (x,t) \in \mathrm{R}^n \times (0,T],$$

with given distributions in the right-hand sides of the equation and initial conditions is studied. Our task is to determinate a pair of functions: a generalized solution u (continuous in time variable in general sense) and unknown continuous minor coefficient $r(t)$. The unique solvability of the problem is established.

Key words and phrases: generalized function, fractional derivative, inverse problem, Green vector-function.

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INTRODUCTION

The existence and uniqueness theorems were proved, and the representation (in terms of the Green function) of classical solution of a time- and a time-space-fractional Cauchy problem was obtained, for example, in [1, 3–5, 14]. The unique solvability of a time-space-fractional Cauchy problem in spaces of distributions was proved in [8, 10].

Inverse problems for such equations arise in many branches of science and engineering. The inverse boundary value problems for determination of a leading coefficient, or a part of the right-hand side, or an order of a diffusion-wave equation, or an unknown boundary condition, were studied, for example, in [2, 6, 11, 12, 15].

In the present paper we prove the existence and uniqueness of a solution (u, r) of the inverse Cauchy problem

$$u_t^{(\alpha)} - r(t)u_t^{(\beta)} + a^2(-\Delta)^{\gamma/2}u = F_0(x)g(t), \quad (x,t) \in \mathrm{R}^n \times (0,T], \tag{1}$$

$$u(x,0) = F_1(x), \quad u_t(x,0) = F_2(x), \ x \in \mathrm{R}^n, \tag{2}$$

$$\big(u(\cdot,t), \varphi_0(\cdot)\big) = F(t), \quad t \in (0,T] \tag{3}$$

with the Riemann-Liouville fractional derivatives $u_t^{(\alpha)}, u_t^{(\beta)}$, where F_0, F_1, F_2 are given distributions, F, g, φ_0 are given smooth functions, the symbol (f, φ) stands for the value of the distribution f on the test function φ, a^2 is a positive constant, $(-\Delta)^{\gamma/2}u$ is defined with the use of the Fourier transform as follows

$$F[(-\Delta)^{\gamma/2}u] = |\lambda|^\gamma F[u],$$

and the following assumption holds:

(L) $\qquad \alpha \in (1,2), \beta \in (0,1), \gamma > \alpha, \quad \min\{n, 2, \gamma\} > (n-1)/2.$

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1 NOTATIONS AND AUXILIARY RESULTS

Denote the set of natural numbers by symbol N. Let $Z_+ := N \cup \{0\}$, $Q := R^n \times (0, T]$, $n \in N$. Let $\mathcal{E}(R^n) := C^\infty(R^n)$ and $\mathcal{D}(R^n)$ be the space of infinitely differentiable functions compactly supported in R^n. $\mathcal{D}(\bar{Q})$ is the space of infinitely differentiable functions having compact supports with respect to space variables and such that $(\frac{\partial}{\partial t})^k v|_{t=T} = 0$, $k \in Z_+$, $\mathcal{D}^k(R^n)$ is the space of functions from $C^k(R^n)$ having compact supports, $\|\varphi\|_{D^k(R^n)} := \max\limits_{|\kappa| \le k} \max\limits_{x \in supp \varphi} |D^\kappa \varphi(x)|$, where $\kappa = (\kappa_1, \ldots, \kappa_n)$, $\kappa_j \in Z_+$, $j \in \{1, \ldots, n\}$, $|\kappa| = \kappa_1 + \cdots + \kappa_n$, $D^\kappa \varphi(x) := \frac{\partial^{|\kappa|}\varphi(x)}{\partial x_1^{\kappa_1} \ldots \partial x_n^{\kappa_n}}$, while $\mathcal{D}'(R^n)$, $\mathcal{E}'(R^n)$ and $\mathcal{D}'(\bar{Q})$ are spaces of linear continuous functionals (distributions) over $\mathcal{D}(R^n)$, $\mathcal{E}(R^n)$ and $\mathcal{D}(\bar{Q})$, respectively. Note that $\mathcal{E}'(R^n)$ is the space of generalized functions with compact supports. Let

$$\mathcal{D}'_+(R) := \{f \in \mathcal{D}'(R) : f = 0, \forall t < 0\},$$
$$\mathcal{D}'_C(Q) = \{v \in \mathcal{D}'(\bar{Q}) : (v(\cdot, t), \varphi(\cdot)) \in C(0, T] \text{ for all } \varphi \in \mathcal{D}(R^n)\}.$$

We denote by $f * g$ the convolution of the generalized functions f and g, and use the function

$$f_\lambda(t) = \begin{cases} \frac{\theta(t) t^{\lambda-1}}{\Gamma(\lambda)}, & \lambda > 0, \\ f'_{1+\lambda}(t), & \lambda \le 0, \end{cases}$$

where $\Gamma(z)$ is the gamma-function, $\theta(t)$ is the Heaviside function. Note that $f_\lambda * f_\mu = f_{\lambda+\mu}$.

Recall that the Riemann-Liouville derivative of order $\beta > 0$ is defined as

$$v_t^{(\beta)}(x, t) = f_{-\beta}(t) * v(x, t),$$

and the Caputo fractional derivative is defined in [3] by

$$D_t^\beta v(x, t) = \frac{1}{\Gamma(1-\beta)} \Big[\frac{\partial}{\partial t} \int_0^t \frac{v(x, \tau)}{(t-\tau)^\beta} d\tau - \frac{v(x, 0)}{t^\beta}\Big], \quad \beta \in (0, 1),$$

$$D_t^\beta v(x, t) = \frac{1}{\Gamma(2-\beta)} \Big[\frac{\partial}{\partial t} \int_0^t \frac{v_\tau(x, \tau)}{(t-\tau)^{\beta-1}} d\tau - \frac{v_t(x, 0)}{(t-\tau)^{\beta-1}}\Big], \quad \beta \in (1, 2).$$

Denote by

$$C_{\alpha,\gamma}(Q) := \{v \in C(Q) : (-\Delta)^{\gamma/2} v, D_t^\alpha v \in C(Q)\},$$
$$C_{\alpha,\gamma}(\bar{Q}) := \{v \in C_{\alpha,\gamma}(Q) \mid v, v_t \in C(\bar{Q})\},$$
$$(Lv)(x, t) := v_t^{(\alpha)}(x, t) + a^2(-\Delta)^{\gamma/2} v(x, t),$$
$$(L^{reg} v)(x, t) := D_t^\alpha v(x, t) + a^2(-\Delta)^{\gamma/2} v(x, t),$$
$$(\widehat{L} v)(x, t) := f_{-\alpha}(t) \hat{*} v(x, t) + a^2(-\Delta)^{\gamma/2} v(x, t), \quad (x, t) \in Q,$$

where $f_{-\alpha}(t) \hat{*} v(x, t) = \big(f_{-\alpha}(\tau), v(x, t+\tau)\big)$, $v \in \mathcal{D}(\bar{Q})$. The Green formula holds [8]:

$$\int_Q v(y, \tau)(\widehat{L}\psi)(y, \tau) dy d\tau = \int_Q (L^{reg} v)(y, \tau) \psi(y, \tau) dy d\tau$$

$$- \int_{R^n} v(y, 0) dy \int_0^T f_{2-\alpha}(\tau) \psi_\tau(y, \tau) d\tau + \int_{R^n} v_t(y, 0) dy \int_0^T f_{2-\alpha}(\tau) \psi(y, \tau) d\tau,$$

for all $v \in C_{\alpha,\gamma}(\bar{Q})$, $\psi \in \mathcal{D}(\bar{Q})$.

Assumptions:

(A1) $F_0, F_1, F_2 \in \mathcal{E}'(\mathrm{R}^n)$, $t^\varepsilon g(t)$ is a continuous function on $[0,T]$ for some $\varepsilon \in (0, \alpha/2)$;

(A2) $F, F^{(\beta)} \in C(0,T]$, $\inf\limits_{t\in(0,T]} |F^{(\beta)}(t)| = f = const > 0$, $t^\varepsilon F^{(\alpha)}(t)$ is a continuous function on $[0,T]$ for some $\varepsilon \in (0,\alpha/2)$, $\varphi_0 \in \mathcal{D}(\mathrm{R}^n)$.

Definition 1. *A pair of functions $(u, r) \in \mathcal{D}'_C(Q) \times C(0,T]$ satisfying the identity*

$$(u, \widehat{L}\psi) = \int_0^T g(t)\big(F_0(\cdot), \psi(\cdot,t)\big)dt + \int_0^T r(t)\big(u_t^{(\beta)}(\cdot,t), \psi(\cdot,t)\big)dt + \sum_{j=1}^{2} \big(F_j(x) f_{j-\alpha}(t), \psi(x,t)\big) \quad (4)$$

for all $\psi \in \mathcal{D}(\bar{Q})$ and the condition (3) *is called a solution of the problem* (1)–(3).

We use the Green function method to prove the solvability of this problem.

Definition 2. *A vector-function $\big(G_0(x,t), G_1(x,t), G_2(x,t)\big)$ such that under rather regular g_0, g_1, g_2 the function*

$$u(x,t) = \int_0^t d\tau \int_{\mathrm{R}^n} G_0(x-y, t-\tau) g_0(y,\tau)dy + \sum_{j=1}^{2} \int_{\mathrm{R}^n} G_j(x-y,t) g_j(y)dy, \quad (x,t) \in \bar{Q} \quad (5)$$

is a classical (from $C_{\alpha,\gamma}(\bar{Q})$) solution of the Cauchy problem

$$L^{reg} u(x,t) = g_0(x,t), \quad (x,t) \in Q,$$
$$u(x,0) = g_1(x), \; u_t(x,0) = g_2(x), \quad x \in \mathrm{R}^n,$$

is called a Green vector-function of this problem.

Denote by

$$(\widehat{\mathcal{G}}_0\varphi)(y,\tau) := \int_\tau^T \int_{\mathrm{R^n}} G_0(x-y, t-\tau)\varphi(x,t)dxdt,$$
$$(\widehat{\mathcal{G}}_j\varphi)(y) := \int_0^T \int_{\mathrm{R^n}} G_j(x-y,t)\varphi(x,t)dxdt, \quad j = 1,2.$$

Lemma 1 ([8]). *The following relations hold:*

$$G_j(x,t) = \big(f_{j-\alpha}(\tau), G_0(x, t-\tau)\big), \quad (x,t) \in Q, \quad j = 1,2, \quad (6)$$

$$(\widehat{\mathcal{G}}_0(\widehat{L}\psi))(y,\tau) = \psi(y,\tau), \quad (y,\tau) \in \bar{Q},$$
$$(\widehat{\mathcal{G}}_j(\widehat{L}\psi))(y) = \big(f_{j-\alpha}(\tau), \psi(y,\tau)\big), \quad y \in \mathrm{R}^n, \quad j = 1,2, \text{ for all } \psi \in \mathcal{D}(\bar{Q}). \quad (7)$$

Lemma 2 ([1,4]). *The Green vector-function of the Cauchy problem* (1), (2) *exists.*

We also use the notations

$$(\widehat{G_j}\varphi)(y,t) := \int_{\mathrm{R}^n} G_j(x-y,t)\varphi(x)\, dx, \;\; j = 0,1,2.$$

Lemma 3. *For all $k \in \mathbb{Z}_+$, multi-index κ, $|\kappa| = k$, $\varphi \in \mathcal{D}(\mathbb{R}^n)$ we have*

$$D_y^\kappa(\widehat{G_j}\varphi) \in C(Q), \qquad j = 0, 1, 2,$$

and for all $\varepsilon \in (0,1)$ the following estimates hold:

$$\begin{aligned}
&\left|D_y^\kappa(\widehat{G_0}\varphi)(y,t)\right| \leq c_k t^{\alpha-\varepsilon-1}\|\varphi\|_{\mathcal{D}^k(\mathbb{R}^n)},\\
&\left|D_y^\kappa(\widehat{G_1}\varphi)(y,t)\right| \leq c_k(1 + |\ln t|)\|\varphi\|_{\mathcal{D}^k(\mathbb{R}^n)},\\
&\left|D_y^\kappa(\widehat{G_2}\varphi)(y,t)\right| \leq c_k\|\varphi\|_{\mathcal{D}^k(\mathbb{R}^n)}, \quad (y,t) \in Q.
\end{aligned}$$

Hereinafter b_i, c_i, $i \in \mathbb{Z}_+$, are positive constants.

Proof. Lemma can be proved with the use of the estimates of the Green vector-function components, which were obtained in [8] by using the properties of the H-functions of Fox [7, 13]. □

Theorem 1. *Assume that (L), (A1) hold. Then there exists a unique solution $u \in \mathcal{D}'_C(Q)$ of the problem (1), (2) with $r(t) = 0$, $t \in [0,T]$. It is defined by*

$$\Big(u(\cdot,t), \varphi(\cdot)\Big) = h_\varphi(t) \ \text{ for all } \varphi \in \mathcal{D}(\mathbb{R}^n),\ t \in (0,T], \tag{8}$$

where

$$h_\varphi(t) = \sum_{j=1}^{2}\Big(F_j(\cdot), (\widehat{G}_j\varphi)(\cdot,t)\Big) + \int_0^t g(\tau)\Big(F_0(\cdot), (\widehat{G}_0\varphi)(\cdot, t-\tau)\Big)d\tau, \ \ t \in (0,T].$$

Proof. A distribution from $\mathcal{E}'(\mathbb{R}^n)$ has a finite order of the singularity. So, there exist $k_0, k_1, k_2 \in \mathbb{Z}_+$ and the functions $g_{0\kappa}, g_{1\kappa}, g_{2\kappa} \in L_1(\mathbb{R}^n)$ such that

$$\big(F_j, \varphi\big) = \sum_{|\kappa| \leq k_j}\int_{\mathbb{R}^n} g_{j\kappa}(y) D^\kappa\varphi(y)dy \quad \text{ for all } \varphi \in \mathcal{D}(\mathbb{R}^n), \quad j = 0, 1, 2. \tag{9}$$

It means that $s(F_j) \leq k_j$, $j = 0, 1, 2$.

Using (9) and Lemma 3, similarly to [9], we show that the function (8) belongs to $\mathcal{D}'_C(Q)$, and using (7), show that it satisfies the equality (4) with $r(t) = 0$, $t \in [0,T]$. The uniqueness of a solution can be proved as in [9]. □

2 THE EXISTENCE AND UNIQUENESS THEOREMS FOR THE INVERSE PROBLEM

As we know from the Theorem 1, under assumptions (L), (A1) the solution $u \in \mathcal{D}'_C(Q)$ of the Cauchy problem (1), (2) satisfies the equation

$$\big(u(\cdot,t), \varphi(\cdot)\big) = h_\varphi(t) + \int_0^t r(\tau)\Big(u_t^{(\beta)}(\cdot,t), (\widehat{G}_0\varphi)(\cdot,t-\tau)\Big)d\tau, \ \ \varphi \in \mathcal{D}(\mathbb{R}^n), t \in (0,T], \tag{10}$$

and $h_\varphi \in C(0,T]$ for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$. Conversely, any solution $u \in \mathcal{D}'_C(Q)$ of (10) is the solution of the problem (1), (2).

From the equation (1) we obtain

$$\big(u_t^{(\alpha)}(\cdot,t),\varphi_0(\cdot)\big)=a^2\big(u(\cdot,t),(-\Delta)^{\gamma/2}\varphi_0(\cdot)\big)+r(t)\big(u_t^{(\beta)}(\cdot,t),\varphi_0\big)+g(t)\big(F_0,\varphi_0\big).$$

Using (3) and (A2) find

$$r(t)=\big[F^{(\alpha)}(t)-a^2\big(u(\cdot,t),(-\Delta)^{\gamma/2}\varphi_0(\cdot)\big)-g(t)\big(F_0,\varphi_0\big)\big]\,[F^{(\beta)}(t)]^{-1},\quad t\in(0,T]. \tag{11}$$

Denote by $H(u,t)$ the right-hand side of (11), substitute it in (10) instead of $r(t)$. We obtain the nonlinear operator equation

$$\big(u(\cdot,t),\varphi(\cdot)\big)=h_\varphi(t)+\int_0^t H(u,\tau)\big(u(\cdot,t),(\widehat{G}_0\varphi)(\cdot,t-\tau)\big)d\tau,\ \ \varphi\in\mathcal{D}(\mathrm{R}^n),\ t\in(0,T], \tag{12}$$

relatively unknown function $u\in\mathcal{D}'{}_C(Q)$. Conversely, if $u\in\mathcal{D}'{}_C(Q)$ is a solution of (12), r is defined by (11) then, by the Theorem 1, the pair (u,r) satisfies the problem (1)–(3). So, under assumptions (L), (A1), (A2) a pair $(u,r)\in\mathcal{D}'_C(Q)\times C(0,T]$ is a solution of the problem (1)–(3) if and only if the function $u\in\mathcal{D}'{}_C(Q)$ is a solution of (12) and $r(t)$ is defined by (11).

Theorem 2. *Assume that (L), (A1), (A2) hold. Then there exist $T^*\in(0,T]$ ($Q^*=\mathrm{R}^n\times(0,T^*]$, respectively) and the solution $(u,r)\in\mathcal{D}'_C(Q^*)\times C(0,T^*]$ of the problem (1)–(3): the function u is a solution of (12), r is defined by (11).*

Proof. By the Theorem 1 the right-hand side of (12) is continuous on $(0,T]$. It is enough to prove the solvability of the equation (12) in $\mathcal{D}'_C(Q)$. Using (9) and Lemma 3, for all $\varepsilon\in(0,1)$, $\varphi\in\mathcal{D}^K(\mathrm{R}^n)$ with $K\in\mathrm{Z}_+$, $K\geq\max\{k_0,k_1,k_2\}$, where $s(F_j)\leq k_j$, $j=0,1,2$, we obtain

$$t^\varepsilon\Big|\int_0^t g(\tau)\big(F_0(\cdot),(\widehat{\mathcal{G}_0}\varphi)(\cdot,t,\tau)\big)d\tau\Big|\leq b_0t^\alpha\|\varphi\|_{\mathcal{D}^K(\mathrm{R}^n)}, \tag{13}$$

$$t^\varepsilon|h_\varphi(t)|\leq\big[t^\alpha b_0+b_1\big]\ \|\varphi\|_{\mathcal{D}^K(\mathrm{R}^n)}. \tag{14}$$

Let $\mathrm{R}>0$, $\varepsilon\in(0,\alpha/2)$,

$$M_{R,\varepsilon}=M_{R,\varepsilon}(Q)=\Big\{v\in\mathcal{D}'{}_C(Q):\|v\|_\varepsilon=\sup_{t\in(0,T]}\ \sup_{\varphi\in D^K(\mathrm{R}^n)}\frac{t^\varepsilon\big|\big(v(\cdot,t),\varphi(\cdot)\big)\big|}{\|\varphi\|_{D^K(\mathrm{R}^n)}}\leq R\Big\}.$$

Define the operator $P:\mathcal{D}'_C(Q)\to\mathcal{D}'_C(Q)$ as follows

$$\big((Pv)(\cdot,t),\varphi(\cdot)\big)=h_\varphi(t)+\int_0^t H(v,\tau)\big(v(\cdot,t),(\widehat{G}_0\varphi)(\cdot,t-\tau)\big)d\tau,\quad\varphi\in D^K(\mathrm{R}^n). \tag{15}$$

We use the Banach principle to prove the solvability of the equation (12), that is

$$u=Pu,\quad u\in M_{R,\varepsilon}(Q)\subset\mathcal{D}'_C(Q).$$

At the beginning we show that there exist $R>0$, $T^*\in(0,T]$, $Q^*=\mathrm{R}^n\times(0,T^*]$ and $M^*_{R,\varepsilon}=M_{R,\varepsilon}(Q^*)$ such that $P:M^*_{R,\varepsilon}\to M^*_{R,\varepsilon}$.

For every $v \in M_{R,\varepsilon}$ we have

$$\tau^{\varepsilon}\big|\big(v(\cdot,\tau), a^2(-\Delta)^{\gamma/2}\varphi_0(\cdot)\big)\big| \le R\|(-\Delta)^{\gamma/2}\varphi_0\|_{\mathcal{D}^K(\mathrm{R}^n)} := b_2 R,$$

and therefore

$$\tau^{\varepsilon}|H(v,\tau)| \le \frac{B+b_2R}{f}, \text{ where } B = \sup_{\tau\in(0,T]} \tau^{\varepsilon}|F^{(\alpha)}(\tau) - g(\tau)(F_0,\varphi_0)|.$$

From here, taking into account (13), (14) and Lemma 3, for all $v \in M_{R,\varepsilon}$, $\varphi \in \mathcal{D}(\mathrm{R}^n)$ we obtain

$$\begin{aligned}
\frac{t^{\varepsilon}\big|\big((Pv)(\cdot,t),\varphi(\cdot)\big)\big|}{\|\varphi\|_{\mathcal{D}^K(\mathrm{R}^n)}} &\le t^{\alpha}b_0 + b_1 + \frac{(B+b_2R)R}{f}\int_0^t \frac{\|(\widehat{\mathcal{G}}_0\varphi)(\cdot,t-\tau)\|_{\mathcal{D}^K(\mathrm{R}^n)}\tau^{-\varepsilon}d\tau}{\|\varphi\|_{\mathcal{D}^K(\mathrm{R}^n)}} \\
&\le t^{\alpha}b_0 + b_1 + \frac{(B+b_2R)R}{f}\int_0^t c_K(t-\tau)^{\alpha-\varepsilon-1}\tau^{-\varepsilon}d\tau \\
&\le t^{\alpha-2\varepsilon}(q_0R^2+q_1R+q_2)+b_1,
\end{aligned}$$

where q_j ($j \in \{0,1,2\}$) are positive constants.

To realize the inequality

$$t^{\alpha-2\varepsilon}(q_0R^2+q_1R+q_2)+b_1 \le R \text{ for all } t \in [0,T^*] \tag{16}$$

with some $T^* \in (0,T]$, we will at first choose $R \ge 2b_1$ and $t_0 \in (0,T]$ such that

$$q_2t^{\alpha-2\varepsilon}+b_1 \le R/2 \text{ for all } t \in [0,t_0].$$

Then (16) follows from the inequality

$$(q_0+q_1)t^{\alpha-2\varepsilon}R \le \frac{1}{2} \text{ for all } t \in [0,T^*] \tag{17}$$

for some $R \ge \max\{1, 2b_1\}$, where $T^* = \min\{t_0, 1/[2(q_0+q_1)R]^{1/(\alpha-2\varepsilon)}\}$. We have proved the existence R, T^* such that $P : M^*_{R,\varepsilon} \to M^*_{R,\varepsilon}$.

Now we show that P is the contraction operator on $M^*_{R,\varepsilon}$. For $v_1, v_2 \in M^*_{R,\varepsilon}$, $\varphi \in \mathcal{D}(\mathrm{R}^n)$ and $t \in [0,T^*]$ we have

$$\begin{aligned}
&\frac{t^{\varepsilon}\big|\big((Pv_1)(\cdot,t) - \big((Pv_2)(\cdot,t),\varphi(\cdot)\big)\big|}{\|\varphi\|_{\mathcal{D}^K(\mathrm{R}^n)}} = \frac{t^{\varepsilon}}{\|\varphi\|_{\mathcal{D}^K(\mathrm{R}^n)}}\int_0^t \Big|H(v_2,\tau)\big(v_1(\cdot,t)-v_2(\cdot,t),(\widehat{G}_0\varphi)(\cdot,t-\tau)\big) \\
&\quad + \big(H(v_1,\tau)-H(v_2,\tau)\big)\big(v_1(\cdot,t),(\widehat{G}_0\varphi)(\cdot,t-\tau)\big)\Big|d\tau \\
&\quad \le \frac{(B+b_2R)t^{\varepsilon}}{f}\int_0^t \frac{\big|\big(v_1(\cdot,t)-v_2(\cdot,t),(\widehat{G}_0\varphi)(\cdot,t-\tau)\big)\big|}{\|(\widehat{G}_0\varphi)(\cdot,t-\tau)\|_{\mathcal{D}^K(\mathrm{R}^n)}}\frac{\|(\widehat{G}_0\varphi)(\cdot,t-\tau)\|_{\mathcal{D}^K(\mathrm{R}^n)}}{\|\varphi\|_{\mathcal{D}^K(\mathrm{R}^n)}}\tau^{-\varepsilon}d\tau \\
&\quad + \frac{a^2t^{\varepsilon}R\|(-\Delta)^{\gamma/2}\varphi_0\|_{\mathcal{D}^K(\mathrm{R}^n)}}{f}\int_0^t \frac{\big|\big(v_1(\cdot,\tau)-v_2(\cdot,\tau),(-\Delta)^{\gamma/2}\varphi_0(\cdot)\big)\big|}{\|(-\Delta)^{\gamma/2}\varphi_0\|_{\mathcal{D}^K(\mathrm{R}^n)}}\frac{\|(\widehat{G}_0\varphi)(\cdot,t-\tau)\|_{\mathcal{D}^K(\mathrm{R}^n)}}{\|\varphi\|_{\mathcal{D}^K(\mathrm{R}^n)}}d\tau \\
&\quad \le \frac{(B+2b_2R)}{f}\cdot\|v_1-v_2\|_{\varepsilon}\frac{\int_0^t \|(\widehat{G}_0\varphi)(\cdot,t-\tau)\|_{\mathcal{D}^K(\mathrm{R}^n)}\tau^{-\varepsilon}d\tau}{\|\varphi\|_{\mathcal{D}^K(\mathrm{R}^n)}} \le (2q_0R+q_1)t^{\alpha-2\varepsilon}\|v_1-v_2\|_{\varepsilon}.
\end{aligned}$$

If $(-\Delta)^{\gamma/2}\varphi_0(x) \equiv 0$, $x \in \mathrm{R}^n$, then $\big(v_1(\cdot,t)-v_2(\cdot,t), (-\Delta)^{\gamma/2}\varphi_0(\cdot)\big) = 0$ for all $t \in [0, T^*]$, and the factor 2 is absent in the obtained expression.

For $t \in [0, T^*]$ we have

$$(2q_0R+q_1)t^{\alpha-2\varepsilon} \le \frac{2q_0R+q_1}{2(q_0+q_1)R} \le \frac{2q_0+q_1}{2(q_0+q_1)} < 1.$$

So, P is the contraction operator on $M_{R,\varepsilon}(Q^*)$, and by the Banach theorem we obtain the solvability of the equation (12) in $M^*_{R,\varepsilon} \subset \mathcal{D}'_C(Q^*)$. □

Theorem 3. *Under conditions $F^{(\beta)} \in C(0,T]$, $\inf\limits_{t\in(0,T]} |F^{(\beta)}(t)| \neq 0$ a solution $(u,r) \in \mathcal{D}'_C(Q) \times C(0,T]$ of the problem (1)–(3) is unique.*

Proof. Take two solutions (u_1, r_1), $(u_2, r_2) \in \mathcal{D}'_C(Q) \times C(0,T]$ of the problem (1)–(3) and substitute them in (1), (2). Putting $u = u_1 - u_2$, $r = r_1 - r_2$ obtain the Cauchy problem for the equation

$$u_t^{(\alpha)} = a^2(-\Delta)^{\gamma/2}u + r_2 u_t^{(\beta)} + r {u_1}_t^{(\beta)} \tag{18}$$

with zero initial conditions. By the definition of solution

$$\big(u, \widehat{L}\psi\big) = \int_0^T \Big[r_2(t)\big(u_t^{(\beta)}(\cdot,t), \psi(\cdot,t)\big) + r(t)\big({u_1}_t^{(\beta)}(\cdot,t), \psi(\cdot,t)\big)\Big]dt \ \text{ for all } \ \psi \in \mathcal{D}(\bar{Q}).$$

According to [8], for each $\varrho \in \mathcal{D}(\bar{Q})$ there exists $\psi = \widehat{\mathcal{G}}_0\varrho \in \mathcal{D}(\bar{Q}_0)$ such that $\widehat{L}\psi = \varrho$ in Q. Then for each $\varrho \in \mathcal{D}(\bar{Q})$ we have

$$\int_0^T \Big(u(\cdot,t), \varrho(\cdot,t)\Big)dt = \int_0^T \Big(r_2(t)u_t^{(\beta)}(\cdot,t) + r(t){u_1}_t^{(\beta)}(\cdot,t), (\widehat{\mathcal{G}}_0\varrho)(\cdot,t)\Big)dt. \tag{19}$$

From the over-determination condition (3), by using (11), we find

$$a^2\Big(u(z,t), (-\Delta)^{\gamma/2}\varphi_0(z)\Big) = -r(t)F^{(\beta)}(t), \quad t \in (0,T], \tag{20}$$

and then, from (19), for all $\varrho \in \mathcal{D}(\bar{Q})$ we obtain the equation

$$\int_0^T \Big(u_t^{(\beta)}(\cdot,t), \varrho(\cdot,t) - r_2(t)(\widehat{\mathcal{G}}_0\varrho)(\cdot,t) + \frac{(-\Delta)^{\gamma/2}\varphi_0(\cdot)w_\varrho(t)}{F^{(\beta)}(t)}\Big)dt = 0, \tag{21}$$

where

$$\begin{aligned} w_\varrho(t) &= a^2\big({u_1}_t^{(\beta)}(\cdot,t), (\widehat{\mathcal{G}}_0\varrho)(\cdot,t)\big) \\ &= a^2\big(f_{-\beta}(t) * u_1(\cdot,t), (\widehat{\mathcal{G}}_0\varrho)(\cdot,t)\big) = a^2\big(u_1(\cdot,t), f_{-\beta}(t) \hat{*} (\widehat{\mathcal{G}}_0\varrho)(\cdot,t)\big) \end{aligned}$$

is the known function from $C(0,T]$,

$$\varrho(\cdot,t) - r_2(t)(\widehat{\mathcal{G}}_0\varrho)(\cdot,t) + \frac{(-\Delta)^{\gamma/2}\varphi_0(\cdot)w_\varrho(t)}{F^{(\beta)}(t)} \in \mathcal{D}(\mathrm{R}^n), \quad t \in (0,T]$$

is the continuous function in $t \in (0, T]$. So, for each $\varphi \in \mathcal{D}(\mathrm{R}^n)$, $\mu \in \mathcal{D}(0, T]$, $\mu(T) = 0$ there exists a unique solution $\varrho \in \mathcal{D}(\bar{Q})$ of the second type Volterra integral equation

$$\varrho(x,t) - r_2(t)(\widehat{\mathcal{G}}_0\varrho)(x,t) + \frac{(-\Delta)^{\gamma/2}\varphi_0(x)w_\varrho(t)}{F^{(\beta)}(t)} = \varphi(x)\mu(t), \quad (x,t) \in \bar{Q},$$

with integrable kernel. Then (21) implies that

$$\int_0^T \Big(u_t^{(\beta)}(\cdot,t), \varphi(\cdot)\Big)\mu(t)dt = 0 \ \text{ for all } \ \varphi \in \mathcal{D}(\mathrm{R}^n), \ \ \mu \in \mathcal{D}(0,T], \ \ \mu(T) = 0.$$

By the Dubua-Rejmon lemma we obtain

$$\big(u_t^{(\beta)}(\cdot,t), \varphi(\cdot)\big) = 0 \ \text{ for all } \ \varphi \in \mathcal{D}(\mathrm{R}^n), \ \ t \in (0,T].$$

Therefore, $u_t^{(\beta)} = 0$, i.e. $f_{-\beta}(t) * u(x,t) = 0$, i.e. $f_\beta(t) * f_{-\beta}(t) * u(x,t) = 0$, i.e. $u = 0$ in $\mathcal{D}'_C(Q)$, and (20) implies that $r(t) = 0$, $t \in (0, T]$. $\square$

3 CONCLUSIONS

The inverse Cauchy problem for a time-space-fractional telegraph equation with given distributions in the right-hand sides has been studied. We have determinated a generalized solution u of direct Cauchy problem and unknown, depending on time variable, continuous minor coefficient r of the equation. The existence of a solution $(u, r) \in \mathcal{D}'_C(Q^*) \times C(0, T^*]$ is obtained for some $T^* \in (0, T]$. The uniqueness of a solution $(u, r) \in \mathcal{D}'_C(Q) \times C(0, T]$ is obtained for arbitrary $T > 0$.

Let $\mathcal{D}'_C(\bar{Q}) = \{v \in \mathcal{D}'(\bar{Q}) : (v(\cdot,t), \varphi(\cdot)) \in C[0,T] \ \text{ for all } \ \varphi \in \mathcal{D}(\mathrm{R}^n)\}$. The Green vector-function of the Cauchy problem for the operator $D_t^\alpha - A(x, D)$, where $A(x, D)$ is an elliptic differential expression of the second order with infinitely differentiable coefficients, has the exponential descending at infinity. So, unlike the case of the proposed problem (1)–(3), under assumptions $F_0, F_1, F_2 \in \mathcal{E}'(\mathrm{R}^n)$, $g \in C[0, T]$, $F, F^{(\beta)}, F^{(\alpha)} \in C[0, T]$, $F^{(\beta)}(t) \neq 0$, $t \in [0, T]$ and the compatibility conditions

$$(F_1, \varphi_0) = F(0), \quad (F_2, \varphi_0) = F'(0),$$

there exist $T^* \in (0, T]$ and the solution $(u, r) \in \mathcal{D}'_C(\bar{Q}^*) \times C[0, T^*]$ of the problem (1)–(3) with the operator $-A(x, D)$ instead of $a^2(-\Delta)^{\gamma/2}$.

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Досліджуємо обернену задачу Коші для рівняння

$$u_t^{(\alpha)} - r(t)u_t^{(\beta)} + a^2(-\Delta)^{\gamma/2}u = F_0(x)g(t), \quad (x,t) \in \mathrm{R}^n \times (0,T],$$

з дробовими похідними та заданими узагальненими функціями в правих частинах рівняння і початкових умов. Наше завдання полягає у визначенні пари функцій: узагальненого розв'язку u (неперервного за часом в узагальненому сенсі) та невідомого молодшого коефіцієнта $r(t)$. У статті встановлено однозначну розв'язність задачі.

Ключові слова і фрази: узагальнена функція, дробова похідна, обернена задача, вектор функція Гріна.

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HYPERCYCLIC OPERATORS ON ALGEBRA OF SYMMETRIC ANALYTIC FUNCTIONS ON ℓ_p

In the paper, it is proposed a method of construction of hypercyclic composition operators on $H(\mathbb{C}^n)$ using polynomial automorphisms of $\mathbb{C}^n$ and symmetric analytic functions on ℓ_p. In particular, we show that a "symmetric translation" operator is hypercyclic on a Fréchet algebra of symmetric entire functions on ℓ_p which are bounded on bounded subsets.

Key words and phrases: hypercyclic operators, functional spaces, symmetric functions.

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INTRODUCTION

The theory of hypercyclicity studies the long-term behavior of continuous operators on topological spaces. Let X be a Fréchet (linear complete metric) space.

Definition 1. *A continuous linear operator $T : X \to X$ is called hypercyclic if there is a vector $x_0 \in X$ for which the orbit under T, $\mathrm{Orb}(T, x_0) = \{x_0, Tx_0, T^2x_0, \dots\}$ is dense in X. Every such vector x_0 is called a hypercyclic vector of T.*

The classical Birkhoff's theorem [6] asserts that any operator of composition with translation $x \mapsto x + a$, $T_a \colon f(x) \mapsto f(x + a)$ is hypercyclic on a space of entire functions $H(\mathbb{C})$ on a complex plane $\mathbb{C}$ if $a \neq 0$. The Birkhoff's translation T_a has also been regarded as a differentiation operator

$$T_a(f) = \sum_{n=0}^{\infty} \frac{a^n}{n!} D^n f.$$

A generalization of Birkhoff's theorem was proved by Godefroy and Shapiro in [9]. They showed that if $\varphi(z) = \sum\limits_{|\alpha|\geq 0} c_\alpha z^\alpha$ is a non-constant entire function of exponential type on $\mathbb{C}^n$, then the operator

$$f \mapsto \sum_{|\alpha|\geq 0} c_\alpha D^\alpha f, \qquad f \in H(\mathbb{C}^n), \tag{1}$$

is hypercyclic. Moreover, in [9], it is proved that any continuous linear operator T on $H(\mathbb{C}^n)$, which commutes with translations and is not a scalar multiple of the identity, can be expressed by (1) and so is hypercyclic as well.

Let us recall that an operator C_Φ on $H(\mathbb{C}^n)$ is said to be a *composition operator* if $C_\Phi f(x) = f(\Phi(x))$ for some analytic map $\Phi \colon \mathbb{C}^n \to \mathbb{C}^n$. It is known that only translation operator T_a for

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some $a \neq 0$ is a hypercyclic composition operator on $H(\mathbb{C})$ [5]. However, if $n > 1$, $H(\mathbb{C}^n)$ supports more hypercyclic composition operators. Bernal-González [4] established some necessary and sufficient conditions for a composition operator by an affine map to be hypercyclic.

In [14], it was proposed a method of construction of hypercyclic composition operators on $H(\mathbb{C}^n)$, which can not be described by formula (1), using symmetric analytic functions on ℓ_1. The purpose of this paper is a generalization of the method for the space ℓ_p, $1 < p < \infty$. Also similarly to [14], we show that a symmetric translation operator is hypercyclic on a Fréchet algebra $H^n_{bs}(\ell_p)$ of symmetric entire functions on ℓ_p which are bounded on bounded subsets. More about hypercyclic composition operators the reader can find in [13].

In Section 1, we discuss some relationship between polynomial automorphisms on $\mathbb{C}^n$ and an operation of changing of polynomial bases in an algebra of symmetric analytic functions on the Banach space of summing sequences, ℓ_p. In Section 2, we prove the hypercyclicity of a special operator on the algebra of symmetric analytic functions on ℓ_p which plays the role of translation in this algebra. We consider, in the third section, an algebra which is the completion of the space of symmetric polynomials on ℓ_p endowed with the uniform topology on bounded subsets and we prove hypercyclicity of our special operator on this algebra.

Let us recall a well known Kitai-Gethner-Shapiro's theorem which is also known as the Hypercyclicity Criterion.

Theorem 1 (Hypercyclicity Criterion). *Let X be a separable complete linear metric space and $T\colon X \to X$ be a linear and continuous operator. Suppose there exist X_0, Y_0 dense subsets of X, a sequence (n_k) of positive integers and a sequence of mappings (possibly nonlinear, possibly not continuous) $S_n\colon Y_0 \to X$ so that*

1. *$T^{n_k}(x) \to 0$ for every $x \in X_0$ as $k \to \infty$,*
2. *$S_{n_k}(y) \to 0$ for every $y \in Y_0$ as $k \to \infty$,*
3. *$T^{n_k} \circ S_{n_k}(y) = y$ for every $y \in Y_0$.*

Then T is hypercyclic.

The operator T is called the operator that satisfy the *Hypercyclicity Criterion for full sequence* if we can chose $n_k = k$.

For details of the theory of analytic functions on Banach spaces we refer the reader to Dineen's book [8]. Note that an analogue of the Godefroy-Shapiro's theorem for a special class of entire functions on Banach space with separable dual was proved by Aron and Bés in [2]. Current state of theory of symmetric analytic functions on Banach spaces can be found in [1, 10]. A detailed survey of hypercyclic operators is given by Grosse-Erdmann in [3, 11, 12].

1 Algebra of symmetric functions

Let X be a Banach space with a symmetric basis $(e_i)_{i=1}^{\infty}$. A function g on X is called *symmetric* if for every $x = \sum_{i=1}^{\infty} x_i e_i \in X$, $g(x) = g\Big(\sum_{i=1}^{\infty} x_i e_i\Big) = g\Big(\sum_{i=1}^{\infty} x_i e_{\sigma(i)}\Big)$ for an arbitrary permutation σ on the set $\{1, ..., m\}$ for any positive integer m. The sequence of homogeneous polynomials $(P_j)_{j=1}^{\infty}$, $\deg P_k = k$ is called a *homogeneous algebraic basis* in the algebra of symmetric polynomials, if for every symmetric polynomial P of degree n on X there exists a polynomial q on $\mathbb{C}^n$ such that $P(x) = q(P_1(x), \ldots, P_n(x))$.

We denote by $\mathcal{P}_s(\ell_p)$ algebra symmetric continuous polynomials. Let $\lceil p \rceil$ be the smallest integer that is greater than or equal to p. In [10], it is proved that the polynomials

$$F_k\left(\sum_{i=1}^{\infty} a_i e_i\right) = \sum_{i=1}^{\infty} a_i^k \tag{2}$$

for $k = \lceil p \rceil, \lceil p \rceil + 1, \ldots$ form an algebraic basis in $\mathcal{P}_s(\ell_p)$.

So, there are no symmetric polynomials of degree less than $\lceil p \rceil$ in $\mathcal{P}_s(\ell_p)$ and if $\lceil p_1 \rceil = \lceil p_2 \rceil$, then $\mathcal{P}_s(\ell_{p_1}) = \mathcal{P}_s(\ell_{p_2})$. Thus, without loss of generality we can consider $\mathcal{P}_s(\ell_p)$ only for integer values of p. Throughout, we will assume that p is an integer, $1 \leq p < \infty$.

Corollary 1 ([1]). *Given $(\xi_1, \ldots, \xi_n) \in \mathbb{C}^n$, there is $x \in \ell_p^{n+p-1}$ such that*

$$F_p(x) = \xi_1, \ldots, F_{n+p-1}(x) = \xi_n.$$

This result shows that any $P \in \mathcal{P}_s(\ell_p)$ has a unique representation in terms of $\{F_k\}$, in sense that if $q \in \mathcal{P}(\mathbb{C}^n)$ for some n is such that $P(x) = q(F_p(x), \ldots, F_{n+p}(x))$, and if $q' \in \mathcal{P}(\mathbb{C}^m)$ for some m is such that $P(x) = q'(F_p(x), \ldots, F_{m+p}(x))$, with, say, $n \leq m$, then $q'(\xi_1, \cdots, \xi_m) = q(\xi_1, \cdots, \xi_n)$.

Let us denote by $\mathcal{P}_s^n(\ell_p)$, $n \geq p$, the subalgebra of $\mathcal{P}_s(\ell_p)$ generated by $\{F_p, \ldots, F_n\}$.

Denote by $H_{bs}^n(\ell_p)$ an algebra of entire symmetric functions on ℓ_p which is topologically generated by polynomials $F_p, \ldots, F_n$. It means that $H_{bs}^n(\ell_p)$ is the completion of the algebraic span of $F_p, \ldots, F_n$ in the uniform topology on bounded subsets. We say that polynomials $Q_p, \ldots, Q_n$ (not necessary homogeneous) form an *algebraic basis* in $H_{bs}^n(\ell_p)$ if they topologically generate $H_{bs}^n(\ell_p)$. Evidently, if $(Q_j)_{j=1}^{\infty}$ is a homogeneous algebraic basis in $\mathcal{P}_s(\ell_p)$, then $(Q_p, \ldots, Q_n)$ is an algebraic basis in $H_{bs}^n(\ell_p)$.

2 SYMMETRIC TRANSLATION

In this section, we construct a special operator on the algebra of symmetric analytic functions on ℓ_p. We start with an evident statement, which actually is a very special case of the Universal Comparison Principle (see [11, Proposition 4]).

Proposition 1. *Let T be a hypercyclic operator on X and A be an isomorphism of X. Then $A^{-1}TA$ is hypercyclic.*

We will say that $A^{-1}TA$ is a *similar* operator to T. If $T = C_\alpha$ is a composition operator on $H(\mathbb{C}^n)$ and $A = C_\Phi$ is a composition by an analytic automorphism Φ of $\mathbb{C}^n$, then $A^{-1}TA = C_{\Phi\circ\alpha\circ\Phi^{-1}}$ is a composition operator too. If A is a composition with a polynomial automorphism, we will say that $A^{-1}TA$ is *polynomially similar* to T. Now we consider operators which are similar to the translation composition $T_a \colon f(x) \mapsto f(x+a)$ on $H(\mathbb{C}^n)$.

Let us denote by $\mathcal{F}_p^n$ the mapping from ℓ_p to $\mathbb{C}^{n+1-p}$, $n \geq p$, given by

$$\mathcal{F}_p^n : x \mapsto (F_p(x), \ldots, F_n(x)).$$

It is known (see [1]) that the map

$$C_{\mathcal{F}_p^n} \colon f(t_1, \ldots, t_n) \mapsto f(F_p(x), \ldots, F_n(x))$$

is a topological isomorphism from the algebra $H(\mathbb{C}^{n+1-p})$ to the algebra $H^n_{bs}(\ell_p)$.

Easy to see that for symmetric function $f(x)$ on ℓ_p the function $f(x+y)$ is not symmetric for some fixed $y \in \ell_p$. The space of symmetric function is not invariant respect to certain translation operator $f(x) \mapsto f(x+y)$. We propose another translation on ℓ_p, which keep the space of symmetric analytic functions.

Let $x, y \in \ell_p$, $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$. We put

$$x \bullet y := (x_1, y_1, x_2, y_2, \ldots).$$

We note the basic properties of symmetric translation.

1. If $x = \sigma_1(u)$ i $y = \sigma_2(v)$ for some permutations σ_1, σ_2 then $x \bullet y = \sigma(u \bullet v)$ for some permutation σ on $\mathbb{N}$.
2. $\|x \bullet y\|^p = \|x\|^p + \|y\|^p$.
3. For any natural $n \geq p$
$$F_n(x \bullet y) = F_n(x) + F_n(y). \tag{3}$$

We define

$$\mathcal{T}_y(f)(x) := f(x \bullet y)$$

and will say that $x \mapsto x \bullet y$ is the *symmetric translation* and the operator $\mathcal{T}_y$ is the *symmetric translation operator*. It is clear that if f is a symmetric function, then $f(x \bullet y)$ is a symmetric function for any fixed y. In [7], it is proved that $\mathcal{T}_y$ is a topological isomorphism from the algebra of symmetric analytic functions to itself.

Let $g \in H^n_s(\ell_p)$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$. Set for $f = (\mathcal{F}^{\mathbf{F}}_n)^{-1} g$

$$\mathcal{D}^\alpha g := \mathcal{F}^{\mathbf{F}}_n D^\alpha (\mathcal{F}^{\mathbf{F}}_n)^{-1} g = \left(\frac{\partial^{\alpha_1}}{\partial t_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial t_n^{\alpha_n}} f \right) (F_1(\cdot), \ldots, F_{p+n-1}(\cdot)).$$

Theorem 2. *Let $y \in \ell_p$ such that $(F_p(y), \ldots, F_{p+n-1}(y))$ is a nonzero vector in $\mathbb{C}^n$. Then the symmetric translation operator $\mathcal{T}_y$ is hypercyclic on $H^n_{bs}(\ell_p)$. Moreover, every operator $\mathcal{A}$ on $H^n_s(\ell_p)$ which commutes with $\mathcal{T}_y$ and is not a scalar multiple of the identity is hypercyclic and can be represented by*

$$\mathcal{A}(g) = \sum_{|\alpha| \geq 0} c_\alpha \mathcal{D}^\alpha g, \tag{4}$$

where c_α are coefficients of a non-constant entire function of exponential type on $\mathbb{C}^n$.

Proof. Let $a = (F_p(y), \ldots, F_{p+n-1}(y)) \in \mathbb{C}^n$. If $g \in H^n_{bs}(\ell_p)$, then

$$g(x) = C_{\mathcal{F}^n_p}(f)(x) = f(F_p(x), \ldots, F_{p+n-1}(x))$$

for some $f \in H^n_s(\ell_1)$ and property (3) implies that

$$\begin{aligned} \mathcal{T}_y(g)(x) &= g(x \bullet y) = f(F_p(x \bullet y), \ldots, F_{p+n-1}(x \bullet y)) \\ &= f(F_p(x) + F_p(y), \ldots, F_{p+n-1}(x) + F_{p+n-1}(y)) \\ &= C_{\mathcal{F}^n_p}((f)(t+a)) = C_{\mathcal{F}^n_p}(T_a(f)(t)). \end{aligned}$$

Since the set $(T^k_a(f))_{k=1}^\infty$ is dense in $H(\mathbb{C}^n)$, then set $(\mathcal{T}^k_y(g))_{k=1}^\infty = (C_{\mathcal{F}^n_p}(T^k_a(f)))_{k=1}^\infty$ is dense in $H^n_{bs}(\ell_p)$. So, the symmetric translation of operator $\mathcal{T}_y$ is hypercyclic on $H^n_{bs}(\ell_p)$. Since $\mathcal{T}_y(g)(x) = \mathcal{F}^{\mathbf{F}}_n T_a (\mathcal{F}^{\mathbf{F}}_n)^{-1}(g)(x)$, the proof of (4) follows from Proposition 1 and the Godefroy-Shapiro Theorem. $\square$

A given algebraic basis $\mathbf{R}$ on $H_s^n(\ell_p)$ we set

$$T_{\mathbf{R},y} := (\mathcal{F}_n^{\mathbf{R}})^{-1}\mathcal{T}_y\mathcal{F}_n^{\mathbf{R}} \quad \textit{and} \quad D_{\mathbf{R}}^{\alpha} := (\mathcal{F}_n^{\mathbf{R}})^{-1}\mathcal{D}^{\alpha}\mathcal{F}_n^{\mathbf{R}}.$$

Corollary 2. *Let $\mathbf{R}$ be an algebraic basis on $H_s^n(\ell_p)$ and let $y \in \ell_p$ such that $(F_p(y),\ldots,F_{p+n-1}(y)) \neq 0$. Then the operator $T_{\mathbf{R},y}$ is hypercyclic on $H(\mathbb{C}^n)$. Moreover, every operator A on $H(\mathbb{C}^n)$ which commutes with $T_{\mathbf{R},y}$ and is not a scalar multiple of the identity is hypercyclic and can be represented by the form*

$$A(f) = \sum_{|\alpha|\geq 0} c_\alpha D_{\mathbf{R}}^{\alpha} f, \tag{5}$$

where c_α as in (1).

We need the next proposition.

Proposition 2 ([14])**.** *Let $\Phi = (\Phi_1,\ldots,\Phi_n)$ be a polynomial automorphism on $\mathbb{C}^n$. Then $(\Phi_1(\mathbf{R}),\ldots,\Phi_n(\mathbf{R}))$ is an algebraic basis in $H_s^n(\ell_p)$ for an arbitrary algebraic basis $\mathbf{R} = (R_1,\ldots,R_n)$.*

Conversely, if $(\Phi_1(\mathbf{R}),\ldots,\Phi_n(\mathbf{R}))$ is an algebraic basis for some algebraic basis $\mathbf{R} = (R_1,\ldots,R_n)$ in $H_s^n(\ell_p)$ and a polynomial map Φ on $\mathbb{C}^n$, then Φ is a polynomial automorphism.

Note that due to Proposition 2 the transformation $(\mathcal{F}_n^{\mathbf{R}})^{-1}\mathcal{T}_y\mathcal{F}_n^{\mathbf{R}}$ is nothing else than a composition with $\Phi \circ (I + a) \circ \Phi^{-1}$, where $\Phi(F_p,\ldots,F_{p+n-1}) = (R_p,\ldots,R_{p+n-1})$ and $a = (F_p(y),\ldots,F_{p+n-1}(y))$. Conversely, every polynomially similar operator to the translation can be represented by the form $(\mathcal{F}_n^{\mathbf{R}})^{-1}\mathcal{T}_y\mathcal{F}_n^{\mathbf{R}}$ for some algebraic basis of symmetric polynomials $\mathbf{R}$. This observation can be helpful in order to construct some examples of such operators.

The next algebraic bases of $\mathcal{P}_s(\ell_p)$ is useful for us: $(G_k^{(p)})_{k=1}^{\infty}$, where

$$G_k(x) = G_k^{(1)}(x) = \sum_{i_1<\cdots<i_k} x_{i_1}\cdots x_{i_k}$$

and $G_k^{(p)}(x)$ can be obtained from Newton's formula (see [16, §53]), putting $F_1(x) = F_2(x) = \cdots = F_{p-1}(x) = 0$. So, we get ([15])

$$\begin{aligned} nG_n^{(p)} &= (-1)^{p+1}F_p(x)G_{n-p}^{(p)}(x) + (-1)^{p+2}F_{p+1}(x)G_{n-p-1}^{(p)}(x) \\ &\quad + \cdots + (-1)^{n-p+1}F_{n-p}(x)G_p^{(p)}(x) + (-1)^{n+1}F_n(x), \end{aligned}$$

where $n > p$, $G_0^{(p)}(x) \equiv 1$, $F_0(x) \equiv 1$ and $G_1^{(p)}(x) = G_2^{(p)}(x) = \cdots = G_{p-1}^{(p)}(x) = 0$, $F_1(x) = F_2(x) = \cdots = F_{p-1}(x) = 0$. By another words, in (2) the terms $F_r(x)G_{q-r}^{(p)}(x) = 0$, if $r < p$ and $q - r < p$, where $p \leq r \leq n-p$, $p \leq q-r \leq n-p$.

Let us compute how looks the operator $T_{\mathbf{R},y}$ for $\mathbf{R} = \mathbf{G}$. We observe first that

$$G_m^{(p)}(x \bullet y) = \sum_{j+k=m} G_j^{(p)}(x)G_k^{(p)}(y), \qquad p \leq m \leq p+n-1,$$

where for the sake of convenience we take $G_0^{(p)} \equiv 1$. Thus

$$\begin{aligned}\mathcal{T}_y\mathcal{F}_n^{\mathbf{G}} f(t_1,\dots,t_n) &= \mathcal{T}_y f(G_p^{(p)}(x),\dots,G_{p+n}^{(p)}(x)) = f(G_p^{(p)}(x\bullet y),\dots,G_{p+n}^{(p)}(x\bullet y))\\ &= f\Big(G_p^{(p)}(x)+G_p^{(p)}(y),\dots,\sum_{i+k=m} G_i^{(p)}(x)G_k^{(p)}(y),\dots,\sum_{i+k=p+n-1} G_i^{(p)}(x)G_k^{(p)}(y)\Big).\end{aligned}$$

Therefore,

$$T_{\mathbf{G},y} f(t_1,\dots,t_n) = f\Big(t_p+b_p,\dots,\sum_{j+k=m} t_j b_k,\dots,\sum_{j+k=p+n-1} t_j b_k\Big), \tag{6}$$

where $t_1 = 0,\dots,t_{p-1} = 0$, $b_1 = 0,\dots,b_{p-1} = 0$, and $b_j = G_j^{(p)}(y)$ for $1 \le j \le p+n-1$.

Godefroy and Shapiro proved that any continuous linear operator T on $H(\mathbb{C}^n)$, which commutes with translations and is not a scalar multiple of the identity, can be generated by (1). Composition with an affine map still does not commute with T_a. Indeed, by (6),

$$T_a \circ T_{\mathbf{G},y} f(t_1,\dots,t_n) = f\Big(t_p+b_p+a_p,\dots,\sum_{j=0}^{p+n-1} t_j b_{p+n-1-j} + a_{p+n-1}\Big);$$

$$T_{\mathbf{G},y} \circ T_a f(t_1,\dots,t_n) = f\Big(t_p+b_p+a_p,\dots,\sum_{j=0}^{p+n-1} (t_j+a_j) b_{p+n-1-j}\Big),$$

where $a_0 = 1$. Evidently, $T_a \circ T_{\mathbf{G},y} \ne T_{\mathbf{G},y} \circ T_a$ for some $a \ne 0$ whenever $b \ne (0,\dots,0,b_{p+n-1})$.

3 THE CASE OF SPACE $H_{bs}(\ell_p)$

Note that T_a satisfies the Hypercyclicity Criterion for full sequence [9] and so the symmetric shift $\mathcal{T}_y$ on $H_s^n(\ell_p)$ satisfies the Hypercyclicity Criterion for full sequence provided $(F_p(y),\dots,F_{p+n-1}(y)) \ne 0$.

We will establish our result about hypercyclic operators on the space of symmetric entire functions on ℓ_p. But before this, we need the following general auxiliary statement, which might be of some interest by itself.

Lemma 1 ([14]). *Let X be a Fréchet space and $X_1 \subset X_2 \subset \cdots \subset X_m \subset \cdots$ be a sequence of closed subspaces such that $\bigcup_{m=1}^{\infty} X_m$ is dense in X. Let T be an operator on X such that $T(X_m) \subset X_m$ for each m each restriction $T|_{X_m}$ satisfies the Hypercyclicity Criterion for full sequence on X_m. Then T satisfies the Hypercyclicity Criterion for full sequence on X.*

We denote by $H_{bs}(\ell_p)$ a Fréchet algebra of symmetric entire functions on ℓ_p which are bounded on bounded subsets. This algebra is the completion of the space of symmetric polynomials on ℓ_p endowed with the uniform topology on bounded subsets.

Theorem 3. *The symmetric translation operator $\mathcal{T}_y$ is hypercyclic on $H_{bs}(\ell_p)$ for every $y \ne 0$.*

Proof. Since $y \ne 0$, $F_{m_0}(y) \ne 0$ for some m_0 [1]. So, $\mathcal{T}_y$ is hypercyclic (and satisfies the Hypercyclicity Criterion for full sequence) on $H_s^m(\ell_p)$ whenever $m \ge m_0$. The set $\bigcup_{m=m_0}^{\infty} H_s^m(\ell_p)$ contains the space of all symmetric polynomials on ℓ_p and so it is dense in $H_{bs}(\ell_p)$. Also $H_s^m(\ell_p) \subset H_s^n(\ell_p)$, if $n > m$. Hence, by Lemma 1, $\mathcal{T}_y$ is hypercyclic. $\square$

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Можировська З.Г. *Гіперциклічні оператори на алгебрі симетричних аналітичних функцій на ℓ_p*. // Карпатські матем. публ. — 2016. — Т.8, №1. — С. 127–133.

В статті запропоновано метод побудови гіперциклічних операторів композиції на просторі $H(\mathbb{C}^n)$ з використанням поліноміальних автоморфізмів на $\mathbb{C}^n$ і симетричних аналітичних функцій на ℓ_p. Зокрема, в роботі показано гіперциклічність оператора "симетричного зсуву" на алгебрі Фреше симетричних цілих функцій на ℓ_p, які є обмеженими на обмежених підмножинах.

Ключові слова і фрази: гіперциклічні оператори, функціональні простори.

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ON THE DIMENSION OF VERTEX LABELING OF k-UNIFORM DCSL OF k-UNIFORM CATERPILLAR

A distance compatible set labeling (dcsl) of a connected graph G is an injective set assignment $f : V(G) \to 2^X$, X being a nonempty ground set, such that the corresponding induced function $f^{\oplus} : E(G) \to 2^X \setminus \{\varnothing\}$ given by $f^{\oplus}(uv) = f(u) \oplus f(v)$ satisfies $| f^{\oplus}(uv) |= k^f_{(u,v)} d_G(u,v)$ for every pair of distinct vertices $u, v \in V(G)$, where $d_G(u,v)$ denotes the path distance between u and v and $k^f_{(u,v)}$ is a constant, not necessarily an integer. A dcsl f of G is k-uniform if all the constant of proportionality with respect to f are equal to k, and if G admits such a dcsl then G is called a k-uniform dcsl graph. The *k-uniform dcsl index* of a graph G, denoted by $\delta_k(G)$ is the minimum of the cardinalities of X, as X varies over all k-uniform dcsl-sets of G. A *linear extension* $\mathbf{L}$ of a partial order $\mathbf{P} = (P, \preceq)$ is a linear order on the elements of P, such that $x \preceq y$ in $\mathbf{P}$ implies $x \preceq y$ in $\mathbf{L}$, for all $x, y \in P$. The dimension of a poset $\mathbf{P}$, denoted by $dim(\mathbf{P})$, is the minimum number of linear extensions on $\mathbf{P}$ whose intersection is '$\preceq$'. In this paper we prove that $dim(\mathcal{F}) \leq \delta_k(P_n^{+k})$, where $\mathcal{F}$ is the range of a k-uniform dcsl of the k-uniform caterpillar, denoted by P_n^{+k} $(n \geq 1, k \geq 1)$ on '$n(k+1)$' vertices.

Key words and phrases: k-uniform dcsl index, dimension of a poset, lattice.

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Introduction

Acharya [1] introduced the notion of vertex *set-valuation* as a set-analogue of number valuation. For a graph $G = (V, E)$ and a nonempty set X, Acharya defined a *set-valuation* of G as an injective *set-valued* function $f : V(G) \to 2^X$, and defined a *set-indexer* $f^{\oplus} : E(G) \to 2^X \setminus \{\varnothing\}$ as a *set-valuation* such that the function given by $f^{\oplus}(uv) = f(u) \oplus f(v)$ for every $uv \in E(G)$ is also injective, where 2^X is the set of all subsets of X and '$\oplus$' is the binary operation of taking the symmetric difference of subsets of X.

Acharya and Germina [2], introduced the particular kind of set-valuation for which a metric, especially the cardinality of the symmetric difference, associated with each pair of vertices is k (where k be a constant) times that of the distance between them in the graph [2]. In other words, determine those graphs $G = (V, E)$ that admit an injective set-valued function $f : V(G) \to 2^X$, where 2^X is the power set of a nonempty set X, such that, for each pair of distinct vertices u and v in G, the cardinality of the symmetric difference $f(u) \oplus f(v)$ is k times

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that of the usual path distance $d_G(u,v)$ between u and v in G, where k is a non-negative constant. They in [2] called such a *set-valuation* f of G a *k-uniform distance-compatible set-labeling* (*k-uniform dcsl*) of G, and the graph G which admits k-uniform dcsl, a *k-uniform distance-compatible set-labeled graph* (*k-uniform dcsl graph*) and the non empty set X corresponding to f, a *k-uniform dcsl-set*. The *k-uniform dcsl index* [4] of a graph G, denoted by $\delta_k(G)$ is the minimum of the cardinalities of X, as X varies over all k-uniform dcsl-sets of G.

Consider a *partially ordered set* or a *poset* $\mathbf{P}$ as a structure $(P, \preceq)$ where P is a nonempty set and '$\preceq$' is a partial order relation on P. We denote $(x, y) \in \mathbf{P}$ by $x \preceq y$, and identify the ground set of a poset with the whole poset. Two elements of $\mathbf{P}$ standing in the relation of $\mathbf{P}$ are called *comparable*, otherwise they are *incomparable*. We denote the incomparable elements x and y of $\mathbf{P}$ by $x \parallel y$. A poset is a *chain* if it contains no incomparable pair of elements, and in this case, the partial order is a *linear order*. A poset is an *antichain* if all of its pairs are incomparable. The length of a chain is one less than the number of elements in the chain. An element $p \in \mathbf{P}$ of a finite poset is on *level k*, if there exists a sequence of elements $p_0, p_1, \ldots, p_k = p$ in $\mathbf{P}$ such that $p_0 \preceq p_1 \preceq ldots \preceq p_k = p$ and any other such sequences in $\mathbf{P}$ has length less than or equal to k. The size of a largest chain in a poset $\mathbf{P}$ is called the *height* of the poset, denoted by $height(\mathbf{P})$ or $h(\mathbf{P})$, and that of a largest antichain is called its *width*, denoted by $width(\mathbf{P})$ or $w(\mathbf{P})$. A *Hasse diagram* of a poset $(P, \preceq)$ is a drawing in which the points of P are placed so that if y *covers* x (we say, z *covers* y if and only if $y \prec z$ and $y \preceq x \preceq z$ implies either $x = y$ or $x = z$), then y is placed at a higher level than x and joined to x by a line segment. A poset $\mathbf{P}$ is *connected*, if its Hasse diagram is connected as a graph. A *Cover graph or Hasse graph* of a poset $(P, \preceq)$ is the graph with vertex set P such that $x, y \in P$ are adjacent if and only if one of them covers the other.

Let $\mathbf{P} = (P, \preceq_P)$ and $\mathbf{Q} = (Q, \preceq_Q)$ be two partially ordered sets. A mapping f from the poset $\mathbf{P}$ to the poset $\mathbf{Q}$ is called *order preserving* if for every two elements x and y of P, $x \preceq_P y$ implies $f(x) \preceq_Q f(y)$. A poset $\mathbf{Q}$ is a subposet of $\mathbf{P}$ if $Q \subseteq P$, and $\preceq_Q$ is the restriction of $\preceq_P$ to $Q \times Q$. i.e., for $a, b \in Q$, $a \preceq_Q b$ if and only if $a \preceq_P b$. Two posets $\mathbf{P}$ and $\mathbf{Q}$ are called *isomorphic* if there is a one to one order preserving mapping Φ from the poset $\mathbf{P}$ onto the poset $\mathbf{Q}$ such that for every two elements x and y of P, $x \preceq_P y$ in $\mathbf{P}$ if and only if $\Phi(x) \preceq_Q \Phi(y)$ in $\mathbf{Q}$. The poset $\mathbf{Q}$ is said to be *embedded* or *contained* in $\mathbf{P}$, denoted by $\mathbf{Q} \sqsubseteq \mathbf{P}$, if $\mathbf{Q}$ is isomorphic to a subposet of $\mathbf{P}$. Let $\mathbf{R}$ and $\mathbf{S}$ are two partial orders (with respect to $\preceq$) on the same set X, we call $\mathbf{S}$ an *extension* of $\mathbf{R}$ if $\mathbf{R} \subseteq \mathbf{S}$, i.e., $x \preceq y$ in $\mathbf{R}$ implies $x \preceq y$ in $\mathbf{S}$ for all $x, y \in X$. In particular if $\mathbf{S}$ is a chain, then we call it as a *linear extension* of $\mathbf{R}$. For convenience, let $\mathbf{L} : [x_1, x_2, \ldots, x_n]$ denote linear order on $\{x_1, x_2, \ldots, x_n\}$ in which $x_1 \preceq x_2 \preceq \cdots \preceq x_n$.

Definition 1 ([8]). *A set $\mathcal{R} = \{\mathbf{L}_1, \mathbf{L}_2, \ldots, \mathbf{L}_k\}$ of linear extensions of $\mathbf{P}$ is a **realizer** of $\mathbf{P}$ if for every incomparable pair $x, y \in \mathbf{P}$, there are $\mathbf{L}_i, \mathbf{L}_j \in \mathcal{R}$ with $x \preceq y$ in $\mathbf{L}_i$ and $x \succeq y$ in $\mathbf{L}_j$ for $1 \leq i \neq j \leq k$. The **dimension** of $\mathbf{P}$ (denoted by $dim(\mathbf{P})$) is the minimum cardinality of a realizer.*

There are equivalent definitions for $dim(\mathbf{P})$. It is defined as the minimum k for which there are linear extensions $\mathbf{L}_1, \ldots, \mathbf{L}_k$ such that $\mathbf{P} = \mathbf{L}_1 \cap \mathbf{L}_2 \cap \cdots \cap \mathbf{L}_k$, where the intersection is taken over the sets of relations of $\mathbf{L}_i$, for $1 \leq i \leq k$. Another characterization of dimension, in terms of coordinates, is obtained by using an embedding of $\mathbf{P}$ into R^t (called t-dimensional poset) [11]. Let R^t denotes the poset of all t-tuples of real numbers, partially ordered by inequality in each coordinate: $(a_1, a_2, \ldots, a_t) \leq (b_1, b_2, \ldots, b_t)$ if and only if $a_i \leq b_i$, for $i = 1, 2, \ldots, t$. Then the dimension of a poset $\mathbf{P}$ is the minimum number t such that $\mathbf{P}$ is embedded in $\mathbf{R^t}$, denoted as $\mathbf{P} \sqsubseteq \mathbf{R^t}$. For more results on dimension of poset one may see [7, 9, 12, 13].

A poset $(L, \preceq)$ is a *lattice* if every pair of elements $x, y \in L$, has a *least upper bound* (*lub*), denoted by $x \vee y$ (called join), and a *greatest lower bound* (*glb*), denoted by $x \wedge y$ (called meet). In general, a lattice is denoted by $(L, \preceq)$. Throughout this paper lattice (and poset) means lattice (and poset) under set inclusion $\subseteq$. Unless otherwise mentioned, for all the terminology in graph theory and lattice theory, the reader is asked to refer, respectively [5, 6].

This paper initiates a study on the dimension of vertex labeling of k-uniform dcsl of k-uniform caterpillar, and prove that $dim(\mathcal{F}) \leq \delta_k(P_n^{+k})$, where $\mathcal{F}$ is the range of a k-uniform dcsl of the k-uniform caterpillar, denoted by P_n^{+k} $(n \geq 1, k \geq 1)$ on '$n(k+1)$' vertices that forms a poset under set inclusion $\subseteq$.

Following are the definitions and results used in this paper.

Definition 2 ([10]). *The height-2 poset H_n on $2n$ elements $a_1, \ldots, a_n, b_1, \ldots, b_n$ is the poset of height two consisting of two antichains $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$ such that $b_i \preceq a_j$ in H_n exactly if $i = j$, and $j = i - 1$.*

Proposition 1 ([10]). *For $n \geq 2$, $dim(H_n) = 2$.*

Proposition 2 ([10]). *Let $\mathcal{F}$ be the range of a vertex labeling of 1-uniform dcsl path $P_n (n > 2)$, which is embedded in H_n, then $dim(\mathcal{F}) = 2$.*

Definition 3 ([10]). *A width-2 poset W_n is the poset $(\{a_1, \ldots, a_n, b_1, \ldots, b_n\}, \preceq)$ of width two consisting of two chains $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$ such that $a_{i-1} \preceq a_i$ for $2 \leq i \leq n$, $b_i \preceq b_{i+1}$ for $1 \leq i \leq n-1$, $a_1 \preceq b_i$ for $1 \leq i \leq n$, and for $2 \leq i \leq n$ and $1 \leq j \leq n$, $a_i \mid\mid b_j$.*

Proposition 3 ([10]). *For $n \geq 2$, $dim(W_n) = 2$.*

Proposition 4 ([10]). *Let $\mathcal{F}$ be the range of a vertex labeling of 1-uniform dcsl path $P_n (n > 2)$, which is embedded in W_n, then $dim(\mathcal{F}) = 2$.*

Lemma 1 ([3]). *$\delta_d(P_n) = n - 1$, for $n > 2$.*

Lemma 2 ([10]). *$\delta_k(P_n) = k(n-1)$, for $n > 2$.*

1 MAIN RESULTS

Since the existence of vertex labeling of 1-uniform dcsl graph is not unique, the problem of determining posets which embeds the vertex labeling of 1-uniform dcsl of k-uniform caterpillar is same as determining the existence of different vertex labels f of 1-uniform dcsl of k-uniform caterpillar whose corresponding range, say $\mathcal{F} = Range(f)$ forms a poset under set inclusion $\subseteq$. Thus, there is a one to one correspondence between the vertex labeling f of 1-uniform dcsl of k-uniform caterpillar and its corresponding poset $\mathcal{F}$. Thus, it is always possible to find a 1-uniform dcsl f of a graph G so that $\mathcal{F} = Range(f)$ forms a poset under set inclusion $\subseteq$. Hence, $\mathcal{F}$ contains the vertex labeling f of 1-uniform dcsl graph G as an embedding of itself. Hence, the problem of determining the 1-uniform dcsl vertex labeling f of a graph G is equivalent in determining the poset $\mathcal{F}$ which embeds the 1-uniform dcsl vertex labeling f of the same graph G.

Definition 4. *Let $\mathbf{P} = (\{a_1, \dots, a_n\}, \preceq)$ be a poset. We define k-uniform extended poset or, simply, k-extended poset of $\mathbf{P}$, denoted by $\mathbf{P}^k$ as*

$$(\{a_1, a_1^1, a_1^2, \dots, a_1^k, a_2, a_2^1, a_2^2, a_2^k, \dots, a_n, a_n^1, a_n^2, \dots, a_n^k\}, \preceq),$$

which is an extension of $\mathbf{P}$, and for $1 \leq i \leq n$, each $k(\geq 1)$ elements a_i^1, a_i^2, …, a_i^k of $\mathbf{P}^k$ covers only a_i. We call $\mathbf{P}$ as an underline poset of $\mathbf{P}^k$.

Remark 1. *It is interesting to note the following in a k-extended posets.*

(i) If there exist any two distinct elements which belong to the same level in $\mathbf{P}^k$, then they are incomparable.

(ii) For each $k(\geq 1)$ elements a_i^1, a_i^2, …, a_i^k of $\mathbf{P}^k$ covers only a_i, where $1 \leq i \leq n$. This implies that there exist no element in $\mathbf{P}^k$ that covers any one of the k elements $a_i^1, a_i^2, \dots, a_i^k$. Hence, the k elements $a_i^1, a_i^2, \dots, a_i^k$ are maximal elements of $\mathbf{P}^k$. Thus, they are the nk maximal elements, namely, a_i^j in $\mathbf{P}^k$, $1 \leq i \leq n$ and $1 \leq j \leq k$.

Proposition 5. *For any poset $\mathbf{P}$ (finite and connected) of size greater than 1, the k-extended poset $\mathbf{P}^k(k \geq 1)$ of $\mathbf{P}$, does not form a lattice.*

Proof. If possible let, $\mathbf{P}^k$ forms a lattice, then $\mathbf{P}^k$ has unique glb and unique lub, say g and l respectively. Since l is the lub of $\mathbf{P}^k$, $x \preceq l$, for every $x \in \mathbf{P}^k$, which in turn implies one of the element from the maximal elements $a_n^1, a_n^2, \dots, a_n^k$ of $\mathbf{P}^k$ should be equal to l, say, a_n^1. Hence for $2 \leq i \leq n$, we have $a_n^i \preceq l$ which is a contradiction as remarked in Remark 1. $\square$

Proposition 6. *Let $\mathbf{P}$ be a linear order as of the form: $a_{i-1} \preceq a_i$, for $2 \leq i \leq n$, then the dimension of k-extended poset $\mathbf{P}^k(k \geq 1)$ of $\mathbf{P}$ is 2.*

Proof. Let $\mathcal{R} = \{\mathbf{L}_1, \mathbf{L}_2\}$ be linear extensions of $\mathbf{P}^k$, where
$\mathbf{L}_1 : [a_1, a_1^1, \dots, a_1^k, a_2, a_2^1, \dots, a_2^k, \dots, a_n, a_n^1, \dots, a_n^k]$ and
$\mathbf{L}_2$: $[a_1, \dots, a_n, a_n^k, \dots, a_n^1, a_{n-1}^k, \dots, a_{n-1}^1, \dots, a_1^k, \dots, a_1^1]$.
Then $\mathcal{R}$ is a realizer of $\mathbf{P}^k$, and hence $dim(\mathbf{P}^k) \leq 2$. We prove that there is no proper subset $\mathcal{S}$ of $\mathcal{R}$ which realizes $\mathbf{P}^k$. For, if there is a proper subset $\mathcal{S}$ of $\mathcal{R}$ which realizes $\mathbf{P}^k$, then, the only one member in $\mathcal{S}$ give rise to the poset $\mathbf{P}^k$, and hence, all the elements of $\mathbf{P}^k$ are comparable, which is a contradiction. Hence $dim(\mathbf{P}^k) = 2$. $\square$

Since the graph P_n^{+k} is the extension of P_n, the k-extended poset can embed the vertex labeling of a 1-uniform dcsl k-uniform caterpillar only when its corresponding underline poset embed the vertex labeling of a 1-uniform dcsl path.

Now, we aim to determine the dimension of k-extended posets which embeds the vertex labeling of a 1-uniform dcsl of a k-uniform caterpillar.

Proposition 7. *Let $\mathbf{P}$ be a linear order as $a_{i-1} \preceq a_i$, for $2 \leq i \leq n$, then the k-extended poset $\mathbf{P}^k$ embeds the vertex labeling of a 1-uniform dcsl of the k-uniform caterpillar.*

Proof. Let $G = P_n^{+k}$ be the k-uniform caterpillar with $n(k+1)$ vertices, where $n \geq 2$ and $k \geq 1$. Let $V(G) = \{v_i, v_i^j \mid 1 \leq i \leq n, 1 \leq j \leq k\}$, where v_i are the internal vertices and v_i^j are the pendant vertices which are adjacent to v_i.

First we claim that there exist a vertex labeling f of a 1-uniform dcsl of the k-uniform caterpillar, whose range is suitable for the embedding of k-extended poset $\mathbf{P}^k$. Let $X = \{1, 2, \ldots, n(k+1) - 1\}$. Define $f : V(G) \to 2^X$ such that $f(v_1) = \varnothing$ and $f(v_j) = \{1, 2, \ldots, j-1\}, 2 \leq j \leq n$. For, $1 \leq i \leq n$ and $1 \leq j \leq k$,

$$f(v_i^j) = f(v_i) \cup \{(n-1) + (i-1)k + j\} = \{1, 2, \ldots, i-1, (n-1) + (i-1)k + j\}.$$

Case 1: When $u = v_l$ and $v = v_m$, $l = 1$ and $2 \leq m \leq n$. Then,

$$|f(v_l) \oplus f(v_m)| =| \varnothing \oplus \{1, 2, \ldots, m-1\} |=| \{1, 2, \ldots, m-1\} |= m - l = d(v_l, v_m).$$

Case 2: When $u = v_l$ and $v = v_m$, $l \neq m, 2 \leq l,\ m \leq n$. Then,

$$\begin{aligned}|f(v_l) \oplus f(v_m)| &=| \{1, 2, \ldots, l-1\} \oplus \{1, 2, \ldots, m-1\} | \\ &=| \{l, l+1, \ldots, m-1\} |= m - l = d(v_l, v_m), \quad 2 \leq l < m \leq n.\end{aligned}$$

Case 3: When $u = v_l$ and $v = v_m^j$, $l = 1, 2 \leq m \leq n$ and $1 \leq j \leq k$. Then,

$$\begin{aligned}|f(v_l) \oplus f(v_m^j)| &=| \varnothing \oplus \{1, 2, \ldots, m-1, (n-1) + (m-1)k + j\} | \\ &=| \{1, 2, \ldots, m-1, (n-1) + (m-1)k + j\} |= m = d(v_l, v_m^j).\end{aligned}$$

Case 4: When $u = v_l$ and $v = v_m^j$, $l \neq m, 2 \leq l,\ m \leq n$ and $1 \leq j \leq k$. Then,

$$\begin{aligned}|f(v_l) \oplus f(v_m^j)| &=| \{1, 2, \ldots, l-1\} \oplus \{1, 2, \ldots, m-1, (n-1) + (m-1)k + j\} | \\ &=| \{l, l+1, \ldots, m-1, (n-1) + (m-1)k + j\} | \\ &= m - l + 1 = d(v_l, v_m^j), \quad 2 \leq l < m \leq n \text{ and } 1 \leq j \leq k.\end{aligned}$$

Case 5: When $u = v_l^i$ and $v = v_m^j$, $l = 1, 2 \leq m \leq n$ and $1 \leq i, j \leq k$. Then,

$$\begin{aligned}|f(v_l^i) \oplus f(v_m^j)| &=| \{(n-1) + (l-1)k + i\} \\ &\oplus \{1, \ldots, m-1, (n-1) + (m-1)k + j\} | \\ &=| \{1, \ldots, m-1, (n-1) + (m-1)k + j, (n-1) + (l-1)k + i\} |= m + 1 = d(v_l^i, v_m^j).\end{aligned}$$

Case 6: When $u = v_l^i$ and $v = v_m^j$, $l \neq m, 2 \leq l,\ m \leq n$ and $1 \leq i, j \leq k$. Then,

$$\begin{aligned}f(v_l^i) \oplus f(v_m^j)| &=| \{1, \ldots, l-1, (n-1) + (l-1)k + i\} \\ &\oplus \{1, \ldots, m-1, (n-1) + (m-1)k + j\} | \\ &=| \{(n-1) + (l-1)k + i, l, l+1, ., m-1, (n-1) + (m-1)k + j\} | \\ &= m - l + 2 = d(v_l^i, v_m^j), \quad 2 \leq l < m \leq n \text{ and } 1 \leq i \leq j \leq k.\end{aligned}$$

Hence, for any distinct $u, v \in V(G)$, $|f(u) \oplus f(v)|$ = d(u, v). Thus, f is a 1-uniform dcsl of G.

Now, to prove, $\mathcal{F} \sqsubseteq \mathbf{P}^k$, where $\mathcal{F}$ is the range of f which forms a poset under '$\subseteq$' and $\mathbf{P}$ a linear order as $a_{i-1} \preceq a_i$, $2 \leq i \leq n$. Define $\Phi : \mathcal{F} \to \mathbf{P}^k$ as follows.

Case 1. On the internal vertices v_i of $V(G)$, define $\Phi(f(v_i)) = a_i$.
Case 2. On the pendant vertices v_i^j of $V(G)$, define $\Phi(f(v_i^j)) = a_i^j$.

In Case 1, the corresponding vertex labels of a pair of internal vertices are comparable where as in Case 2, for any pair of pendant vertices the corresponding vertex labels are incomparable. Hence, $f(v_i) \subseteq f(v_j)$ in $\mathcal{F}$ if and only if $a_i \preceq a_j$ in $\mathbf{P}^k$ and $f(v_i^r) \parallel f(v_i^s)$ in $\mathcal{F}$ if and only if $a_i^r \parallel a_i^s$ in $\mathbf{P}^k$. Also, $f(v_i) \subseteq f(v_i^j)$ in $\mathcal{F}$ if and only if $a_i \preceq a_i^j$ in $\mathbf{P}^k$ and $f(v_i) \parallel f(v_{i-1}^s)$ in $\mathcal{F}$ if and only if $a_i \parallel a_{i-1}^s$ in $\mathbf{P}^k$. Therefore, $\mathcal{F} \sqsubseteq \mathbf{P}^k$. □

Using Proposition 6 and Proposition 7, we have the following result.

Proposition 8. *Let $\mathcal{F}$ be the range of a 1-uniform dcsl of the k-uniform caterpillar such that $\mathcal{F} \sqsubseteq \mathbf{P}^k$, where $\mathbf{P}$ is a linear order of finite length. Then $dim(\mathcal{F}) = 2$.*

Remark 2. *From Proposition 2 and Proposition 4, we have seen that the height-2 poset, H_n and width-2 poset, W_n on '$2n$' elements embeds the vertex labeling of a 1-uniform dcsl path. Choosing these posets as underline posets defined on 'n' elements, the corresponding k-extended posets embedding, restricted to height-2 poset and width-2 poset on n elements, give two subposets, namely min height poset (denoted by Min_n) and avg height poset(denoted by Avg_n), respectively. Further, the poset Min_n end up with $b_{\lceil \frac{n}{2} \rceil}$, when n is odd; $a_{\frac{n}{2}}$ if n is even. Hence, $Min_n \sqsubseteq H_n$. For the poset Avg_n, $Avg_n \sqsubseteq W_n$. For, without loss of generality, consider the poset as $(\{a_1, \ldots, a_{\lceil \frac{n}{2} \rceil = h}, b_1, \ldots, b_{n-h}\}, \preceq)$ of width two consisting of two chains $A = \{a_1, \ldots, a_h\}$ and $B = \{b_1, \ldots, b_{n-h}\}$ such that a_{i-1} preceqa_i for $2 \leq i \leq h$, $b_i \preceq b_{i+1}$ for $1 \leq i \leq n-h-1$, $a_1 \preceq b_i$ for $1 \leq i \leq n-h$, and for $2 \leq i \leq h$ and $1 \leq j \leq n-h$, $a_i \,||\, b_j$. In particular, if the underline poset is of linear order, then it posses maximum height and by Proposition 6, the k-extended poset of it has dimension 2.*

Proposition 9. *For a k-extended poset Min_n, $dim(Min_n^k) = 2$.*

Proof. We define the linear extensions $\mathbf{L}_1$ and $\mathbf{L}_2$ of Min_n^k, in two cases.
Case 1: When n is even. Consider,

$$\mathbf{L}_1 : [b_1, b_1^1, \ldots, b_1^k, b_2, b_2^1, \ldots, b_2^k, \ldots, b_{\frac{n}{2}}, b_{\frac{n}{2}}^1, \ldots, b_{\frac{n}{2}}^k, a_1, a_1^1, \ldots, a_1^k, a_2, a_2^1, \ldots, a_2^k, \ldots, a_{\frac{n}{2}}, a_{\frac{n}{2}}^1, \ldots, a_{\frac{n}{2}}^k] \text{ and}$$
$$\mathbf{L}_2 : [b_{\frac{n}{2}}, a_{\frac{n}{2}}, b_{\frac{n}{2}-1}, a_{\frac{n}{2}-1}, \ldots, b_1, a_1, a_{\frac{n}{2}}^k, \ldots, a_{\frac{n}{2}}^1, a_{\frac{n}{2}-1}^k, \ldots, a_{\frac{n}{2}-1}^1, \ldots, a_1^k, \ldots, a_1^1, b_{\frac{n}{2}}^k, \ldots, b_{\frac{n}{2}}^1, b_{\frac{n}{2}-1}^k, \ldots, b_{\frac{n}{2}-1}^1, \ldots, b_1^k, \ldots, b_1^1].$$

Since, these extensions intersect to yield the partial order on Min_n^k, $dim(Min_n^k) \leq 2$.
Case 2: When n is odd. Consider,

$$\mathbf{L}_1 : [b_{\lceil \frac{n}{2} \rceil}, b_{\lceil \frac{n}{2} \rceil}^1, \ldots, b_{\lceil \frac{n}{2} \rceil}^k, b_{\lceil \frac{n}{2} \rceil-1}, b_{\lceil \frac{n}{2} \rceil-1}^1, \ldots, b_{\lceil \frac{n}{2} \rceil-1}^k, \ldots, b_1, b_1^1, \ldots, b_1^k, a_{\lceil \frac{n}{2} \rceil-1}, a_{\lceil \frac{n}{2} \rceil-1}^1, \ldots, a_{\lceil \frac{n}{2} \rceil-1}^k, \ldots, a_1, a_1^1, \ldots, a_1^k] \text{ and}$$
$$\mathbf{L}_2 : [b_1, a_1, b_2, a_2, \ldots, b_{\lceil \frac{n}{2} \rceil-1}, a_{\lceil \frac{n}{2} \rceil-1}, b_{\lceil \frac{n}{2} \rceil}, a_1^k, \ldots, a_1^1, a_2^k, \ldots, a_2^1, \ldots, a_{\lceil \frac{n}{2} \rceil-1}^k, \ldots, a_{\lceil \frac{n}{2} \rceil-1}^1, b_1^k, \ldots, b_1^1, b_2^k, \ldots, b_2^1, \ldots, b_{\lceil \frac{n}{2} \rceil}^k, \ldots, b_{\lceil \frac{n}{2} \rceil}^1].$$

Clearly, these extensions produces a realizer of Min_n^k, hence $dim(Min_n^k) \leq 2$. Following as in the proof of Proposition 6, the dimension cannot be less than 2. Therefore, $dim(Min_n^k) = 2$. □

Proposition 10. *The k-extended poset Min_n^k embeds the vertex labeling of a 1-uniform dcsl of the k-uniform caterpillar.*

Proof. Let $V(P_n^k) = \{v_1, v_1^1, \ldots, v_1^k, v_2, v_2^1, \ldots, v_2^k \ldots, v_n, v_n^1, \ldots, v_n^k\}$, where v_i are the internal vertices and v_i^j are the pendant vertices which are adjacent to v_i.

Let $X = \{1, 2, \ldots, w, \ldots, n, \ldots, m = n(k+1) - 1\}$, where $w = \lceil \frac{|V(P_n)|}{2} \rceil$.

We claim that there exists a poset $\mathcal{F}$ which can be obtained from a vertex labeling of 1-uniform dcsl caterpillar, that suits for the embedding of Min_n^k.

Define $f : V(P_n^k) \to 2^X$, on internal vertices, by

$$f(v_1) = \{1, 2, \ldots, w-1\}, f(v_2) = \{1, 2, \ldots, w-1, w\}, f(v_3) = \{2, \ldots, w-1, w\},$$
$$f(v_4) = \{2, \ldots, w-1, w, w+1\}, f(v_5) = \{3, \ldots, w, w+1\}, \ldots, f(v_n) = \{w, w+1, \ldots, n-1\},$$

when n is odd; otherwise, $f(v_n) = \{w, w+1, \ldots, n\}$. In general, for $1 \leq i \leq n$,

$$f(v_i) = \begin{cases} \{ \frac{i+1}{2}, \frac{i+1}{2}+1, \ldots, \frac{i+1}{2}+w-2 \}, & \text{if } i \text{ is odd} \\ \{ \frac{i}{2}, \frac{i}{2}+1, \ldots, \frac{i}{2}+w-1 \}, & \text{otherwise,} \end{cases}$$

and on pendant vertices, vertex labeling is same, as in Proposition 7.

Case 1: When $u = v_i$ and $v = v_{i+1}$, where i is odd. Then,

$$\begin{aligned} |f(v_i) \oplus f(v_{i+1})| &= \mid \{\frac{i+1}{2}, \ldots, \frac{i+1}{2} + w - 2\} \oplus \{\frac{i+1}{2}, \ldots, \frac{i+1}{2} + w - 1\} \mid \\ &= \mid \{\frac{i+1}{2} + w - 1\} \mid = 1 = d(v_i, v_{i+1}). \end{aligned}$$

Case 2: When $u = v_{i+1}$ and $v = v_i$, where i is even. Then,

$$\begin{aligned} |f(v_{i+1}) \oplus f(v_i)| &= \mid \{\frac{i+2}{2}, \ldots, \frac{i+2}{2} + w - 2\} \oplus \{\frac{i}{2}, \ldots, \frac{i}{2} + w - 1\} \mid \\ &= \mid \{\frac{i}{2}\} \mid = 1 = d(v_{i+1}, v_i). \end{aligned}$$

Case 3: When $u = v_l$ and $v = v_m$, $l \neq m$, $1 \leq l,\ m \leq n$ and both l and m are odd. Then,

$$\begin{aligned} |f(v_l) \oplus f(v_m)| &= \mid \{\frac{l+1}{2}, \ldots, \frac{l+1}{2} + w - 2\} \oplus \{\frac{m+1}{2}, \ldots, \frac{m+1}{2} + w - 2\} \mid \\ &= \mid \{\frac{l+1}{2}, \ldots, \frac{m+1}{2} + w - 2\} \mid = m - l = d(v_l, v_m), \quad 1 \leq l < m \leq n. \end{aligned}$$

Case 4: When $u = v_l$ and $v = v_m$, $l \neq m$, $1 \leq l,\ m \leq n$ and both l and m are even. Then,

$$\begin{aligned} |f(v_l) \oplus f(v_m)| &= \mid \{\frac{l}{2}, \ldots, \frac{l}{2} + w - 1\} \oplus \{\frac{m}{2}, \ldots, \frac{m}{2} + w - 1\} \mid \\ &= \mid \{\frac{l}{2}, \ldots, \frac{m}{2} + w - 1\} \mid = m - l = d(v_l, v_m), \quad 1 \leq l < m \leq n. \end{aligned}$$

Case 5: When $u = v_i$ and $v = v_i^j$, $1 \leq i \leq n$ and $1 \leq j \leq k$. Then,

$$|f(v_i) \oplus f(v_i^j)| = \mid \{n + (i-1)k + (j-1)\} \mid = 1 = d(v_i, v_i^j).$$

Case 6: When $u = v_i$ and $v = v_{i+1}^j$, $1 \leq j \leq k$ and i is odd. Then,

$$\begin{aligned}
|f(v_i) \oplus f(v_{i+1}^j)| =& \mid \{\frac{i+1}{2}, \ldots, \frac{i+1}{2} + w - 2\} \\
& \oplus \{\frac{i+1}{2}, \ldots, \frac{i+1}{2} + w - 1, n + (i)k + (j-1)\} \mid \\
& = \mid \{\frac{i+1}{2} + w - 1, n + (i)k + (j-1)\} \mid = 2 = d(v_i, v_{i+1}^j).
\end{aligned}$$

Case 7: $u = v_{i+1}$ and $v = v_i^j$, $1 \leq j \leq k$ and i is even. Then,

$$\begin{aligned}
|f(v_{i+1}) \oplus f(v_i)| =& \mid \{\frac{i+2}{2}, \frac{i+2}{2} + 1, \ldots, \frac{i+2}{2} + w - 2\} \\
& \oplus \{\frac{i}{2}, \frac{i}{2} + 1, \ldots, \frac{i}{2} + w - 1, n + (i-1)k + (j-1)\} \mid \\
& = \mid \{\frac{i}{2}, n + (i-1)k + (j-1)\} \mid = 2 = d(v_{i+1}, v_i^j).
\end{aligned}$$

Case 8: When $u = v_l$ and $v = v_m^j$, $l \neq m$, $1 \leq l,\ m \leq n$, $1 \leq j \leq k$ and both l and m are odd. Then,

$$\begin{aligned}
|f(v_l) \oplus f(v_m^j)| =& \mid \{\frac{l+1}{2}, \frac{l+1}{2} + 1, \ldots, \frac{l+1}{2} + w - 2\} \\
& \oplus \{\frac{m+1}{2}, \frac{m+1}{2} + 1, \ldots, \frac{m+1}{2} + w - 2, n + (m-1)k + (j-1)\} \mid \\
& = \mid \{\frac{l+1}{2}, \ldots, \frac{m+1}{2} + w - 2, n + (m-1)k + (j-1)\} \mid = m - l + 1 = d(v_l, v_m^j), \\
& 1 \leq l < m \leq n \text{ and } 1 \leq j \leq k.
\end{aligned}$$

Case 9: When $u = v_l$ and $v = v_m^j$, $l \neq m$, $1 \leq l,\ m \leq n$, $1 \leq j \leq k$ and both l and m are even. Then,

$$\begin{aligned}
|f(v_l) \oplus f(v_m^j)| =& \mid \{\frac{l}{2}, \frac{l}{2} + 1, \ldots, \frac{l}{2} + w - 1\} \\
& \oplus \{\frac{m}{2}, \frac{m}{2} + 1, \ldots, \frac{m}{2} + w - 1, n + (m-1)k + (j-1)\} \mid \\
& = \mid \{\{\frac{l}{2}, \ldots, \frac{m}{2} + w - 1, n + (m-1)k + (j-1)\} \mid = m - l + 1 = d(v_l, v_m^j), \\
& 1 \leq l < m \leq n \text{ and } 1 \leq j \leq k.
\end{aligned}$$

Case 10: When $u = v_i^r$ and $v = v_{i+1}^s$, $1 \leq r,\ s \leq k$ and i is odd. Then,

$$\begin{aligned}
|f(v_i^r) \oplus f(v_{i+1}^s)| =& \mid \{\frac{i+1}{2}, \ldots, \frac{i+1}{2} + w - 2, n + (i-1)k + (r-1)\} \\
& \oplus \{\frac{i+1}{2}, \ldots, \frac{i+1}{2} + w - 1, n + (i)k + (s-1)\} \mid \\
& = \mid \{n + (i-1)k + (r-1), \frac{i+1}{2} + w - 1, n + (i)k + (s-1)\} \mid = 3 = d(v_i^r, v_{i+1}^s).
\end{aligned}$$

Case 11: $u = v^r_{i+1}$ and $v = v^s_i$, $1 \le r,\ s \le k$ and i is even. Then,

$$\begin{aligned}|f(v^r_{i+1}) \oplus f(v^s_i)| &= \mid \{\frac{i+2}{2},\dots,\frac{i+2}{2}+w-2, n+(i)k+(r-1)\}\\ &\oplus \{\frac{i}{2},\dots,\frac{i}{2}+w-1, n+(i-1)k+(j-1)\} \mid\\ &= \mid \{\frac{i}{2}, n+(i)k+(r-1), n+(i-1)k+(s-1)\} \mid = 3 = d(v^r_{i+1}, v^s_i).\end{aligned}$$

Case 12: When $u = v^i_l$ and $v = v^j_m$, $l \ne m$, $1 \le l,\ m \le n$, $1 \le i,\ j \le k$ and both l and m are odd. Then,

$$\begin{aligned}|f(v^i_l) \oplus f(v^j_m)| &= \mid \{\frac{l+1}{2},\dots,\frac{l+1}{2}+w-2, n+(l-1)k+(i-1)\}\\ &\oplus \{\frac{m+1}{2},\dots,\frac{m+1}{2}+w-2, n+(m-1)k+(j-1)\} \mid\\ &= \mid \{\{\frac{l+1}{2},\dots,\frac{m+1}{2}+w-2, n+(l-1)k+(i-1), n+(m-1)k+(j-1)\} \mid\\ &= m-l+2 = d(v^i_l, v^j_m), \quad 1 \le l < m \le n \text{ and } 1 \le i,j \le k.\end{aligned}$$

Case 13: When $u = v^i_l$ and $v = v^j_m$, $l \ne m$, $1 \le l,\ m \le n$, $1 \le i,\ j \le k$ and both l and m are even. Then,

$$\begin{aligned}|f(v^i_l) \oplus f(v^j_m)| &= \mid \{\frac{l}{2},\dots,\frac{l}{2}+w-1, n+(l-1)k+(i-1)\}\\ &\oplus \{\frac{m}{2},\dots,\frac{m}{2}+w-1, n+(m-1)k+(j-1)\} \mid\\ &= \mid \{\{\frac{l}{2},\dots,\frac{m}{2}+w-1, n+(l-1)k+(i-1), n+(m-1)k+(j-1)\} \mid\\ &= m-l+2 = d(v^i_l, v^j_m), \quad 1 \le l < m \le n \text{ and } 1 \le i,j \le k.\end{aligned}$$

Thus, for any distinct $u, v \in V(P^k_n)$, $|f(u) \oplus f(v)|$ = d(u,v) and hence f admits 1-uniform dcsl. Also, to prove $\mathcal{F} \sqsubseteq Min^k_n$, where $\mathcal{F}$ is the range of f, which forms a poset, we define $\Phi : \mathcal{F} \to Min^k_n$ as follows in two different cases.

Case 1. On the internal vertices v_i of $V(P^k_n)$. $\Phi(f(v_i)) = \begin{cases} a_{\frac{i}{2}}, & \text{if } i \text{ is even,}\\ b_{\lceil \frac{i}{2} \rceil}, & \text{otherwise.}\end{cases}$

Case 2. On the pendant vertices v^j_i of $V(P^k_n)$. $\Phi(f(v^j_i)) = \begin{cases} a^j_{\frac{i}{2}}, & \text{if } i \text{ is even,}\\ b^j_{\lceil \frac{i}{2} \rceil}, & \text{otherwise.}\end{cases}$

In Case 1, the internal vertex labeling of $V(P^k_n)$, exhibits the embedding of $\mathcal{F}$ into the underline poset of Min^k_n; and in Case 2, the pendent vertex labeling of $V(P^k_n)$, exhibits the embedding of $\mathcal{F}$ into the outermost labeling of an underline set of Min^k_n. Thus, all together, we get $\mathcal{F} \sqsubseteq Min^k_n$. □

Analogously, from Proposition 9 and Proposition 10, we have.

Proposition 11. *Let $\mathcal{F}$ be the range of a 1-uniform dcsl of the k-uniform caterpillar such that $\mathcal{F} \sqsubseteq Min_n^k$. Then $dim(\mathcal{F}) = 2$.*

Proposition 12. *For the k-extended poset Avg_n^k, $dim(Avg_n^k) = 2$.*

Proof. Let us take the linear extensions of Avg_n^k as

$$\begin{aligned}\mathbf{L}_1 :&[a_1, a_1^1, \dots, a_1^k, a_2, a_2^1, \dots, a_2^k, \dots, a_h, a_h^1, \dots, a_h^k, b_1, b_1^1, \dots, b_1^k, b_2, b_2^1, \dots, b_2^k, \dots, b_{n-h},\\ &b_{n-h}^1, \dots, b_{n-h}^k] \text{ and}\\ \mathbf{L}_2 :&[a_1, b_1, b_2, \dots, b_{n-h}, a_2, \dots, a_h, b_{n-h}^k, \dots, b_{n-h}^1, b_{n-h-1}^k, \dots, b_{n-h-1}^1, \dots, b_1^k, \dots, b_1^1,\\ &a_h^k, \dots, a_h^1, a_{h-1}^k, \dots, a_{h-1}^1, \dots, a_1^k, \dots, a_1^1].\end{aligned}$$

Then dimension of Avg_n^k is at most 2. Again, as in Proposition 6 the dimension cannot be less than 2. Hence $dim(Avg_n^k) = 2$. □

Proposition 13. *The k-extended poset Avg_n^k embeds the vertex labeling of a 1-uniform dcsl of the k-uniform caterpillar.*

Proof. Let $v_1, v_1^1, \dots, v_1^k, v_2, v_2^1, \dots, v_2^k, \dots, v_n, v_n^1, \dots,$ and v_n^k be the vertices of $V(P_n^k)$.

Let $X = \{1, 2, \dots, h, \dots, n, \dots, m = n(k+1) - 1\}$, where $h = \lceil \frac{|V(P_n)|}{2} \rceil$. To prove the existence of a poset $\mathcal{F}$ from a vertex labeling of 1-uniform dcsl of the k-uniform caterpillar, that suits for the embedding of Avg_n^k, define $f : V(P_n^k) \to 2^X$, on internal vertices, by

$$\begin{aligned}&f(v_j) = \{1, \dots, n-h-(j-1)\},\ 1 \le j \le n-h, \quad f(v_{n-h+1}) = \varnothing,\\ &f(v_{n-h+i}) = \{n-h+1, \dots, n-h+(i-1)\},\ 2 \le i \le h\end{aligned}$$

and we consider the vertex labeling on pendant vertices which is same as mentioned in Proposition 7.

Case 1: When $u = v_l$ and $v = v_m$, $l \neq m$, $1 \le l \le n-h$ and $m = n-h+1$. Then,

$$\begin{aligned}|f(v_l) \oplus f(v_m)| &=| \{1, \dots, n-h-(l-1)\} \oplus \varnothing |\\ &=| \{1, \dots, n-h-(l-1)\} |= n-h-(l-1) = d(v_l, v_m).\end{aligned}$$

Case 2: When $u = v_l$ and $v = v_m$, $l \neq m$, $n-h+2 \le l \le n$ and $m = n-h+1$. Then,

$$\begin{aligned}|f(v_l) \oplus f(v_m)| &=| \{n-h+1, \dots, l-1\} \oplus \varnothing |\\ &=| \{n-h+1, \dots, l-1 = n-h+(l-m)\} |= l-m = d(v_l, v_m).\end{aligned}$$

Case 3: When $u = v_l$ and $v = v_m$, $l \neq m$, $1 \le l \le n-h$ and $n-h+2 \le m \le n$. Then,

$$\begin{aligned}|f(v_l) \oplus f(v_m)| &=| \{1, \dots, n-h-(l-1)\} \oplus \{n-h+1, \dots, m-1\} |\\ &=| \{1, \dots, n-h-(l-1), n-h+1, \dots, m-1\} |= m-l = d(v_l, v_m).\end{aligned}$$

Case 4: When $u = v_l$ and $v = v_m^j$, $l \neq m$, $1 \le l \le n-h$, $m = n-h+1$ and $1 \le j \le k$. Then,

$$\begin{aligned}|f(v_l) \oplus f(v_m^j)| &=| \{1, \dots, n-h-(l-1)\} \oplus \{n-1+(m-1)k+j\} |\\ &=| \{1, \dots, n-h-(l-1), n-1+(m-1)k+j\} |= m-l+1 = d(v_l, v_m^j).\end{aligned}$$

Case 5: When $u = v_l$ and $v = v_m^j$, $l \neq m$, $n-h+2 \leq l \leq n$, $m = n-h+1$ and $1 \leq j \leq k$. Then,

$$\begin{aligned}|f(v_l) \oplus f(v_m^j)| &= | \{n-h+1, \ldots, l-1\} \oplus \{n-1+(m-1)k+j\} | \\ &= | \{n-h+1, \ldots, l-1, n-1+(m-1)k+j\} | = l-m+1 = d(v_l, v_m^j).\end{aligned}$$

Case 6: When $u = v_l$ and $v = v_m^j$, $l \neq m$, $1 \leq l \leq n-h$, $n-h+2 \leq m \leq n$ and $1 \leq j \leq k$. Then,

$$\begin{aligned}|f(v_l) \oplus f(v_m^j)| &= | \{1, \ldots, n-h-(l-1)\} \oplus \{n-h+1, \ldots, m-1, n-1+(m-1)k+j\} | \\ &= | \{1, \ldots, n-h-(l-1), n-h+1, \ldots, m-1, n-1+(m-1)k+j\} | \\ &= m-l+1 = d(v_l, v_m^j).\end{aligned}$$

Case 7: When $u = v_l^i$ and $v = v_m^j$, $l \neq m$, $1 \leq l \leq n-h$, $m = n-h+1$ and $1 \leq i,\ j \leq k$. Then,

$$\begin{aligned}|f(v_l^i) \oplus f(v_m^j)| &= | \{1, \ldots, n-h-(l-1), n-1+(l-1)k+i\} \oplus \{n-1+(m-1)k+j\} | \\ &= | \{1, \ldots, n-h-(l-1), n-1+(l-1)k+i, n-1+(m-1)k+j\} | \\ &= m-l+2 = d(v_l^i, v_m^j).\end{aligned}$$

Case 8: When $u = v_l^i$ and $v = v_m^j$, $l \neq m$, $n-h+2 \leq l \leq n$, $m = n-h+1$ and $1 \leq i,\ j \leq k$. Then,

$$\begin{aligned}&|f(v_l^i) \oplus f(v_m^j)| = | \{n-h+1, \ldots, l-1, n-1+(l-1)k+i\} \oplus \{n-1+(m-1)k+j\} | \\ &= | \{n-h+1, \ldots, l-1, n-1+(l-1)k+i, n-1+(m-1)k+j\} | = l-m+2 = d(v_l^i, v_m^j).\end{aligned}$$

Case 9: When $u = v_l^i$ and $v = v_m^j$, $l \neq m$, $1 \leq l \leq n-h$, $n-h+2 \leq m \leq n$ and $1 \leq j \leq k$. Then,

$$\begin{aligned}&|f(v_l^i) \oplus f(v_m^j)| = | \{1, \ldots, n-h-(l-1), n-1+(l-1)k+i\} \\ &\oplus \{n-h+1, \ldots, m-1, n-1+(m-1)k+j\} | \\ &= | \{1, \ldots, n-h-(l-1), n-1+(l-1)k+i, n-h+1, \ldots, m-1, n-1+(m-1)k+j\} | \\ &= m-l+2 = d(v_l^i, v_m^j).\end{aligned}$$

Thus, for any distinct vertices $u, v \in V(P_n^k)$, $|f(u) \oplus f(v)|$ = d(u, v), and hence f admits 1-uniform dcsl.

Finally, to prove $\mathcal{F} \sqsubseteq Avg_n^k$, where $\mathcal{F}$ is the range of f, which forms a poset, define $\Psi : \mathcal{F} \to Avg_n^k$ as follows.

Case 1. On the internal vertices v_i of $V(P_n^k)$. $\Psi(f(v_i)) = \begin{cases} b_i, & \text{when } 1 \leq i \leq n-h, \\ a_{i-(n-h)}, & \text{otherwise.} \end{cases}$

Case 2. On the pendant vertices v_i^j of $V(P_n^k)$. $\Phi(f(v_i^j)) = \begin{cases} b_i^j, & \text{when } 1 \leq i \leq n-h, \\ a_{i-(n-h)}^j, & \text{otherwise.} \end{cases}$

In Case 1, we can identify the internal vertex labeling of $V(P_n^k)$, as the embedding of $\mathcal{F}$ into the underline poset of Avg_n^k. In Case 2, the pendent vertex labeling of $V(P_n^k)$, list the embedding of $\mathcal{F}$ into the outermost labeling of an underline set of Avg_n^k. Thus, from Case 1 and Case 2, we get $\mathcal{F} \sqsubseteq Avg_n^k$. □

The following result follows from Proposition 12 and Proposition 13.

Proposition 14. *Let $\mathcal{F}$ be the range of vertex labeling of a 1-uniform dcsl k-uniform caterpillar such that $\mathcal{F} \sqsubseteq Avg_n^k$. Then $dim(\mathcal{F}) = 2$.*

Theorem 1 ([7]). *If $\mathbf{T}$ is a tree*[1]*, then $dim(\mathbf{T}) \leq 2$ unless $\mathbf{T}$ contains one or more of the trees J_1 and J_2 or their duals as subposets.*

Theorem 2. *Let $\mathcal{F}$ be the poset. Then there exists a 1-uniform dcsl f (the vertex labeling of a k-uniform caterpillar) such that $\mathcal{F} = Range(f) = \{f(v) \mid v \in V(P_n^k)\}$, where $n > 2$ and $k \geq 1$, and $dim(\mathcal{F}) = 2$.*

Proof. Let f be a vertex labeling of 1-uniform dcsl k-uniform caterpillar on '$n(k+1)$' vertices, where $n > 2$ and $k \geq 1$, other than the labeling which is mentioned in Proposition 7, Proposition 10 and Proposition 13, respectively, and let $\mathcal{F}$ be the range of f. Hence, $\mathcal{F} = Range(f) = \{f(v) \mid v \in V(P_n^k)\}$, is a poset.

We prove that $dim(\mathcal{F}) = 2$.

Since the Hasse diagram of $\mathcal{F}$ is a tree, from Theorem 1, we have $dim(\mathcal{F}) \leq 2$. But, $dim(\mathcal{F})$ is never less than 2. For, if it is of dimension 1, then the Hasse diagram of it resembles a path, which is not possible. Hence, $dim(\mathcal{F}) = 2$. □

Recall that [3] the minimum cardinality of the underlying set X such that G admits a 1-uniform dcsl is called the 1-uniform dcsl index $\delta_d(G)$ of G. Following discussion is an attempt to establish the relationship between the 1-uniform dscl index of a k-uniform caterpillar and the dimension of the poset $\mathcal{F} = Range(f) = \{f(v) \mid v \in V(P_n^k)\}$, where $n \geq 1$ and $k \geq 1$.

Lemma 3. *The 1-uniform dcsl index of P_n^k $(n \geq 1,\ k \geq 1)$ is $n(k+1) - 1$.*

Proof. Let $V(P_n^k) = \{v_1, v_1^1, \ldots, v_1^k, v_2, v_2^1, \ldots, v_2^k, \ldots, v_n, v_n^1, \ldots, v_n^k\}$, and let f be the dcsl labeling of P_n^k with the underlying set as X. First, we claim that $\mid X \mid \geq n(k+1) - 1$. By Lemma 1, the 1-uniform dcsl index of P_n is $n - 1$, and hence for the internal vertices of P_n^k, the dcsl index is $n - 1$. For the remaining 'nk' vertices (pendant vertices), we need to have atleast 'nk' subsets of X other than the subsets which has already been labeled for the internal vertices. Hence, the cardinality of X is atleast $nk + n - 1$. By Proposition 7, the vertex labeling of 1-uniform dcsl of P_n^k with underlying set X is of cardinality $n(k+1) - 1$. Hence, $\delta_d(P_n^k) = n(k+1) - 1$. □

In Propositions 7, 10 and 13, the existence of different vertex labeling of 1-uniform dcsl of k-uniform caterpillar and their embedding in respective posets have been established.

In the following theorem we determine the bounds of the poset $\mathcal{F}$, where $\mathcal{F} = Range(f) = \{f(v) \mid v \in V(P_n^k)\}$.

Theorem 3. *Let $\mathcal{F}$ be the poset which is the range of a 1-uniform dcsl of the k-uniform caterpillar, with respect to set inclusion '$\subseteq$'. Then, $dim(\mathcal{F}) \leq \delta_d(P_n^k)$.*

Proof. Let f be a 1-uniform dcsl of $P_n^k (n \geq 1,\ k \geq 1)$, such that $\mathcal{F} = \{f(v) \mid v \in V(P_n^k)\}$ forms a poset with respect to set inclusion '$\subseteq$'. Depending on the number of vertices of $V(P_n^k)$, we prove the theorem for the following four cases.

[1] we call a poset is a tree if its Hasse diagram is a tree in the graph theoretic sense.

Case 1: When $n = 1$ and $k = 1$. In this case, the poset $\mathcal{F}$ is isomorphic to a poset which is a chain of length 1, and hence $dim(\mathcal{F}) = 1$. But by Lemma 3, $\delta_d(P_1^1) = 1$. Thus, we have $dim(\mathcal{F}) = \delta_d(P_n^k)$.

Case 2: When $n = 2$ and $k = 1$. By Lemma 3, we have $\delta_d(P_2^1) = 3$. Also $\mathcal{F}$ is isomorphic to any of the four posets namely, a poset which is a chain of length 3, poset Avg_4, poset $\hat{Avg}_4$ or poset $\mathbf{P}^1$, where $\mathbf{P}$ is a chain of length 1. If $\mathcal{F}$ is isomorphic to chain of length 3, then $dim(\mathcal{F}) = 1$, and hence $dim(\mathcal{F}) < \delta_d(P_n^k)$. If $\mathcal{F} \cong Avg_4$, then by Proposition 14, $dim(\mathcal{F}) = 2$, and hence $dim(\mathcal{F}) < \delta_d(P_n^k)$. Since, for a poset $\mathbf{P}$, $dim(\mathbf{P}) = dim(\hat{\mathbf{P}})$ (see [7]), so if $\mathcal{F} \cong \hat{Avg}_4$, then $dim(\mathcal{F}) = dim(\hat{\mathcal{F}}) = dim(Avg_4) = 2$. Thus, $dim(\mathcal{F}) < \delta_d(P_n^k)$. If $\mathcal{F} \cong \mathbf{P}^1$, where $\mathbf{P}$ is a chain of length 1, then by Proposition 8, $dim(\mathcal{F}) = 2$, and hence, $dim(\mathcal{F}) < \delta_d(P_n^k)$.

Case 3: When $n \geq 3$ and $k \geq 1$. In this case, we prefer k-extended posets that embeds $\mathcal{F}$, as it is not easy to predict all the variations of the poset $\mathcal{F}$. Thus, based on the underline posets of the k-extended posets, since by Lemma 3, $\delta_d(P_n^k) = n(k+1) - 1$, it is enough to consider the following subcases under Case 3.

Case 3.1: If the underline poset is a linear order of finite length, say $\mathbf{L} : a_{i-1} \preceq a_i$, for $2 \leq i \leq n$, then by Proposition 8, $dim(\mathcal{F}) = 2$. Hence $\delta_d(P_n^k) > dim(\mathcal{F})$.

Case 3.2: If the underline poset is isomorphic to Min_n, then by Proposition 11, $dim(\mathcal{F}) = 2$. Hence $dim(\mathcal{F}) < \delta_d(P_n^k)$.

Case 3.3: If the underline poset is isomorphic to Avg_n, then by Proposition 14, $dim(\mathcal{F}) = 2$. Hence $dim(\mathcal{F}) < \delta_d(P_n^k)$.

Case 4: When the poset $\mathcal{F}$ is not isomorphic to either $\mathbf{P}^k$, Min_n^k or Avg_n^k. We have from Theorem 2, $dim(\mathcal{F}) = 2$ and, by Lemma 3, $\delta_d(P_n^k) = n(k+1) - 1$, hence $dim(\mathcal{F}) < \delta_d(P_n^k)$. Thus in all the cases we get $dim(\mathcal{F}) \leq \delta_d(P_n^k)$. □

Theorem 4. *The k-uniform caterpillar P_n^k admits a k-uniform dcsl.*

Proof. Consider $G = P_n^k$ with $n(k+1)$ vertices, say $v_1, v_1^1, \ldots, v_1^k, v_2, v_2^1, \ldots, v_2^k, \ldots, v_n, v_n^1, \ldots,$ and v_n^k. Let $X = \{1, 2, \ldots, h, \ldots, n, \ldots, n(k+1) - 1, \ldots, k(n(k+1) - 1)\}$.

Define $f : V(G) \to 2^X$ by $f(v_1) = \varnothing$, $f(v_i) = \{1, 2, \ldots, (i-1)k\}$ for $2 \leq i \leq n$, and for $1 \leq i \leq k$,

$$\begin{aligned}
&f(v_1^i) = f(v_1) \cup \{(n-1)k + (i-1)k + 1, \ldots, (n-1)k + (i-1)k + k\}, \\
&f(v_2^i) = f(v_2) \cup \{(n-1)k + k^2 + (i-1)k + 1, \ldots, (n-1)k + k^2 + (i-1)k + k\} \text{ and} \\
&f(v_n^i) = f(v_n) \cup \\
&\qquad \{(n-1)k + (n-1)k^2 + (i-1)k + 1, \ldots, (n-1)k + (n-1)k^2 + (i-1)k + k\}.
\end{aligned}$$

In general, for $1 \leq i \leq n$ and $1 \leq j \leq k$,
$f(v_i^j) = f(v_i) \cup \{(n-1)k + (i-1)k^2 + (j-1)k + 1, \ldots, (n-1)k + (i-1)k^2 + (j-1)k + k\}$.
Case 1: When $u = v_l$ and $v = v_m$, $l = 1$ and $2 \leq m \leq n$. Then,

$$\begin{aligned}
|f(v_l) \oplus f(v_m)| &= \mid \varnothing \oplus \{1, 2, \ldots, (m-1)k\} \mid \\
&= \mid \{1, 2, \ldots, (m-1)k\} \mid = (m-1)k = kd(v_l, v_m).
\end{aligned}$$

Case 2: When $u = v_l$ and $v = v_m$, $l \neq m$, $2 \leq l,\ m \leq n$. Then,

$$\begin{aligned}
|f(v_l) \oplus f(v_m)| &= \mid \{1, 2, \ldots, (l-1)k\} \oplus \{1, 2, \ldots, (m-1)k\} \mid \\
&= \mid \{(l-1)k + 1, \ldots, (m-1)k\} \mid = (m-l)k = kd(v_l, v_m), \quad 2 \leq l < m \leq n.
\end{aligned}$$

Case 3: When $u = v_l$ and $v = v_m^j$, $l = 1, 2 \leq m \leq n$ and $1 \leq j \leq k$. Then,

$$
\begin{aligned}
&|f(v_l) \oplus f(v_m^j)| \\
&=| \varnothing \oplus \{1,2,\ldots,(m-1)k,(n+j-2)k+(m-1)k^2+1,\ldots,(n+j-2)k+(m-1)k^2+k\} | \\
&=| \{1,2,\ldots,(m-1)k,(n+j-2)k+(m-1)k^2+1,\ldots,(n+j-2)k+(m-1)k^2+k\} | \\
&= (m-l+1)k = kd(v_l, v_m^j).
\end{aligned}
$$

Case 4: When $u = v_l$ and $v = v_m^j$, $l \neq m, 2 \leq l,\ m \leq n$ and $1 \leq j \leq k$. Then,

$$
\begin{aligned}
&|f(v_l) \oplus f(v_m^j)| \\
&=| \{1,2,\ldots,(l-1)k\} \oplus \{1,2,\ldots,(m-1)k,(n+j-2)k+(m-1)k^2+1,\ldots, \\
&(n+j-2)k+(m-1)k^2+k\} | \\
&=| \{(l-1)k+1,\ldots,(m-1)k,(n+j-2)k+(m-1)k^2+1,\ldots, \\
&(n+j-2)k+(m-1)k^2+k\} | \\
&= (m-l+1)k = kd(v_l, v_m^j), \quad 2 \leq l < m \leq n \text{ and } 1 \leq j \leq k.
\end{aligned}
$$

Case 5: When $u = v_l^i$ and $v = v_m^j$, $l = 1, 2 \leq m \leq n$ and $1 \leq i,j \leq k$. Then,

$$
\begin{aligned}
&|f(v_l^i) \oplus f(v_m^j)| \\
&=| \{(n-1)k+(i-1)k+1,\ldots,(n-1)k+(i-1)k+k\} \oplus \{1,\ldots,(m-1)k, \\
&(n-1)k+(m-1)k^2+(j-1)k+1,\ldots,(n-1)k+(m-1)k^2+(j-1)k+k\} | \\
&=| \{1,\ldots,(m-1)k,(n-1)k+(m-1)k^2+(j-1)k+1,\ldots, \\
&(n-1)k+(m-1)k^2+(j-1)k+k,(n-1)k+(i-1)k+1,\ldots,(n-1)k+(i-1)k+k\} | \\
&= (m-l+2)k = kd(v_l^i, v_m^j).
\end{aligned}
$$

Case 6: When $u = v_l^i$ and $v = v_m^j$, $l \neq m, 2 \leq l,\ m \leq n$ and $1 \leq i,j \leq k$. Then,

$$
\begin{aligned}
&|f(v_l^i) \oplus f(v_m^j)| \\
&=| \{1,\ldots,(l-1)k,(n-1)k+(l-1)k^2+(i-1)k+1,\ldots, \\
&(n-1)k+(l-1)k^2+(i-1)k+k\} \oplus \{1,\ldots,(m-1)k,(n-1)k+(m-1)k^2+ \\
&(j-1)k+1,\ldots,(n-1)k+(m-1)k^2+(j-1)k+k\} | \\
&=| \{(n-1)k+(l-1)k^2+(i-1)k+1,\ldots,(n-1)k+(l-1)k^2+(i-1)k+k, \\
&(l-1)k+1,\ldots,(m-1)k,(n-1)k+(m-1)k^2+(j-1)k+1,\ldots, \\
&(n-1)k+(m-1)k^2+(j-1)k+k\} | \\
&= (m-l+2)k = kd(v_l^i, v_m^j), \quad 2 \leq l < m \leq n \text{ and } 1 \leq i \leq j \leq k.
\end{aligned}
$$

Hence, for any distinct $u, v \in V(G)$, $|f(u) \oplus f(v)| = kd(u,v)$. Which shows that f admits k-uniform dcsl. □

Lemma 4. *For* $n \geq 1, k \geq 1$, $\delta_k(P_n^k) = k(n(k+1)-1)$.

Proof. Let $V(P_n^k) = \{v_1, v_1^1, \ldots, v_1^k, v_2, v_2^1, \ldots, v_2^k, \ldots, v_n, v_n^1, \ldots, v_n^k\}$, and let f be the dcsl labeling of P_n^k with the underlying set as X. By Lemma 2, the 1-uniform dcsl index of P_n is $k(n-1)$, which implies that for internal vertices of P_n^k, the required dcsl index is $k(n-1)$, where as for remaining 'nk' vertices (pendant vertices), we need at least 'k^2n' subsets of X other than the subsets which has already been labeled. Hence the cardinality of X is atleast $k^2n + k(n-1)$. Since by Theorem 4, P_n^k is a k-uniform dcsl with underlying set X of cardinality $k(n(k+1)-1)$, thus we have, $\delta_k(P_n^k) = k(n(k+1)-1)$. □

Theorem 5 ([4]). *If G is k-uniform dcsl, and m is a positive integer, then G is mk-uniform dcsl.*

It has been already established in [4] that path admits arbitrary k-uniform dcsl labeling and k-uniform dcsl index, $\delta_k(P_n)$ is k times that of 1-uniform dcsl index. In this paper, this result is extended to a k-uniform caterpillar, and we prove that the k-uniform dcsl index, $\delta_k(P_n^k)$ is k times that of the 1-uniform dcsl index of k-uniform caterpillar. It is interesting to note that the range of any arbitrary k-uniform dcsl of a k-uniform caterpillar, P_n^k need not form a connected poset. However, there always exists a k-uniform dcsl of P_n^k, whose range is a connected poset. Hence, the Hasse diagram (or poset) which embeds the vertex labeling of 1-uniform dcsl P_n^k, can also embed the vertex labeling of k-uniform dcsl P_n^k. Hence, for such postes the dimension corresponding to 1-uniform dcsl P_n^k and the dimension corresponding to k-uniform dcsl P_n^k are same. Thus, we have the following theorem.

Theorem 6. *If $\mathcal{F}$ is the range of a k-uniform dcsl of the k-uniform caterpillar P_n^k $(n \geq 1, k \geq 1)$, that forms a poset with respect to set inclusion '$\subseteq$', then, $dim(\mathcal{F}) \leq \delta_k(P_n^k)$.*

Proof. Proof is immediate from Theorem 5, Lemma 4 and Theorem 3. □

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Наґесвара Рао К., Ґерміна К.А., Шаіні П. *Про розмірність маркування вершин k-однорідного dcsl k-однорідного графа* // Карпатські матем. публ. — 2016. — Т.8, №1. — С. 134–149.

Сумісне з відстанню множинне маркування (dcsl) зв'язного графа G є ін'єктивним відображенням $f : V(G) \to 2^X$, де X є непорожною базовою множиною такою, що відповідна індукована функція $f^{\oplus} : E(G) \to 2^X \setminus \{\varnothing\}$, задана рівністю $f^{\oplus}(uv) = f(u) \oplus f(v)$, задовольняє $| f^{\oplus}(uv) |= k^f_{(u,v)} d_G(u,v)$ для довільної пари різних вершин $u, v \in V(G)$, де $d_G(u,v)$ позначає відстань між u і v та $k^f_{(u,v)}$ є числом, не обов'язково цілим. Сумісне з відстанню множинне маркування f графа G є k-однорідним, якщо всі коефіцієнти пропорційності відносно f рівні k, і якщо G допускає таке маркування, то G називають k-однорідним dcsl графом. *k-однорідний dcsl індекс* графа G, що позначається $\delta_k(G)$, є мінімальним серед потужностей X, де X пробігає всі k-однорідні dcsl-множини графа G. *Лінійне розширення* $\mathbf{L}$ часткового порядку $\mathbf{P} = (P, \preceq)$ є лінійним порядком на елементах із P таким, що з $x \preceq y$ в $\mathbf{P}$ слідує, що $x \preceq y$ в $\mathbf{L}$ для всіх $x, y \in P$. Розмірність множини $\mathbf{P}$, яка позначається $dim(\mathbf{P})$, є мінімальним числом лінійних розширень на $\mathbf{P}$, перетин яких є '$\preceq$'. У цій статті ми доводимо, що $dim(\mathcal{F}) \leq \delta_k(P_n^{+k})$, де $\mathcal{F}$ є образом k-однорідного dcsl k-однорідного графа, позначеного P_n^{+k} $(n \geq 1, k \geq 1)$ на '$n(k+1)$' вершинах.

Ключові слова і фрази: k-однорідний dcsl індекс, розмірність множини з частковим порядком, решітка.

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APPROXIMATION RELATIONS ON THE POSETS OF PSEUDOMETRICS AND OF PSEUDOULTRAMETRICS

We show that non-trivial "way below" and "way above" relations on the posets of all pseudometrics and of all pseudoultrametrics on a fixed set X are possible if and only if the set X is finite.

Key words and phrases: pseudometric, pseudoultrametric, way below, way above.

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INTRODUCTION

It turned out (see [1]) that partial orders are closely related to topologies, in particular, a "decent" ordering of a set determines quite natural and useful topologies, e.g., Scott topology, upper/lower topology, Lawson topology etc. For these topologies to have nice properties, the original order has to satisfy certain requirements, mostly related to approximation relations.

Recall that a poset $(D, \leq)$ is directed (resp. filtered) if for all $d_1, d_2 \in D$ there is $d \in D$ such that $d_1, d_2 \leq d$ (resp. $d_1, d_2 \geq d$).

Definition 1. *An element x_0 is called to be way below an element x_1 (or approximates x_1 from below) in a poset $(X, \leq)$ (denoted $x_0 \ll x_1$) if for every non-empty directed subset $D \subset X$ such that $x_1 \leq \sup D$ there is an element $d \in D$ such that $x_0 \leq d$.*

Definition 2. *An element x_0 is called to be way above an element x_1 (or approximates x_1 from above) in a poset $(X, \leq)$ (denoted $x_0 \gg x_1$) if for every non-empty filtered subset $D \subset X$ such that $x_1 \geq \inf D$ there is an element $d \in D$ such that $x_0 \geq d$.*

Obviously $x_0 \ll x_1$ or $x_0 \gg x_1$ imply respectively $x_0 \leq x_1$ or $x_0 \geq x_1$ (see more in [1]).

A poset is called continuous (dually continuous) if each element is the least upper bound of all elements approximating it from below (resp. the greatest lower bound of all elements approximating it from above).

We are going to apply the above apparatus to the set of all pseudometrics on a fixed set, and to its subset that consists of all pseudoultrametrics. Ultrametrics (or non-Archimedean metrics [2]) are studied since the beginning of XX century, cf. a review in [3]. They found numerous applications, e.g., in computer science.

Monotone families of (pseudo-)ultrametrics were studied in [4], but approximation relations were out of the scope of the latter paper.

The following notion is a natural mixture of ones of ultrametric and pseudometric.

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Definition 3. *A mapping $d : X \times X \to \mathbb{R}$, that satisfies the conditions:*

- $d(x,y) \geq 0$ *for all* $x,y \in X$ *(nonnegativeness);*
- $d(x,x) = 0$ *for all* $x \in X$ *(identity);*
- $d(x,y) = d(y,x)$ *for all* $x,y \in X$ *(symmetry);*
- $d(x,y) \leq \max\{d(y,z), d(z,x)\}$ *for all* $x,y,z \in X$ *(strong triangle inequality);*

is called a pseudoultrametric on the set X*.*

It is just a pseudometric such that the usual triangle inequality $d(x,y) \leq d(y,z) + d(z,x)$ holds in a stronger form.

The main results of this paper are somewhat disappointing, but they show that, to obtain meaningful theory of approximation, narrower classes of pseudometrics should be considered.

1 Poset of pseudometrics

We denote by $Ps(X)$ the set of all pseudometrics on a set X. The partial order on $Ps(X)$ is defined pointwise: a pseudometric d_1 precedes a pseudometric d_2 (written $d_1 \leq d_2$) if $d_1(x,y) \leq d_2(x,y)$ holds for all points $x,y \in X$.

Obviously the trivial pseudometric $d \equiv 0$ is the least element of $Ps(X)$, hence $Ps(X)$ is bounded from below. The greatest lower bound for two pseudometrics is described with the following statement.

Lemma 1. *For* $d_1, d_2 \in Ps(X)$ *the function*

$$d_*(x,y) = \inf\left\{ \sum_{k=0}^{n-1} \{\min\{d_1(t_k,t_{k+1}), d_2(t_k,t_{k+1})\}\} | n \in \mathbb{N}, x = t_0, \{t_1, ..., t_{n-1}\} \subset X, t_n = y \right\}$$

is the infimum of d_1*,* d_2 *in the set* $Ps(X)$*.*

Proof. Properties of symmetry and identity clearly hold for d_*. To verify the triangle inequality

$$d_*(x,y) \leq d_*(x,z) + d_*(z,y),$$

recall that (after renumbering points in the second sum)

$$\begin{aligned}
d_*(x,z) + d_*(z,y) = \inf\{ &\sum_{k=1}^{m} \{\min\{d_1(t_{k-1},t_k), d_2(t_{k-1},t_k)\}| \\
& m \in \mathbb{N}, t_0, t_1, \ldots, t_m \in X, x = t_0, t_m = z\} \\
& + \inf\{ \sum_{k=m+1}^{n} \min\{d_1(t_{k-1},t_k), d_2(t_{k-1},t_k)\}| \\
& m,n \in \mathbb{N}, 1 \leqslant m \leqslant n-1, t_m, \ldots, t_{n-1}, t_n \in X, t_m = z, t_n = y\} \\
& \geq \inf\{ \sum_{k=1}^{n} \{\min\{d_1(t_{k-1},t_k), d_2(t_{k-1},t_k)\}| \\
& m,n \in \mathbb{N}, 1 \leqslant m \leqslant n-1, t_0, \ldots, t_{n-1}, t_n \in X, t_0 = x, t_m = z, t_n = y\} \\
& \geq \inf\{ \sum_{k=1}^{n} \{\min\{d_1(t_{k-1},t_k), d_2(t_{k-1},t_k)\}| \\
& n \in \mathbb{N}, t_0, \ldots, t_{n-1}, t_n \in X, t_0 = x, t_n = y\} = d_*(x,y).
\end{aligned}$$

Hence $d_* \in Ps(X)$.

The simplest sequence $t_0, t_1, \dots, d_n$ that satisfies the above conditions is $t_0 = x$, $t_1 = y$ (for $n = 1$). It implies $d_*(x,y) \le \min\{d_1(x,y), d_2(x,y)\}$, i.e., d_* is a lower bound of the pseudometrics d_1, d_2.

Show that d_* is the greatest lower bound. For all $x, y \in X$ and $d' \in Ps(X)$ such that $d' \le d_1$, $d' \le d_2$ we obtain

$$\begin{aligned} d'(x,y) &= \inf\{\sum_{k=1}^{n} d'(t_{k-1}, t_k) | n \in \mathbb{N}, t_0, \dots, t_{n-1}, t_n \in X, t_0 = x, t_n = y\} \\ &\leqslant \inf\{\sum_{k=1}^{n} \{\min\{d_1(t_{k-1}, t_k), d_2(t_{k-1}, t_k)\} | n \in \mathbb{N}, t_0, \dots, t_{n-1}, t_n \in X, t_0 = x, t_n = y\} \\ &= d_*(x,y). \end{aligned}$$

□

The least upper bound of pseudometrics d_1, d_2 is the pointwise minimum $d^*(x,y) = \max\{(d_1(x,y), d_2(x,y)\}$ for all $x, y \in X$, thus $Ps(X)$ is a lattice with the least element $d \equiv 0$, but obviously without a greatest element for $|X| > 1$. Being a lattice, $Ps(X)$ is both directed and filtered.

This lattice is not distributive.

Example 1. *Consider, e.g., the set $X = \{x_1, x_2, x_3\}$ and the pseudometrics*

$$d_1(a,b) = \begin{cases} 0, & \{a,b\} = \{x_2, x_3\} \text{ or } a = b, \\ 1 & \text{otherwise,} \end{cases}$$

$$d_2(a,b) = \begin{cases} 0, & \{a,b\} = \{x_1, x_3\} \text{ or } a = b, \\ 1 & \text{otherwise,} \end{cases}$$

$$d_3(a,b) = \begin{cases} 0, & \{a,b\} = \{x_1, x_2\} \text{ or } a = b, \\ 1 & \text{otherwise,} \end{cases}$$

for all $a, b \in X$. Then

$$d_1 \vee d_2(a,b) = \begin{cases} 0, & a = b, \\ 1 & \text{otherwise,} \end{cases} \quad \text{hence } (d_1 \vee d_2) \wedge d_3 = d_3.$$

On the other hand

$$d_1 \wedge d_3 = d_2 \wedge d_3 \equiv 0, \text{ hence } (d_1 \wedge d_3) \vee (d_2 \wedge d_3) \equiv 0.$$

Therefore $(d_1 \vee d_2) \wedge d_3 \neq (d_1 \wedge d_3) \vee (d_2 \wedge d_3)$.

Not having a greatest element, the lattice $Ps(X)$ cannot be complete. Nevertheless, it is straightforward to verify that $Ps(X)$ is a conditionally complete upper semilattice, i.e., each non-empty set D of pseudometrics that is bounded from above by a pseudometric d_0 has a supremum which is calculated pointwise: $(\sup D)(x,y) = \sup\{d(x,y) \mid d \in D\}$ for all $x, y \in X$. The latter supremum exists because the set in the curly braces is bounded by $d_0(x,y)$. The infimum of a set D (which is always bounded from below by $d_0 \equiv 0$) is similar to the one in Lemma 1:

$$(\inf D)(x,y) = \inf\Big\{\sum_{k=1}^{n} \inf\{d(t_{k-1},t_k) \mid d \in D\} \mid n \in \mathbb{N}, x = t_0, \{t_1,...,t_{n-1}\} \subset X, t_n = y\Big\}.$$

Thus $Ps(X)$ is a complete lower semilattice.

Let us start with a simple but important observation.

Lemma 2. *Let pseudometrics d_0, d_1 in X be such that $d_0(x,y) \geq d_1(x,y) > 0$ for some $x,y \in X$. Then neither $d_0 \ll d_1$ nor $d_1 \gg d_0$ is valid.*

Proof. Choose the set $D = \{(1-\frac{1}{n}) \cdot d_1 | n \in \mathbb{N}\}$ of pseudometrics. It is directed, its supremum is equal to d_1, but $(1-\frac{1}{n}) \cdot d_1(x,y) < d_1(x,y) \leqslant d_0(x,y)$, hence $(1-\frac{1}{n})d_1 \not\geqslant d_0$, thus $d_0 \not\ll d_1$. Similarly the set $D' = \{(1+\frac{1}{n}) \cdot d_0 | n \in \mathbb{N}$ is filtered with the greatest lower bound d_0, but neither of its element precedes d_1, hence $d_1 \not\gg d_0$. □

It is easy to see that pseudometrics on a finite set are in the "way below" relation if and only if the above double inequality does not hold for all pairs of points.

Proposition 1. *For pseudometrics d_0 and d_1 on a finite set X the following statements are equivalent:*

(1) $d_0 \ll d_1$ in $Ps(X)$;

(2) $d_1 \gg d_0$ in $Ps(X)$;

(3) for all $x,y \in X$ either $d_0(x,y) = d_1(x,y) = 0$ or $d_0(x,y) < d_1(x,y)$ is valid.

Proof. (1) $\Longrightarrow$ (3) and (2) $\Longrightarrow$ (3) have already been proved. To show (3) $\Longrightarrow$ (1), assume that the condition of the theorem holds for some $d_0, d_1 \in Ps(X)$, and a directed set $D \subset Ps(X)$ is such that $\sup D \geqslant d_1$, hence $\sup\{d(x,y) \mid d \in D\} \geqslant d_1(x,y)$ for all $x,y \in X$. For all pairs $x,y \in X$ such that $d_0(x,y) \geqslant 0$ (and hence $d_1(x,y) > d_0(x,y)$) choose an element $d_{x,y} \in D$ such that $d_{x,y}(x,y) > d_0(x,y)$. The set of the chosen elements of D is finite, D is directed, hence there is $d \in D$ that succeeds all $d_{x,y}$. Obviously $d \geqslant d_0$, thus $d_0 \ll d_1$.

Proof of (3) $\Longrightarrow$ (2) is analogous. □

Unfortunately, for an infinite set X conditions of the latter proposition are necessary but not sufficient.

Example 2. *Consider $X = \mathbb{N}$ with the standard metric $d(x,y) = |x-y|$ and the set of pseudometrics $D = \{d_i | i \in \mathbb{N}\}$,*

$$d_i(x,y) = \begin{cases} |x-y|, & x,y < i; \\ |x-i|, & x < i, y \geq i; \\ |i-y|, & x \geq i, y < i; \\ 0, & x,y \geq i. \end{cases}$$

It is directed because $i \leq j$ implies $d_i \leq d_j$, and $\sup\{d_i \mid i \in \mathbb{N}\} = d$. For the metric $d' = \frac{1}{2}d$ and all points $x,y \in \mathbb{N}$ we have either $d'(x,y) = d(x,y) = 0$ or $d'(x,y) < d(x,y)$ but $d'(i,i+1) = \frac{1}{2} > d_i(i,i+1) = 0$, hence neither of d_i succeeds d'.

We describe a construction of a pseudometric that precedes a given one, and is obtained by "gluing" points. In what follows we denote $d(x,F) = \inf\{d(x,y) \mid y \in F\}$.

Lemma 3. *Let $d \in Ps(X)$ and subset $F \subset X$ be non-empty. Then the function $\acute{d}_F : X \times X \to \mathbb{R}$ that is determined with the formula*

$$\acute{d}_F(x,y) = \min\{d(x,y), d(x,F) + d(y,F)\}, \quad x,y \in X,$$

is a pseudometric on X, and $\acute{d}_F \leq d$. If the set F is bounded, then $d(x,y) - \acute{d}_F(x,y) \leq \operatorname{diam} F$ for all $x,y \in X$.

Proof. Check the prorerties from the definition of pseudometrics for arbitrary $x,y,z \in X$:
(1) $\acute{d}_F(x,y) \geq 0$ because $d(x,y) \geq 0$ i $d(x,F) + d(y,F) \geq 0$.
(2) $\acute{d}_F(x,x) = \min\{d(x,x), d(x,F) + d(x,F)\} = 0$.
(3) $\acute{d}_F(x,y) = \min\{d(x,y), d(x,F) + d(y,F)\} = \min\{d(y,x), d(y,F) + d(x,F)\} = \acute{d}_F(y,x)$.
(4)

$$\begin{aligned}
&\acute{d}_F(x,z) + \acute{d}_F(z,y) \\
&\qquad = \min\{d(x,z), d(x,F) + d(z,F)\} + \min\{d(z,y), d(z,F) + d(y,F)\} \\
&\qquad = \min\big\{d(x,z) + d(z,y), \big(d(x,z) + d(z,F)\big) + d(y,F), \\
&\qquad\qquad \big(d(z,y) + d(z,F)\big) + d(x,F), d(x,F) + d(z,F) + d(z,F) + d(y,F)\big\} \\
&\qquad \geq \min\{d(x,y), d(x,F) + d(y,F), d(y,F) + d(x,F), d(x,F) + d(y,F) + 2d(z,F)\} \\
&\qquad = \min\{d(x,y), d(x,F) + d(y,F)\}.
\end{aligned}$$

Thus $\acute{d}_F$ is a pseudometric.

Now for arbitrary $\varepsilon > 0$ choose $z, z' \in F$ such that $d(x,z) < d(x,F) + \varepsilon$, $d(y,z') < d(y,F) + \varepsilon$. Hence

$$\begin{aligned}
d(x,F) + d(y,F) > d(x,z) + d(y,z') - 2\varepsilon &\geqslant d(x,z) + d(y,z) - d(z,z') - 2\varepsilon \\
&\geqslant d(x,z) + d(y,z) - \operatorname{diam} F - 2\varepsilon \geqslant d(x,y) - \operatorname{diam} F - 2\varepsilon,
\end{aligned}$$

thus

$$\acute{d}_F(x,y) \geqslant d(x,y) - \operatorname{diam} F - 2\varepsilon,$$

then passing to the limit as ε tends to 0 we obtain the required inequality. $\square$

Theorem 1. *For all pseudometrics d_0, d_1 on an infinite set X, $d_0 \gg d_1$ is not valid in $Ps(X)$. If $d_0 \not\equiv 0$, then $d_0 \ll d_1$ also does not hold.*

Proof. Let d_0 be way above d_1. Choose a sequence $x_1, x_2, \dots \in X$ of distinct points and put $\alpha_m = \max\{d_0(x_i, x_j) \mid 1 \leq i,j \leq m\} + m$ for all $m \in \mathbb{N}$. The sequence $(\alpha_m)_{m\in\mathbb{N}}$ is increasing, and the functions

$$\delta_m(a,b) = \begin{cases} 0, & a = b \text{ or } a,b \notin \{x_m, x_{m+1}, \dots\}, \\ \alpha_{\max\{i,j\}}, & a = x_i \neq b = x_j, i,j \geq m, \\ \alpha_i, & a = x_i, i \geq m, b \notin \{x_m, x_{m+1}, \dots\} \\ & \text{ or } b = x_i, i \geq m, a \notin \{x_m, x_{m+1}, \dots\}, \end{cases} \qquad a,b \in X,$$

are pseudometrics and even pseudoultrametrics. It is easy to see that $\delta_1 \geq \delta_2 \geq \dots$, $\inf\{\delta_m \mid m \in \mathbb{N}\} \equiv 0 \leqslant d_1$, but $\delta_m \not\leqslant d_0$ (e.g., $\delta_m(x_m, x_{m+1}) = \alpha_{m+1} \geq d_0(x_m, x_{m+1})$). Therefore $d_0 \not\gg d_1$.

Assume now $d_0 \ll d_1$, $d_0 \not\equiv 0$. Choose a sequence $x_0, x_1, x_2, \dots \in X$ of distinct points such that $d_0(x_0, x_i) > 0$ for all $i \in \mathbb{N}$. Denote $F_i = \{x_0, x_i, x_{i+1}, x_{i+2}, \dots\}$, $i \geq 1$. Let d' be the pseudometric on X:

$$d'(a,b) = \begin{cases} 0, & a,b \notin \{x_0, x_1, \dots\}, \\ |i-j|, & a = x_i, b = x_j, \\ i, & a = x_i, b \notin \{x_0, x_1, \dots\} \\ & \text{ or } a \notin \{x_0, x_1, \dots\}, b = x_i, \end{cases} \qquad x, y \in X.$$

Show that the pseudometric $\rho = d_1 + d' \geqslant d_1$ is the least upper bound of the non-decreasing sequence of pseudometrics $\rho_i = \acute{\rho}_{F_i}$. Clearly $\rho(a, F_i \setminus \{x_0\}) \to \infty$ as $i \to \infty$ for all points $a \in X$, hence $\rho(a, F_i) \to \rho(a, x_0)$, and

$$\acute{\rho}_{F_i}(a,b) \to \min\{\rho(a,b), \rho(a,x_0) + \rho(b,x_0)\} = \rho(a,b).$$

On the other hand, none of ρ_i succeeds d_0 because $\rho_i(x_0, x_i) = 0$ but $d_0(x_0, x_i) > 0$. Therefore d_0 is not way below d_1. □

Thus there is no non-trivial approximation in $Ps(X)$ for infinite X.

2 POSET OF PSEUDOULTRAMETRICS

Consider the subset $PsU(X) \subset Ps(X)$ that consists of all pseudoultrametrics on X, with the restriction of the partial order. It is also a lattice, with the meets (the pairwise infima) calculated pointwise as well, but the formula for the joins (the pairwise suprema) needs to be modified. For $d_1, d_2 \in PsU(X)$ the function

$$d_*(x,y) = \inf\{\max\{\min\{d_1(t_k, t_{k+1}), d_2(t_k, t_{k+1})\} \mid 0 \leq k \leq n-1\} \mid \\ n \in \mathbb{N}, x = t_0, \{t_1, \dots, t_{n-1}\} \subset X, t_n = y\}$$

is the infimum of d_1, d_2 in the set $PsU(X)$. The formula for the infima of arbitrary sets is modified accordingly. The pseudometrics in Example 1 are pseudoultrametrics, hence the lattice $PsU(X)$ is not distributive as well.

Mutatis mutandis we obtain a similar result on approximation relations in $PsU(X)$ for a finite set X.

Proposition 2. *For pseudoultrametrics d_0 and d_1 on a finite set X the following statements are equivalent:*

(1) $d_0 \ll d_1$ in $PsU(X)$;

(2) $d_1 \gg d_0$ in $PsU(X)$;

(3) for all $x, y \in X$ either $d_0(x,y) = d_1(x,y) = 0$ or $d_0(x,y) < d_1(x,y)$ is valid.

Nonetheless, the transfer of Theorem 1 to pseudoultrametrics is not so trivial. We need to modify Lemma 3.

Lemma 4. *Let $d \in PsU(X)$ and subset $F \subset X$ be non-empty. Then the function $\grave{d}_F : X \times X \to \mathbb{R}$ that is determined with the formula*

$$\acute{d}_F(x,y) = \min\{d(x,y), \max\{d(x,F), d(y,F)\}\}, \quad x, y \in X,$$

is a pseudoultrametric on X, and $\grave{d}_F \leq d$. If the set F is bounded, then $d(x,y) \leq \max\{\grave{d}_F(x,y), \operatorname{diam} F\}$ for all $x, y \in X$.

Proof. Only the triangle inequality has to be verified. For arbitrary $x, y, z \in X$:
(4)

$$\begin{aligned}
&\max\{\grave{d}_F(x,z), \grave{d}_F(z,y)\\
&\qquad = \max\big\{\min\{d(x,z), \max\{d(x,F), d(z,F)\}\}, \min\{d(z,y), \max\{d(z,F), d(y,F)\}\}\big\}\\
&\qquad = \min\big\{\max\{d(x,z), d(z,y)\}, \max\{d(x,z), d(z,F), d(y,F)\},\\
&\qquad\qquad \max\{d(z,y), d(z,F), d(x,F)\}, \max\{d(x,F), d(z,F), d(z,F), d(y,F)\}\big\}\\
&\qquad \geq \min\big\{d(x,y), \max\{d(x,F), d(y,F)\}\big\}.
\end{aligned}$$

Thus $\grave{d}_F$ is a pseudoultrametric.

Now for arbitrary $\varepsilon > 0$ choose points $z, z' \in F$ such that $d(x,z) < d(x,F) + \varepsilon$, $d(y,z') < d(y,F) + \varepsilon$. Hence

$$\begin{aligned}
\max\{d(x,F), d(y,F)\} &\geq \max\{d(x,z) - \varepsilon, d(y,z') - \varepsilon\} = \max\{d(x,z), d(y,z')\} - \varepsilon\\
&\geqslant \max\{d(x,z), d(y,z), d(z,z')\} - \varepsilon,
\end{aligned}$$

thus

$$\begin{aligned}
&\max\{\operatorname{diam} F, \grave{d}_F(x,y)\}\\
&\qquad \geq \max\big\{\operatorname{diam} F, \min\{d(x,y), \max\{d(x,z), d(y,z), d(z,z')\} - \varepsilon\}\big\}\\
&\qquad = \min\big\{\max\{\operatorname{diam} F, d(x,y)\}, \max\{\operatorname{diam} F, d(x,z) - \varepsilon, d(y,z) - \varepsilon, d(z,z') - \varepsilon\}\big\}\\
&\qquad \geq \max\{\operatorname{diam} F, d(x,y)\} - \varepsilon
\end{aligned}$$

for all $\varepsilon > 0$, hence $\max\{\operatorname{diam} F, \grave{d}_F(x,y)\} \geq d(x,y)$. □

Now we are ready to prove

Theorem 2. *For all pseudoultrametrics d_0, d_1 on an infinite set X, $d_0 \gg d_1$ is not valid in $PsU(X)$. If $d_0 \not\equiv 0$, then $d_0 \ll d_1$ also does not hold.*

Proof. Recall that the pseudometrics δ_m used in the proof of Theorem 1 are pseudoultrametrics, hence the entire construction is applicable to proof of $d_0 \not\gg d_1$ in $PsU(X)$ as well.

Assume now $d_0 \ll d_1$, $d_0 \not\equiv 0$. Choose a sequence $x_0, x_1, x_2, \dots \in X$ of distinct points such that $d_0(x_0, x_i) > 0$ for all $i \in \mathbb{N}$. Put $\alpha_m = \max\{d_0(x_i, x_j) \mid 0 \leq i, j \leq m\} + m$ for all $m \geq 0$ (hence $\alpha_0 = 0$), and denote $F_i = \{x_0, x_i, x_{i+1}, x_{i+2}, \dots\}$ for all $i \in \mathbb{N}$. The formula

$$d'(a,b) = \begin{cases} 0, & a, b \notin \{x_0, x_1, \dots\} \text{ or } a = b,\\ \alpha_{\max\{i,j\}}, & a = x_i \neq b = x_j,\\ \alpha_i, & a = x_i, b \notin \{x_0, x_1, \dots\}\\ & \text{ or } a \notin \{x_0, x_1, \dots\}, b = x_i, \end{cases} \qquad x, y \in X,$$

defines a pseudoultrametric on X. Then the pseudoultrametric $\rho = \sup\{d_1, d'\} \geqslant d_1$ is the least upper bound of the non-decreasing sequence of pseudoultrametrics $\rho_i = \grave{\rho}_{F_i}$. Observe $\rho(a, F_i \setminus \{x_0\}) \to \infty$ as $i \to \infty$ for all points $a \in X$, hence $\rho(a, F_i) \to \rho(a, x_0)$, and

$$\grave{\rho}_{F_i}(a,b) \to \min\big\{\rho(a,b), \max\{\rho(a,x_0), \rho(b,x_0)\}\big\} = \rho(a,b).$$

Again, $\rho_i(x_0, x_i) = 0$ but $d_0(x_0, x_i) > 0$, hence $\rho_i \geq d_0$ is impossible, which contradicts to $d_0 \ll d_1$ in $PsU(X)$. □

Thus, for an infinite set X the poset $PsU(X)$ is as poor in "way below" and "way above" relations as $Ps(X)$ is.

3 CONCLUSIONS

We have proved that the posets $Ps(X)$ and $PsU(X)$ have no nontrivial approximation relations, hence are not continuous or dually continuous. Therefore we shall restrict our attention to narrower classes of pseudometrics, namely to compact and locally compact pseudoultrametrics. This will be the topic of an upcoming publication.

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Ми доводимо, що нетривіальні відношення апроксимації знизу та апроксимації згори на частково впорядкованих множинах псевдометрик і псевдоультраметрик на фіксованій множині X можливі, якщо і тільки якщо множина X скінченна.

Ключові слова і фрази: псевдометрика, псевдоультраметрика, апроксимація знизу та згори.

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ω-EUCLIDEAN DOMAIN AND LAURENT SERIES

It is proved that a commutative domain R is ω-Euclidean if and only if the ring of formal Laurent series over R is ω-Euclidean domain. It is also proved that every singular matrice over ring of formal Laurent series R_X are products of idempotent matrices if R is ω -Euclidean domain.

Key words and phrases: ω-Euclidean domain, formal Laurent series, idempotent matrices.

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INTRODUCTION

Let R will always denote a commutative domain with nonzero unit element. Let $\varphi : R \to \mathbb{Z}$ be a norm satisfying $\varphi(0) = 0,\ \varphi(a) > 0$ for $a \neq 0$, and $\varphi(ab) \geq \varphi(a)$.

Definition 1. *Domain R is called Euclidean if for any $a, b \in R$ with $b \neq 0$, there exist $q, r \in R$ such that*

$$a = bq + r \quad \text{and} \quad \varphi(r) < \varphi(b).$$

Let $a, b \in R$, $b \neq 0$, and k be an arbitrary positive integer. We talk about *k-term divisibility chain* [7] if there exists a finite sequence of equalities

$$a = bq_1 + r_1, b = r_1q_2 + r_2, \ldots, r_{k-2} = r_{k-1}q_k + r_k. \tag{1}$$

Definition 2. *Domain R is called ω-*Euclidean ring [7] *relatively to norm $\mathbb{N}$, if for every pair of elements $a, b \in R$, $b \neq 0$ can be found $k \in \mathbb{N}$ and such divisibility chain* (1) *of length k that*

$$\varphi(r_k) < \varphi(b).$$

Clearly, 1-Euclidean domain is an Euclidean domain. Now let $R_X = R[[X]][X^{-1}]$ be the ring of formal Laurent series with coefficient in R. P. Samuel in [6] proved that if R_X is euclidean, R is so. Also F. Dress proved the converse in [3]. Also in [1] it is proved similar results are for 2-Euclidean domain.

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MAIN RESULTS

Let R be an integral domain with a norm map $\varphi : R \to \mathbb{Z}$ and let $R_X = R[[X]][X^{-1}]$ be the ring of formal Laurent series with coefficient in R.

For any element

$$f = \sum_{i \geq h} a_i X^i \in R_X,\ a_i \in R,\ h \in \mathbb{Z},\ a_h \neq 0$$

we put a norm map $\psi : R_X \to R$ satisfying $\psi(f) = a_h$ and $\psi(0) = 0$, where a_h be a variable coefficient in the lowest degree.

Proposition 1. *For any $f, g \in R_X$ with $g \neq 0$ we have that $f = gu$ or, $f = gu + v$, where $\psi(g) \nmid \psi(v)$.*

Proof. Let h (resp. k) be the lowest degree of f (resp. g). Set $\psi(f) = \psi(g)q + r$, where $q, r \in R$. Then we can write

$$v = f - qX^{h-k}g = rX^h + \text{higher degree terms}.$$

If $\psi(g) \nmid r$, we get $\psi(g) \nmid r = \psi(v)$.

If $\psi(g) \mid r$, we similarly construct $v_1 = v - q_1 X^{h_1 - k} g$, ($h_1 =$ order of v) and so on. If the process stops after a finite number of steps, we obtain

$$f = gu + v, \qquad \psi(g) \nmid \psi(v).$$

Otherwise the infinite sum

$$u = qX^{h-k} + q_1 X^{h_1 - k} + \cdots + q_n X^{h_n - k} + \cdots$$

is true sense, and we obtain $f = gu$. $\square$

Let a map $\varphi_x : R \to \mathbb{Z}$ by $\varphi_x(f) = \varphi(\psi(f))$. Then we obtain the following.

Theorem 1. *If R is ω-Euclidean domain with respect to φ, then R_X is ω-Euclidean domain with respect to $\varphi_x = \varphi \cdot \psi$.*

Proof. By Proposition 1 for any $f, g \in R_X$ with $g \neq 0$ we have the following:

(1) $f = gu,$ or

(2) $f = gu + v, \quad \psi(g) \nmid \psi(v).$

It is obvious that the case (1), R_X is Euclidean domain and thus R is ω-Euclidean.

In the case of (2) review:

a) if $\varphi(\psi(v)) < \varphi(\psi(g))$, then we have $\varphi_x(v) < \varphi_x(g)$ by definition, R_X is Euclidean domain and thus R is ω-Euclidean;

b) if $\varphi(\psi(v)) \geq \varphi(\psi(g))$, then

$$\psi(v) = \psi(g)q_1 + r_1,\ \psi(g) = r_1 q_2 + r_2, \ldots, r_{k-2} = r_{k-1} q_k + r_k, \tag{2}$$

and $\varphi(r_k) < \varphi(\psi(g))$, because R is ω-Euclidean domain.

Now if we set

$$v - q_1 X^{h_1 - k} g = v_1, \quad (h_1 \text{ - order of } v),$$

we have $f = (u + q_1 X^{h_1 - k})g + v_1$ and $\psi(v_1) = r_1$. If we set

$$g - q_2 X^{k-h_2} v_1 = v_2, \quad (h_2 \text{ - order of } v_1),$$

we have $g = q_2 X^{k-h_2} v_1 + v_2$ and $\psi(v_2) = r_2$. Continuing this process in the k step we get

$$v_{k-2} - q_k X^{h_{k-1}-h_k} v_{k-1} = v_k, \quad (h_k \text{ — order of } v_{k-1}),$$

then $v_{k-2} = q_k X^{h_{k-1}-h_k} v_{k-1} + v_k$ and $\psi(v_k) = r_k$. If $r_k \neq 0$, we obtain

$$f = (u + q_1 X^{h_1-k})g + v_1,\ g = q_2 X^{k-h_2} v_1 + v_2, \ldots, v_{k-2} = q_k X^{h_{k-1}-h_k} v_{k-1} + v_k,$$

and

$$\varphi_x(g) = \varphi(\psi(g)) > \varphi(r_k) = \varphi_x(v_k).$$

If $r_k = 0$, we have $r_{k-2} = r_{k-1} q_k$. Then we have.

If $\varphi(\psi(g)) > \varphi(r_{k-1})$, we obtain $(k-1)$-term divisibility chain, because

$$\varphi(\psi(g)) = \varphi_x(g) > \varphi_x(v_{k-1}) = \varphi(r_{k-1}).$$

On the other hand, since $\varphi(r_{k-1}) \geq \varphi(\psi(g))$, then with (2) we get $\psi(g) = r_{k-1} m$, where $m \in R$. Then $\varphi(m) = 1$.

Hence,

$$r_{k-1} = \psi(g) m^{-1}$$

and

$$\psi(v) = \psi(g) x,$$

for some $x \in R$. This is contradictory to for $\psi(g) \nmid \psi(v)$. $\square$

Theorem 2. *If R_X is ω-Euclidean domain with respect to φ_x, then R is ω-Euclidean domain with respect to φ.*

Proof. Let $a, b \in R$, where $b \neq 0$. Since R_X is ω-Euclidean domain, there exist such $q_1, \ldots, q_n, r_1, \ldots, r_n \in R_X$ that

$$a = bq_1 + r_1, b = r_1 q_2 + r_2, \ \ldots, r_{n-2} = r_{n-1} q_n + r_n, \tag{3}$$

where $\varphi_x(r_n) < \varphi_x(b)$.

Note that

$$q_i = q'_{k_i} X^{k_i} + \text{higher degree terms}, \quad r_i = r'_{s_i} X^{s_i} + \text{higher degree terms}$$

(1) Let $\varphi_x(r_1) < \varphi_x(b)$. If $k_1 < 0$, we have $k_1 = s_1$ and $bq'_{k_1} + r'_{s_1} = 0$, and hence $\varphi_x(r_1) = \varphi(r'_{s_1}) = \varphi(-bq'_{k_1}) \geq \varphi(b) = \varphi_x(b)$. This is a contradiction. Therefore we get $k_1 \geq 0$, then $a = bq'_{k_0} + r'_{s_0}$, $\varphi(r'_{s_0}) = \varphi_x(r_1) < \varphi_x(b) = \varphi(b)$.

(2) Let $\varphi_x(r_1) \geq \varphi_x(b)$. If $s_1 + k_2 < 0$, we get $s_1 + k_2 = s_2$ and $r'_{s_1} q'_{k_2} + r'_{s_2} = 0$ and note that a chain 3 we get $r_n = r_1 x^* + r_2 y^*$ for some $x^*, y^* \in R_X$. Then $\varphi_x(r_n) = \varphi_x(r_1 x^* + r_2 y^*) = \varphi((x^* - q'_{k_2} y^*) r'_{s_1}) \geq \varphi(r'_{s_1}) \geq \varphi_x(b)$.

Hence $\varphi_x(r_n) < \varphi_x(b)$, this is contradiction and we get $s_1 + k_2 \geq 0$. Then we can consider possibility.

Case 1) $r'_{s_2} \neq 0$.

If $k_1 < 0$, we get $bq'_{k_1} + r'_{s_1} = 0$. On the other hand with chain 3 we have $r_n = bx + r_1y$, for some $x, y \in R_X$,

$$\varphi_x(r_n) = \varphi_x(bx + r_1y) = \varphi((x' - q'_{k_1}y')b) \geq \varphi(b) = \varphi_x(b).$$

This is contradiction, because $\varphi_x(r_n) < \varphi_x(b)$. Hence we have $k_1 \geq 0$. The we obtain

$$a = bq'_{k_1} + r'_{s_1}, b = r'_{s_1}q'_{k_2} + r'_{s_2}, \ldots, r'_{s_{n-2}} = r'_{s_{n-1}}q'_{k_n} + r'_{s_n},$$

where $\varphi_x(r_n) = \varphi(r'_{s_n}) < \varphi(b) = \varphi_x(b)$.

Case 2) $r'_{s_2} = 0$.

In this case, we distinguish now two subcases.

1') If $k_1 \geq 0$, it is obvious that

$$a = bq'_{k_1} + r'_{s_1}, b = r'_{s_1}q'_{k_2} + 0,$$

and $\varphi(0) < \varphi(b)$.

2') If $k_1 < 0$ we have $k_1 = s_1 < 0$ and $bq'_{k_1} + r'_{s_1} = 0$.

On the other hand, since $b = r'_{s_1}q'_{k_2}$ we have $r'_{s_1}q'_{k_1}q'_{k_2} + r'_{s_1} = 0$ i $q'_{k_1}q'_{k_2} + 1 = 0$ and hence q'_{k_1}, q'_{k_2} are units. Then we can obtain:

$$\begin{aligned} b &= (r'_{s_1}X^{s_1} + \cdots)(q'_{k_1}X^{k_1} + \cdots) + (r'_{s_2}X^{s_2} + \cdots) = r'_{s_1}q'_{k_2} \\ &\quad + (r'_{s_1}q'_{k_2+1} + r'_{s_1+1}q'_{k_2})X + (r'_{s_1}q'_{k_2+2} + r'_{s_1+1}q'_{k_2+1} + r'_{s_1+2}q'_{k_2})X^2 + \cdots + (r'_{s_2}X^{s_2} + \cdots). \end{aligned}$$

Therefore we get the following equations:

$$\begin{cases} r'_{s_1}q'_{k_2+1} + r'_{s_1+1}q'_{k_2} = 0, \\ r'_{s_1}q'_{k_2+2} + r'_{s_1+1}q'_{k_2+1} + r'_{s_1+2}q'_{k_2} = 0, \\ \cdots\cdots\cdots\cdots \\ r'_{s_1}q'_{k_2+s_2} + r'_{s_1+1}q'_{k_2+s_2-1} + \cdots + r'_{s_1+s_2}q'_{k_2} + r'_{s_2} = 0. \end{cases} \tag{4}$$

Since q'_{k_1} is a unit, we have

$$r'_{s_1+1} = (q'_{k_1})^{-1}r'_{s_1}q'_{k_1+1} = (q'_{k_1})^{-1}q'_{k_1+1}(q'_{k_2})^{-1}b.$$

Hence we get $b \mid r'_{s_1+1}$. Similarly, we have

$$b \mid r'_{s_1+2}, \cdots, r'_{s_1+s_2-1}.$$

Then if $s_1 + s_2 < 0$, we have $bq'_{s_1+s_2} + r'_{s_1+s_2} = 0$ and hence $b \mid r'_{s_1+1}$. By above equations (4), $b \mid r'_{s_2}$ and $\varphi(r'_{s_2}) \geq \varphi(b)$. This is a contradiction with $\varphi(r'_{s_2}) < \varphi(b)$. Therefore we get $s_1 + s_2 \geq 0$.

Now, if $s_1 + s_2 > 0$, there exist an integer h such that $r'_{s_1}q'_{k_2+h} + r'_{s_1+h}q'_{k_2} = 0$ and $b \mid r'_{s_1+h} = r'_0$. Hence we obtain $a = bq'_0 + r'_0 = bq^*$.

If $s_1 + s_2 = 0$, the equation (4) we have

$$r'_{s_1}q'_{k_2+s_2} + \cdots + r'_{s_1+s_2}q'_{k_2} = r'_{s_1}q'_{k_2+s_2} + \cdots + (a - bq'_0)q'_{k_2} + r'_{s_2} = 0.$$

Then we obtain

$$a = bq'_0 + (q'_{k_2})^{-1}(-r'_{s_1}q'_{k_2+s_2} - \cdots - r'_{s_2}) = bq' + (q'_{k_2})^{-1}(-r'_{s_2})$$

and $\varphi((q'_{k_2})^{-1}(-r'_{s_2})) = \varphi(r'_{s_2}) < \varphi(b)$. □

As a consequent we obtain the following.

Theorem 3. *R is ω-Euclidean domain if and only if R_X is ω-Euclidean domain.*

A ring R has IP_n-property, if every square singular matrix of n order over R is a product of idempotent matrices. If this is true for any singular matrix over R, then the ring R has IP-property.

Theorem 4. *Let R is Bezout domain with IP_2-property, then R_X is a domain with IP-property.*

Proof. Let R be Bezout domain with IP_2-property, then R is GE_2-ring [4]. Since the condition GE_2-ring over Bezout domain implies the presence of the infinite divisibility chain for any two elements with R, hence R is ω-Euclidean domain. According to Theorem 1, R_X is ω-Euclidean domain, then from [2] for any two elements of R_X there exists the infinite divisibility chain. Then, according to Theorem 6.2 and Proposition 2.4 of [5] implies that R_X has IP-property. $\square$

Given from theorem 2, consequently the following result is true.

Theorem 5. *Let R_X — ω-Euclidean domain, then R has IP-property.*

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Доведено, що комутативна область є ω-евклідовою тоді і тільки тоді, коли кільце формальних Лоранових рядів є ω-евклідовою областю. Також показано, що довільна особлива матриця над кільцем формальних Лоранових рядів R_X є добутком ідемпотентних матриць, якщо R є ω-евклідове кільце.

Ключові слова і фрази: ω-евклідова область, кільце формальних Лоранових рядів, ідемпотентні матриці.

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COUNTABLE HYPERBOLIC SYSTEMS IN THE THEORY OF NONLINEAR OSCILLATIONS

In this article a model example of a mixed problem for a fourth-order differential equation is reduced to initial-boundary value problem for countable hyperbolic system of first order coherent differential equations.

Key words and phrases: countable hyperbolic system, initial-boundary value problem.

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INTRODUCTION

Many problems from Elasticity Theory, Gas dynamics, Theory of plates and shells reduced to partial higher order differential equations [1, 2, 3] using Fourier method [3] or the method of Principal coordinates [1]. As a result we get a infinite system of ordinary differential equations. The Theory of countable ordinary differential systems is described in the monograph [4]. However, in many cases, particularly in the famous Hadamard's example [5, p.112] about correct solvability of initial problem for Cauchy-Riemann equation, if interpret partial solutions like $u_n = I_n(t)\cos nx$, $v_n = J_n(t)\sin nx$, we get a countable system of partial first order differential equations. Similar systems occur in determining of the generalized solution for hyperbolic first order equations [5, p.132], in the investigation of mathematical models of self-excited oscillator with distributed parameters [6], in many periodic solutions of quasi-linear hyperbolic systems [7] and others. Some questions about the correct solvability of initial-boundary value problems for countable hyperbolic systems of first order differential equations are considered in [8, 9, 10, 13].

1 STATEMENT OF PROBLEM

In the domain $Q = \{(t,x,y) : t \in (0,T), x \in (0,l_1), y \in (0,l_2)\}$ we consider fourth order partial differential equation

$$u_{tt} + B(t,x)(u_{tx} + u_{xyy}) + C(t,x)u_{xx} + u_{yyyy} + 2u_{tyy} = f\Big(t,x,y,u,u_t,u_x,u_{yy}\Big) \qquad (1)$$

with initial

$$\begin{aligned} &u|_{t=0} = \varphi(x,y), \\ &u_t|_{t=0} = \psi(x,y), \quad 0 \le x \le l_1,\ 0 \le y \le l_2, \end{aligned} \qquad (2)$$

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and boundary conditions

$$\begin{aligned}
&u|_{y=0} = u|_{y=l_2} = 0,\\
&\frac{\partial^2 u}{\partial y^2}\Big|_{y=0} = \frac{\partial^2 u}{\partial y^2}\Big|_{y=l_2} = 0, \quad 0 \le x \le l_1,\ 0 \le t \le T,\\
&u|_{x=0} = \mu(t,y),\ u|_{x=l_1} = \nu(t,y), \quad 0 \le y \le l_2,\ 0 \le t \le T,
\end{aligned} \tag{3}$$

where

$$\begin{aligned}
&\mu(0,y) = \varphi(0,y), \quad \nu(0,y) = \varphi(l_1,y), \quad \mu_t'(0,y) = \psi(0,y), \quad \nu_t'(0,y) = \psi(l_1,y),\\
&\varphi(x,0) = \varphi(x,l_2) = 0, \quad \psi(x,0) = \psi(x,l_2) = 0,\\
&\varphi_{yy}''(x,0) = \varphi_{yy}''(x,l_2) = 0, \quad \psi_{yy}''(x,0) = \psi_{yy}''(x,l_2) = 0.
\end{aligned}$$

2 THE REDUCTION EQUATION (1) TO A COUNTABLE SYSTEM OF SECOND ORDER DIFFERENTIAL EQUATIONS

We will search solution of the problem (1)–(3) using separation of variables method, namely in the form of a series

$$u(t,x,y) = v_0(t,x) + \sum_{n=1}^{\infty} \big(v_n(t,x)\cos\alpha_n y + w_n(t,x)\sin\alpha_n y\big), \tag{4}$$

where $\alpha_n = \frac{2\pi n}{l_2}$ (see [12, 13]). Substituting (4) in boundary conditions (3), we obtain $\sum\limits_{n=0}^{\infty} v_n(t,x) = 0$ and $\sum\limits_{n=1}^{\infty} \alpha_n^2 v_n(t,x) = 0$. Suppose, that $v_n(t,x) \equiv 0$ for all $n \in \mathbb{N}$ and $(t,x) \in \Pi^{t,x} = (0,T) \times (0,l_1)$.

Assume that the initial data of the problem (1)–(3) are sufficiently smooth. Let compatibility conditions are fulfilled and the initial data are unambiguous decomposed in a series

$$f\Big(t,x,y,u,\frac{\partial u}{\partial t},\frac{\partial u}{\partial x},\frac{\partial^2 u}{\partial y^2}\Big) = \sum_{n=1}^{\infty} f_n\Big(t,x,w_1,w_2,\dots,\frac{\partial w_1}{\partial t},\frac{\partial w_2}{\partial t},\dots,\frac{\partial w_1}{\partial x},\frac{\partial w_2}{\partial x},\dots\Big)\sin\alpha_n y, \tag{5}$$

$$\varphi(x,y) = \sum_{n=1}^{\infty} \varphi_n(x)\sin\alpha_n y, \quad \psi(x,y) = \sum_{n=1}^{\infty} \psi_n(x)\sin\alpha_n y, \tag{6}$$

$$\mu(t,y) = \sum_{n=1}^{\infty} \mu_n(t)\sin\alpha_n y, \quad \nu(t,y) = \sum_{n=1}^{\infty} \nu_n(t)\sin\alpha_n y. \tag{7}$$

Let $\omega_n = \big(\frac{2\pi n}{l_2}\big)^2$. Substitute equality (4) in equation (1) and conditions (2) and (3). After multiplying received equalities by $\sin\alpha_m y$, $(m = 1,2,\dots)$ and integrating in the interval from 0 to l_2, with considering conditions (5)–(7), we obtain the countable system of second-order differential equations

$$\begin{aligned}
&\frac{\partial^2 w_n}{\partial t^2} + B(t,x)\Big(\frac{\partial^2 w_n}{\partial t\partial x} - \omega_n\frac{\partial w_n}{\partial x}\Big) + C(t,x)\frac{\partial^2 w_n}{\partial x^2} + \omega_n^2 w_n - 2\omega_n\frac{\partial w_n}{\partial t}\\
&\qquad = f_n\Big(t,x,w_1,w_2,\dots,\frac{\partial w_1}{\partial t},\frac{\partial w_2}{\partial t},\dots,\frac{\partial w_1}{\partial x},\frac{\partial w_2}{\partial x},\dots\Big), \quad n \in \mathbb{N},
\end{aligned} \tag{8}$$

with initial and boundary conditions

$$\begin{aligned}
&w_n|_{t=0} = \varphi_n(x),\ \frac{\partial w_n}{\partial t}\Big|_{t=0} = \psi_n(x), \quad 0 \le x \le l_1,\\
&w_n|_{x=0} = \mu_n(t),\ w_n|_{x=l_1} = \nu_n(t), \quad 0 \le t \le T.
\end{aligned}$$

Propose a change of variables $w_n = v_n e^{\omega_n t}$. Then all derivatives will be rewritten in a form

$$\begin{aligned}
&\frac{\partial w_n}{\partial t} = \Big(\frac{\partial v_n}{\partial t} + \omega_n v_n\Big) e^{\omega_n t}, \quad \frac{\partial w_n}{\partial x} = \frac{\partial v_n}{\partial x} e^{\omega_n t},\\
&\frac{\partial^2 w_n}{\partial t^2} = \Big(\frac{\partial^2 v_n}{\partial t^2} + 2\omega_n \frac{\partial v_n}{\partial t} + \omega_n^2 v_n\Big) e^{\omega_n t},\\
&\frac{\partial^2 w_n}{\partial t \partial x} = \Big(\frac{\partial^2 v_n}{\partial t \partial x} + \omega_n \frac{\partial v_n}{\partial x}\Big) e^{\omega_n t}, \quad \frac{\partial^2 w_n}{\partial x^2} = \frac{\partial^2 v_n}{\partial x^2} e^{\omega_n t}.
\end{aligned}$$

As a result, we obtain the countable system of second order differential equations

$$\begin{aligned}
&\frac{\partial^2 v_n}{\partial t^2} + B(t,x)\frac{\partial^2 v_n}{\partial t \partial x} + C(t,x)\frac{\partial^2 v_n}{\partial x^2}\\
&\quad = \tilde{f}_n\Big(t, x, v_1, v_2, \dots, \frac{\partial v_1}{\partial t}, \frac{\partial v_2}{\partial t}, \dots, \frac{\partial v_1}{\partial x}, \frac{\partial v_2}{\partial x}, \dots\Big), \quad n \in \mathbb{N},
\end{aligned}$$

where

$$\begin{aligned}
\tilde{f}_n = e^{-\omega_n t} f_n\Big(&t, x, v_1 e^{\omega_n t}, v_2 e^{\omega_n t}, \dots,\\
&\frac{\partial v_1}{\partial t} e^{\omega_n t} + \omega_n v_1 e^{\omega_n t}, \frac{\partial v_2}{\partial t} e^{\omega_n t} + \omega_n v_2 e^{\omega_n t}, \dots, \frac{\partial v_1}{\partial x} e^{\omega_n t}, \frac{\partial v_2}{\partial x} e^{\omega_n t}, \dots\Big).
\end{aligned}$$

Initial and boundary conditions will be rewritten in a form

$$\begin{aligned}
&v_n|_{t=0} = \varphi_n(x), \quad \frac{\partial v_n}{\partial t}\Big|_{t=0} = \tilde{\psi}_n(x), \qquad 0 \le x \le l_1,\\
&v_n|_{x=0} = \tilde{\mu}_n(t), \quad v_n|_{x=l_1} = \tilde{\nu}_n(t), \qquad 0 \le t \le T,
\end{aligned}$$

where $\tilde{\mu}_n(t) = \mu_n(t) e^{-\omega_n t}$, $\tilde{\nu}_n(t) = \nu_n(t) e^{-\omega_n t}$, $\tilde{\psi}_n(x) = \psi_n(x) - \omega_n \varphi_n(x)$.

3 The reduction to countable system of first order differential equations

Suppose that $\Delta(t,x) = B^2(t,x) - 4C(t,x) > 0$, for all $(t,x) \in \Pi^{t,x}$, so each equation of the system (8) has hyperbolic type. We denote

$$\begin{aligned}
&\lambda_i(t,x) = \frac{B(t,x) + (-1)^i \sqrt{\Delta(t,x)}}{2},\\
&v_{i,n} = \frac{\partial v_n}{\partial t} + \lambda_i \frac{\partial v_n}{\partial x}, \quad i = 1, 2.
\end{aligned}$$

Then

$$\begin{aligned}
&\frac{\partial v_n}{\partial x} = \frac{v_{2,n} - v_{1,n}}{\sqrt{\Delta}},\\
&\frac{\partial v_n}{\partial t} = v_{2,n} - (B + \sqrt{\Delta}) \frac{v_{2,n} - v_{1,n}}{2\sqrt{\Delta}}.
\end{aligned}$$

Due to variables changes, each equation of the system (8) would be equivalent to the system of equations [5, 11]

$$\begin{aligned}
\frac{\partial v_{i,n}}{\partial t} + \lambda_{3-i}\frac{\partial v_{i,n}}{\partial x} &= \frac{1}{\sqrt{\Delta}}\Big(\frac{\partial \lambda_i}{\partial t} + \lambda_{3-i}\frac{\partial \lambda_i}{\partial x}\Big)(v_{2,n} - v_{1,n})\\
&\quad + \tilde{f}_n\Big(t, x, v_1, \dots, v_{2,1} - (B + \sqrt{\Delta})\frac{v_{2,1} - v_{1,1}}{2\sqrt{\Delta}}, \dots, \frac{v_{2,1} - v_{1,1}}{\sqrt{\Delta}}, \dots\Big),\\
\frac{\partial v_n}{\partial t} &= v_{2,n} - (B + \sqrt{\Delta})\frac{v_{2,n} - v_{1,n}}{2\sqrt{\Delta}}, \quad i = 1, 2, \quad n \in \mathbb{N}.
\end{aligned} \tag{9}$$

Suppose, that $\lambda_1(t,x) \geq 0$, $\lambda_2(t,x) \leq 0$ (a sufficient condition is execution the inequality $|B(t,x)| \leq \sqrt{\Delta(t,x)}$). Conduct characteristic $L_1(0,0)$ through the point $(0,0)$ and characteristic $L_2(0,l_1)$ through the point $(0,l_1)$, which are the solutions of Cauchy problems

$$\frac{dx}{dt} = \lambda_1(t,x),\ x(0) = 0,\ \frac{dx}{dt} = \lambda_2(t,x),\ x(0) = l_1.$$

Thus, rectangle $\Pi^{t,x}$ is divided into three parts (see Figure 1).

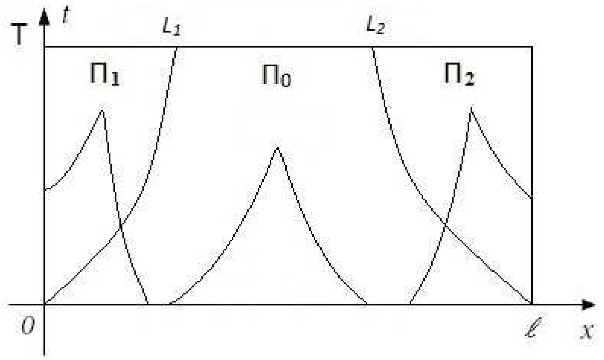

Figure 1: Partition of domain by characteristics with slope $\lambda_1 \geq 0, \lambda_2 \leq 0$.

In subdomain Π_0 for system (9) define the initial conditions

$$v_n|_{t=0} = \varphi_n(x), v_{i,n}|_{t=0} = \tilde{\psi}_n(x) + \lambda_i|_{t=0}\frac{d\varphi_n}{dx}(x), \quad i = 1,2.$$

In Π_1 for v_n and $v_{2,n}$ define the initial conditions, and for $v_{1,n}$ define the boundary conditions on the left side

$$v_n|_{t=0} = \varphi_n(x), \quad v_{2,n}|_{t=0} = \tilde{\psi}_n(x) + \lambda_2|_{t=0}\frac{d\varphi_n}{dx}(x),$$
$$v_{1,n}|_{x=0} = \frac{2\sqrt{\Delta}}{B+\sqrt{\Delta}}\Big|_{x=0}\frac{d\tilde{\mu}_n}{dt}(t) + \Big(1 - \frac{2\sqrt{\Delta}}{B+\sqrt{\Delta}}\Big)\Big|_{x=0} v_{2,n}|_{x=0}.$$

In subdomain Π_2 for v_n and $v_{1,n}$ define the initial conditions, and for $v_{2,n}$ define the boundary conditions on the right side

$$v_n|_{t=0} = \varphi_n(x), \quad v_{1,n}|_{t=0} = \tilde{\psi}_n(x) + \lambda_1|_{t=0}\frac{d\varphi_n}{dx}(x),$$
$$v_{2,n}|_{x=l_1} = \frac{2\sqrt{\Delta}}{\sqrt{\Delta}-B}\Big|_{x=l_1}\frac{d\tilde{\mu}_n}{dt}(t) + \frac{B+\sqrt{\Delta}}{B-\sqrt{\Delta}}\Big|_{x=l_1} v_{1,n}|_{x=l_1}.$$

Remark 3.1. *If the following condition is not fulfilled $\lambda_1 \geq 0, \lambda_2 \leq 0$, there is possible to get such cases:*

i) $\lambda_1 \geq \lambda_2 \geq 0, \lambda_1^2 + \lambda_2^2 \neq 0;$

ii) $\lambda_1 \leq \lambda_2 \leq 0, \lambda_1^2 + \lambda_2^2 \neq 0.$

In the first case, for system (1) *it is necessary to define the boundary conditions in the next form*

$$u|_{x=0} = \mu(t,y), u_x|_{x=0} = \nu(t,y), \quad 0 \leq y \leq l_2,\ 0 \leq t \leq T.$$

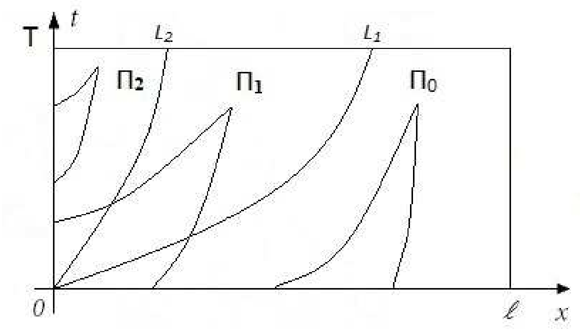

Figure 2: Partition of domain by characteristics with slope $\lambda_1, \lambda_2 > 0$.

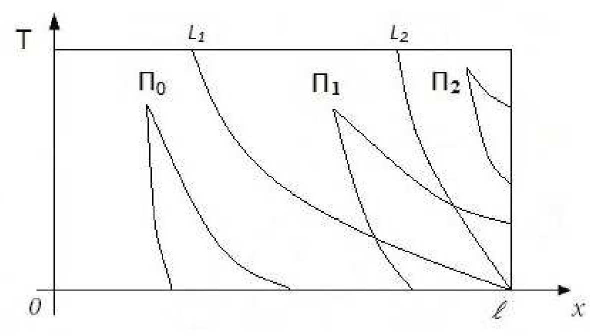

Figure 3: Partition of domain by characteristics with slope $\lambda_1, \lambda_2 < 0$.

Conduct characteristics $L_1(0,0)$ *and* $L_2(0,0)$ *through the point* $(0,0)$, *which are the solutions of Cauchy problems*

$$\frac{dx}{dt} = \lambda_i,\ x(0) = 0, \quad i = 1,2.$$

Thus, rectangle $\Pi^{t,x}$ *is devided into three parts (see Figure 2).*

In subdomain Π_0 *define the initial conditions*

$$v_n|_{t=0} = \varphi_n(x),\ v_{i,n}|_{t=0} = \tilde{\psi}_n(x) + \lambda_i|_{t=0}\frac{d\varphi_n}{dx}(x), \quad i = 1,2.$$

In Π_1 *for* v_n *and* $v_{2,n}$ *define the initial conditions, and for* $v_{1,n}$ *define the boundary conditions on the left side*

$$v_n|_{t=0} = \varphi_n(x), \quad v_{2,n}|_{t=0} = \tilde{\psi}_n(x) + \lambda_2|_{t=0}\frac{d\varphi_n}{dx}(x),$$
$$v_{1,n}|_{x=0} = \frac{d\tilde{\mu}_n}{dt}(t) + \lambda_1|_{x=0}\tilde{\nu}_n(t).$$

In subdomain Π_2 *for* v_n *define the initial conditions, and for* $v_{1,n}$ *and* $v_{2,n}$ *define the boundary conditions on the left side*

$$v_n|_{t=0} = \varphi_n(x), \quad v_{i,n}|_{x=0} = \frac{d\tilde{\mu}_n}{dt}(t) + \lambda_i|_{x=0}\nu_n(t).$$

Similarly, the initial and boundary conditions would be defined in case, when $\lambda_1 \leq \lambda_2 \leq 0$, $\lambda_1^2 + \lambda_2^2 > 0$ *(see Figure 3). In this case for the system* (1) *we have to set the boundary conditions in the following form*

$$u|_{x=l_1} = \mu(t,y),\ u_x|_{x=l_1} = \nu(t,y), \quad 0 \leq y \leq l_2,\ 0 \leq t \leq T.$$

4 EXAMPLE

For example, consider a differential equation

$$u_{tt} - x^2 u_{xx} + u_{yyyy} + 2u_{tyy} = -xu_x + \Big(-\frac{\pi}{6} + \pi y - y^2\Big)u + f(t,x,y), \tag{10}$$

where $f(t,x,y)$ — some polynomial of (t,x,y), with initial conditions

$$u|_{t=0}=0,\ u_t|_{t=0}=\Big(y^5-\frac{5}{2}\pi y^4+\frac{5}{3}\pi^2y^3-\frac{1}{6}\pi^4y\Big)(\pi x-x^2),\quad 0\le x\le\pi,\ 0\le y\le\pi,$$

and homogeneous boundary conditions

$$\begin{aligned}&\frac{\partial^2 u}{\partial y^2}\Big|_{y=0}=\frac{\partial^2 u}{\partial y^2}\Big|_{y=\pi}=0,\\ &u|_{y=0}=u|_{y=\pi}=0,\ u|_{x=0}=u|_{x=\pi}=0,\quad 0\le x\le\pi,\ 0\le y\le\pi,\ 0\le t\le T.\end{aligned}\tag{11}$$

The solution can be sought in the form $u(t,x,y)=\sum\limits_{n=1}^{\infty}w_n(t,x)\sin 2ny$. Functions on the right side of the equation and the initial conditions decomposed in such series

$$\begin{aligned}&y^5-\frac{5}{2}\pi y^4+\frac{5}{3}\pi^2y^3-\frac{1}{6}\pi^4y=-\sum_{n=1}^{\infty}\frac{15}{2n^5}\sin 2ny,\\ &f(t,x,y)=\sum_{n=1}^{\infty}f_n(t,x)\sin 2ny,\\ &-\frac{\pi^2}{6}+\pi y-y^2=-\sum_{m=1}^{\infty}\frac{1}{m^2}\cos 2my,\\ &\Big(-\frac{\pi^2}{6}+\pi y-y^2\Big)u=\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\frac{w_k}{m^2}\delta_n^{k,m}\sin 2ny,\end{aligned}$$

where $\delta_n^{k,m}=\begin{cases}\frac{1}{2}, & \text{if } k+m-n=0,\\ -\frac{1}{2}, & \text{if } (k-m+n)(m-k+n)=0.\end{cases}$

So, we obtain the countable system of second order differential equations

$$\frac{\partial^2 w_n}{\partial t^2}-x^2\frac{\partial^2 w_n}{\partial x^2}+\omega_n^2w_n-2\omega_n\frac{\partial w_n}{\partial t}=-x\frac{\partial w_n}{\partial x}+\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\frac{w_k}{m^2}\delta_n^{k,m}+f_n,\quad n\in\mathbb{N},\tag{12}$$

with initial conditions

$$w_n|_{t=0}=0,\quad \frac{\partial w_n}{\partial t}\Big|_{t=0}=-\frac{15}{2n^5}(\pi x-x^2),\qquad 0\le x\le\pi,\ n\in\mathbb{N},$$

and homogeneous boundary conditions.

Perform a change of variables $w_n=v_ne^{\omega_nt}$. The system (12) will be rewritten in a form

$$\frac{\partial^2 v_n}{\partial t^2}-x^2\frac{\partial^2 v_n}{\partial x^2}+x\frac{\partial v_n}{\partial x}=\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\frac{v_ke^{(\omega_k-\omega_n)t}}{m^2}\delta_n^{k,m}+\frac{f_n}{e^{\omega_nt}},\quad n\in\mathbb{N},$$

with initial and homogeneous boundary conditions

$$v_n|_{t=0}=0,\quad \frac{\partial v_n}{\partial t}\Big|_{t=0}=-\frac{15}{2n^5}(\pi x-x^2),\qquad 0\le x\le\pi,\ n\in\mathbb{N}.$$

In this case $\Delta=4x^2$, that is

$$\begin{aligned}v_{1,n}&=\frac{\partial v_n}{\partial t}+x\frac{\partial v_n}{\partial x},\\ v_{2,n}&=\frac{\partial v_n}{\partial t}-x\frac{\partial v_n}{\partial x}.\end{aligned}$$

As a result, we obtain the countable system of first order differential equations

$$\begin{cases} \dfrac{\partial v_{1,n}}{\partial t} - x\dfrac{\partial v_{1,n}}{\partial x} = \sum\limits_{k=1}^{\infty}\sum\limits_{m=1}^{\infty} \dfrac{v_k e^{(\omega_k-\omega_n)t}}{m^2}\delta_n^{k,m} + \dfrac{f_n}{e^{\omega_n t}}, \\ \dfrac{\partial v_{2,n}}{\partial t} + x\dfrac{\partial v_{2,n}}{\partial x} = \sum\limits_{k=1}^{\infty}\sum\limits_{m=1}^{\infty} \dfrac{v_k e^{(\omega_k-\omega_n)t}}{m^2}\delta_n^{k,m} + \dfrac{f_n}{e^{\omega_n t}}, \\ \dfrac{\partial v_n}{\partial t} = \dfrac{v_{1,n}+v_{2,n}}{2}. \end{cases} \tag{13}$$

Since $\lambda_1 = x > 0, \lambda_2 = -x < 0$, initial and boundary conditions will be rewritten in a form:

$$v_n|_{t=0} = 0,\ v_{1,n}|_{t=0} = -\frac{15}{2n^5}(\pi x - x^2),\ v_{2,n}|_{t=0} = -\frac{15}{2n^5}(\pi x - x^2), \quad (t,x) \in \Pi_0; \tag{14}$$

$$v_n|_{t=0} = 0,\ v_{2,n}|_{t=0} = -\frac{15}{2n^5}(\pi x - x^2),\ v_{1,n}|_{x=0} = -v_{2,n}|_{x=0}, \quad (t,x) \in \Pi_1; \tag{15}$$

$$v_n|_{t=0} = 0,\ v_{1,n}|_{t=0} = -\frac{15}{2n^5}(\pi x - x^2),\ v_{2,n}|_{x=\pi} = -v_{1,n}|_{x=\pi}, \quad (t,x) \in \Pi_2. \tag{16}$$

After solving the problem (13)–(16) (see [9]), we will obtain a system of functions

$$\begin{aligned} v_n &= -\frac{15t}{2n^5 e^{\omega_n t}}(\pi x - x^2), \\ v_{1,n} &= -\frac{15t}{2n^5 e^{\omega_n t}}\big((1-\omega_n t)(\pi x - x^2) + t(\pi x - 2x^2)\big), \\ v_{2,n} &= -\frac{15t}{2n^5 e^{\omega_n t}}\big((1-\omega_n t)(\pi x - x^2) - t(\pi x - 2x^2)\big). \end{aligned}$$

So $w_n = \frac{-15t}{2n^5}(\pi x - x^2)$.

Therefore $u(t,x,y) = \frac{15t}{2}(x^2 - \pi x)\sum\limits_{n=1}^{\infty}\frac{\sin 2ny}{n^5}$ is the exact solution of the problem (10)–(11). In the Figure 4 we can see 3D-graphics of the solution in the case of $t = 0.25$ and $t = 0.5$.

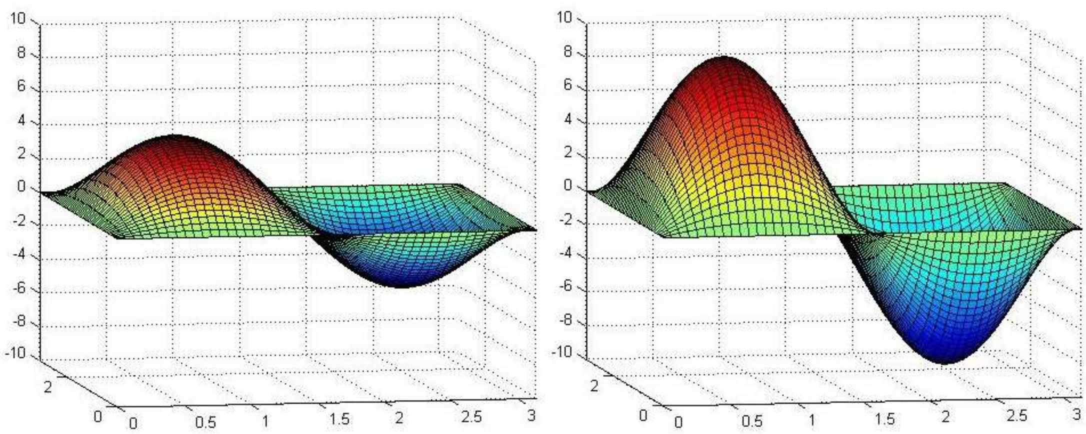

Figure 4: Graphics of solutions at $t = 0.25$ and $t = 0.5$.

Together with the problem (13)–(16), we consider truncated system

$$\begin{cases} \dfrac{\partial v_{1,n}}{\partial t} - x\dfrac{\partial v_{1,n}}{\partial x} = \sum\limits_{k=1}^{N}\sum\limits_{m=1}^{\infty} \dfrac{v_k e^{(\omega_k-\omega_n)t}}{m^2}\delta_n^{k,m} + \dfrac{f_n}{e^{\omega_n t}}, \\ \dfrac{\partial v_{2,n}}{\partial t} + x\dfrac{\partial v_{2,n}}{\partial x} = \sum\limits_{k=1}^{N}\sum\limits_{m=1}^{\infty} \dfrac{v_k e^{(\omega_k-\omega_n)t}}{m^2}\delta_n^{k,m} + \dfrac{f_n}{e^{\omega_n t}}, \\ \dfrac{\partial v_n}{\partial t} = \dfrac{v_{1,n}+v_{2,n}}{2}, \end{cases} \tag{17}$$

with the initial and the boundary conditions (14)–(16). With some suppositions [10], the solutions of the problems (17), (14)–(16) and (13)–(16) will be as close as possible.

Let v_n^N is the solution of the problem (17), (14)–(16) and $u^N(t,x,y) = \sum\limits_{n=1}^{N} w_n^N \sin 2ny$. Figure 5 shows a graph of $\dfrac{\max\limits_{t,x,y}\{|u^N(t,x,y)-u(t,x,y)|\}}{\max\limits_{t,x,y}\{|u(t,x,y)|\}}$.

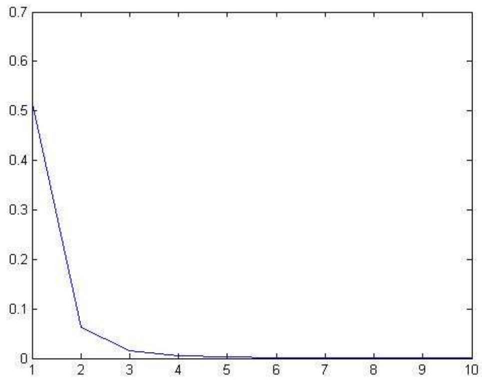

Figure 5: Dependence of difference between exact and approximate solution by N.

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У цій роботі на модельному прикладі мішаної задачі для диференціального рівняння четвертого порядку показано, як таку задачу можа звести до задачі для зліченної гіперболічної системи зв'язних рівнянь першого порядку.

Ключові слова і фрази: зліченна гіперболічна система, мішана задача.

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A CLASS OF JULIA EXCEPTIONAL FUNCTIONS

The class of p-loxodromic functions (meromorphic functions, satisfying the condition $f(qz) = pf(z)$ for some $q \in \mathbb{C}\backslash\{0\}$ and all $z \in \mathbb{C}\backslash\{0\}$) is studied. Each p-loxodromic function is Julia exceptional. The representation of these functions as well as their zero and pole distribution are investigated.

Key words and phrases: p-loxodromic function, the Schottky-Klein prime function, Julia exceptionality.

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INTRODUCTION

Denote $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$, and let $q, p \in \mathbb{C}^*$, $|q| < 1$.

Definition 1. *A meromorphic in $\mathbb{C}^*$ function f is said to be p-loxodromic of multiplicator q if for every $z \in \mathbb{C}^*$*

$$f(qz) = pf(z). \tag{1}$$

Let $\mathcal{L}_{qp}$ denotes the class of p-loxodromic functions of multiplicator q.

The case $p = 1$ has been studied earlier in the works of O. Rausenberger [9], G. Valiron [11] and Y. Hellegouarch [5]. In his work [3, p. 133] which A. Ostrowski [8] called "besonders schöne und überraschende" G. Julia gave an example of a meromorphic in the punctured plane $\mathbb{C}^*$ function satisfying (1) with $p = 1$ for some non-zero q, $|q| \neq 1$, and all $z \in \mathbb{C}^*$. He noted that the family $\{f_n(z)\}$, $f_n(z) = f(q^n z)$ is normal [7] in $\mathbb{C}^*$ because $f_n(z) = f(z)$ for all $z \in \mathbb{C}^*$.

If $p = 1$ the function f is called loxodromic. Loxodromic functions of multiplicator q form a field, which is denoted by $\mathcal{L}_q$. The set $\mathcal{L}_{qp}$ forms an Abelian group with respect to addition.

It is obvious that a ratio of two functions from $\mathcal{L}_{qp}$ is a loxodromic function, and the derivative of the loxodromic function is p-loxodromic with $p = \frac{1}{q}$.

Remark 1. *Every $f \equiv const$ belongs to $\mathcal{L}_q$, but the unique constant function belonging to $\mathcal{L}_{qp}$ is $f \equiv 0$.*

If $f \in \mathcal{L}_{qp}$ and a is a zero of f, then aq^n, $n \in \mathbb{Z}$, are as well. That is, in the case of non-positive q the zeros of f lay on a logarithmic spiral. Let $a = |a|e^{i\alpha}$, $q = |q|e^{i\gamma}$. Then the logarithmic spiral in polar coordinates (r, φ) takes the form

$$\log r - \log |a| = k(\varphi - \alpha),$$

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where $k = \frac{\log |q|}{\gamma}$. The same concerns the poles of f. The image of a logarithmic spiral on the Riemann sphere by the stereographic projection intersects each meridian at the same angle and is called loxodromic curve ($\lambda o \xi o \zeta$ - oblique, $\delta \rho o \mu o \zeta$ - way). That is why we call (following G. Valiron) the function from $\mathcal{L}_q$ loxodromic.

Remark 2. *If $f \in \mathcal{L}_q$ and z is its a-point, $a \in \mathbb{C} \cup \{\infty\}$, then $q^n z, n \in \mathbb{Z}$, are its a-points too. In the case, $f \in \mathcal{L}_{qp}$, the previous considerations are valid only for the zeros and the poles of f.*

It is easy to verify, that $\mathcal{L}_{qp}$ forms the linear spaces over the fields $\mathbb{C}$ and $\mathcal{L}_q$. Also it is clear that $\mathcal{L}_{qp}$ has the following properties.

Proposition. *The linear space $\mathcal{L}_{qp}$ has the following properties.*

1. *The map $D : f(z) \mapsto zf'(z)$ maps $\mathcal{L}_{qp}$ to $\mathcal{L}_{qp}$.*
2. *The map $D_l : f(z) \mapsto z\frac{f'(z)}{f(z)}$ maps $\mathcal{L}_{qp}$ to $\mathcal{L}_q$.*
3. *$f(z) \in \mathcal{L}_{qp} \Rightarrow f(\frac{1}{z}) \in \mathcal{L}_{q\frac{1}{p}}$.*

Let us give nontrivial example of p-loxodromic function of multiplicator q. Put

$$h(z) = \prod_{n=1}^{\infty}(1 - q^n z), \quad 0 < |q| < 1.$$

Definition 2. *The function*

$$P(z) = (1 - z)h(z)h\left(\frac{1}{z}\right) = (1 - z)\prod_{n=1}^{\infty}(1 - q^n z)(1 - \frac{q^n}{z})$$

is called the Schottky-Klein prime function.

This function is holomorphic in $\mathbb{C}^*$ with zero sequence $\{q^n\}$, $n \in \mathbb{Z}$. It was introduced by Schottky [10] and Klein [6] for the study of conformal mappings of doubly-connected domains, see also [2].

It is easy to obtain the following property of P

$$P(qz) = -\frac{1}{z}P(z). \tag{2}$$

Example 1. *Consider the function*

$$f(z) = \frac{P\left(\frac{z}{p}\right)}{P(z)}.$$

Using (2)*, it is easy to show that $f \in \mathcal{L}_{qp}$.*

1 The numbers of zeros and poles of p-loxodromic functions in an annulus

Let $A_q(R) = \{z \in \mathbb{C} : |q|R < |z| \leq R\}$, $R > 0$ and $A_q = A_q(1)$.

Theorem 1. *Let $f \in \mathcal{L}_{qp}$ and the boundary of $A_q(R)$ contains neither zeros nor poles of f. Then f has equal numbers of zeros and poles (counted according to their multiplicities) in every $A_q(R)$.*

Proof. Let $\Gamma_1 = \{z \in \mathbb{C} : |z| = |q|R\}$ and $\Gamma_2 = \{z \in \mathbb{C} : |z| = R\}$ denote the circles bounding $A_q(R)$. Let $n(f)$ be the number of poles of f in $A_q(R)$.

By the argument principle, we have

$$n\left(\frac{1}{f}\right) - n(f) = \frac{1}{2i\pi}\left(\int\limits_{\Gamma_2^+} \frac{f'(z)}{f(z)}dz - \int\limits_{\Gamma_1^+} \frac{f'(\xi)}{f(\xi)}d\xi\right). \tag{3}$$

Setting $\xi = qz$ in the second integral of (3), we obtain

$$n\left(\frac{1}{f}\right) - n(f) = \frac{1}{2i\pi}\int\limits_{\Gamma_2^+} \left(\frac{f'(z)}{f(z)} - q\frac{f'(qz)}{f(qz)}\right)dz. \tag{4}$$

Since $f \in \mathcal{L}_{qp}$, the relation (1) implies

$$f'(qz) = \frac{p}{q}f'(z). \tag{5}$$

Putting (1) and (5) in (4), we obtain the conclusion of the theorem. □

Remark 3. *Every non-constant loxodromic function of multiplicator q has at least two poles (and two zeros) in every annulus $A_q(R)$ [5]. As we see from Example 1, the p-loxodromic function f has the unique pole $z = 1$ in A_q. This is an essential difference between loxodromic and p-loxodromic functions with $p \neq 1$.*

2 REPRESENTATION OF p-LOXODROMIC FUNCTIONS

The representation of loxodromic functions from $\mathcal{L}_q$ was given in [11], [5]. The following theorem gives the representation of a function from $\mathcal{L}_{qp}$.

Let $a_1, ..., a_m$ and $b_1, ..., b_m$ be the zeros and the poles of $f \in \mathcal{L}_{qp}$ in A_q respectively. Denote

$$\lambda = \frac{a_1 \cdot ... \cdot a_m}{b_1 \cdot ... \cdot b_m}. \tag{6}$$

Theorem 2. *The non-identical zero meromorphic in $\mathbb{C}^*$ function f belongs to $\mathcal{L}_{qp}$, $p \neq 1$, if and only if there exists $\nu \in \mathbb{Z}$ such that $\lambda = \frac{p}{q^\nu}$ and f has the form*

$$f(z) = cz^\nu \frac{P\left(\frac{z}{a_1}\right) \cdot ... \cdot P\left(\frac{z}{a_m}\right)}{P\left(\frac{z}{b_1}\right) \cdot ... \cdot P\left(\frac{z}{b_m}\right)}, \tag{7}$$

where c is a constant.

Proof. Firstly, denote

$$M(z) = \frac{P\left(\frac{z}{a_1}\right) \cdot ... \cdot P\left(\frac{z}{a_m}\right)}{P\left(\frac{z}{b_1}\right) \cdot ... \cdot P\left(\frac{z}{b_m}\right)}$$

and consider the function

$$g(z) = \frac{f(z)}{M(z)}.$$

Since the functions f and M have the same zeros and poles, it follows that their ratio g is holomorphic in $\mathbb{C}^*$ function. Let $g(z) = \sum\limits_{n=-\infty}^{+\infty} c_n z^n$ be the Laurant expansion of g in $\mathbb{C}^*$. Using relation (1) and the equality (2), we obtain

$$\lambda g(qz) = pg(z). \tag{8}$$

According to (8), we obtain

$$\lambda \sum_{n=-\infty}^{+\infty} c_n q^n z^n = p \sum_{n=-\infty}^{+\infty} c_n z^n$$

for any $z \in \mathbb{C}^*$. This implies $\lambda c_n q^n = pc_n$ or $c_n(\lambda q^n - p) = 0$. Then there exists at least one $c_\nu \neq 0, \nu \in \mathbb{Z}$, such that

$$c_\nu(\lambda q^\nu - p) = 0. \tag{9}$$

Hence, the relation (9) implies $q^\nu = \dfrac{p}{\lambda}$. We see also that $c_n = 0$ if $n \neq \nu$, so we have $g(z) = c_\nu z^\nu$. Thus, we can conclude

$$f(z) = g(z)M(z) = cz^\nu M(z),$$

where c is a constant.

Secondly, we have $f(z) = cz^\nu M(z)$, $\nu \in \mathbb{Z}$. Show that it belongs to $\mathcal{L}_{qp}$. Thus, $f(qz) = cq^\nu z^\nu M(qz)$. Indeed, using (2), we obtain

$$f(qz) = cq^\nu z^\nu \lambda M(z) = pf(z).$$

This completes the proof.

□

Corollary 1. *Assume $f \in \mathcal{L}_{qp}$, if f is holomorphic in $\mathbb{C}^*$, then $f(z) \equiv 0$ or there exists $k \in \mathbb{Z}\backslash\{0\}$ such that $p = q^k$ and $f(z) = cz^k$, where c is a constant. Conversely, a holomorphic in $\mathbb{C}^*$ function of the form $f(z) = cz^k$, where $k \in \mathbb{Z}\backslash\{0\}$, c is a constant, belongs to $\mathcal{L}_{qp}$.*

3 ZERO AND POLE DISTRIBUTION

Let $\{a_j\}$, $\{b_j\}$, $j \in \mathbb{Z}$ be a couple of sequences in $\mathbb{C}^*$, $p \neq 1$. Put

$$\mu(r) = [\log r / \log |q|] - 1.$$

Note that $\mu(r) = 0$ if $|q| \leq r < 1$. Denote

$$\mathfrak{M}_\nu(r) = \frac{1}{|p|^{\mu(r)}} \times \begin{cases} r^\nu \dfrac{\prod\limits_{1<|a_j|\leq r} \frac{r}{|a_j|}}{\prod\limits_{1<|b_j|\leq r} \frac{r}{|b_j|}}, & r > 1; \\[2ex] r^\nu \dfrac{\prod\limits_{r<|a_j|\leq 1} \frac{|a_j|}{r}}{\prod\limits_{r<|b_j|\leq 1} \frac{|b_j|}{r}}, & 0 < r \leq 1. \end{cases}$$

Theorem 3. *The zero sequence $\{a_j\}$ and the pole sequence $\{b_j\}$ of a non-identical zero meromorphic p-loxodromic function of multiplicator q satisfy the following conditions:*

(i) *the number of a_j and b_j in every annulus of the form $\{z : r < |z| < 2r\}$, $r > 0$ is bounded by an absolute constant;*

(ii) *the difference between the numbers of a_j and b_k in every annulus $\{z : r_1 < |z| < r_2\}$, $0 < r_1 < r_2 < +\infty$ is bounded by an absolute constant;*

(iii) *there exists $C_1 > 0$ such that $\left|\frac{a_j}{b_k} - 1\right| > C_1$ for every $j, k \in \mathbb{Z}$;*

(iv) *the function $\mathfrak{M}_\nu(r)$, where $\nu \in \mathbb{Z}$ such that $\lambda = \frac{p}{q^\nu}$, and λ is given by (6), is bounded for $r > 0$.*

Proof. Let f be a p-loxodromic of multiplicator q function. If f is holomorphic then by Corollary 1 there exists $k \in \mathbb{Z}\backslash\{0\}$ such that $f(z) = cz^k$, and c is a constant. Hence, f has no zeros in $\mathbb{C}^*$. So there is nothing to prove.

Let f be meromorphic. Then by Remark 2 and Theorem 1 it has infinitely many zeros and poles.

(i) First we remark that there exists a unique $n_0 \in \mathbb{Z}_+$ such that $\dfrac{1}{|q|^{n_0}} \leq 2 < \dfrac{1}{|q|^{n_0+1}}$. This n_0 is equal to $\left[\dfrac{\log 2}{\log \frac{1}{|q|}}\right]$.

Since every annulus $\{z : \dfrac{r}{|q|^k} < |z| \leq \dfrac{r}{|q|^{k+1}}\}$, where $k \in \mathbb{Z}$, contains the same number of zeros of f, say m, and

$$(r, 2r] = \left(\bigcup_{k=0}^{n_0-1}\left(\frac{r}{|q|^k}, \frac{r}{|q|^{k+1}}\right]\right) \cup \left(\frac{r}{|q|^{n_0}}, 2r\right]$$

it follows that the annulus $\{z : r < |z| \leq 2r\}$ contains at least $n_0 m$ and less than $(n_0+1)m$ zeros of f. The same is true about the poles of f.

(ii) Similarly as in (i) we can find unique $n_1, n_2 \in \mathbb{Z}$ such that

$$|q|^{n_1+1} < r_1 \leq |q|^{n_1} < |q|^{n_2} < r_2 \leq |q|^{n_2-1}.$$

Hence

$$(r_1, r_2) = (r_1, |q|^{n_1}] \cup \left(\bigcup_{k=n_1}^{n_2-1}(|q|^k, |q|^{k+1}]\right) \cup (|q|^{n_2}, r_2).$$

Every annulus of the form $\{z : |q|^{k+1} < |z| \leq |q|^k\}$, where $k \in \mathbb{Z}$, contains equal amount of zeros and poles of f counted according to their multiplicities (we have denoted this number by m). Therefore the difference between the numbers of zeros and poles of f in the annulus $\{z : r_1 < |z| < r_2\}$ is no greater than $2m$ because of the choice of n_1, n_2.

(iii) Let $a_1, a_2, ..., a_m$ and $b_1, b_2, ..., b_m$ be the zeros and the poles of f in $\{z : |q| < |z| \leq 1\}$ respectively. Then all the zeros of f have the form $\alpha_{\mu,k} = a_k q^\mu$, where $\mu \in \mathbb{Z}$, $k = 1, 2, ..., m$.

The same is true about the poles of f, namely $\beta_{\nu,k} = b_k q^\nu$, where $\nu \in \mathbb{Z}$, $k = 1, 2, ..., m$. So, $\frac{\alpha_{\mu,j}}{\beta_{\nu,k}} = \frac{a_j}{b_k} q^l$, where $l \in \mathbb{Z}$.

It is necessary to show that there exists $C > 0$ such that the inequality

$$\left| \frac{a_j}{b_k} q^l - 1 \right| > C$$

holds for all $j, k \in \{1, 2, ..., m\}$, and $l \in \mathbb{Z}$.

Suppose that for any $\varepsilon > 0$ there exist $j, k \in \{1, 2, ..., m\}$, and $l \in \mathbb{Z}$ such that

$$\left| \frac{a_j}{b_k} q^l - 1 \right| \le \varepsilon. \tag{10}$$

Without loss of generality we can assume that $|l| \le 2$. Indeed, taking into account where a_j, b_k belong to, we have

$$\left| \frac{a_j}{b_k} q^l \right| \le \frac{1}{|q|} |q|^l \le |q|, \qquad l \ge 2.$$

Similarly,

$$\left| \frac{a_j}{b_k} q^l \right| \ge |q| |q|^l \ge \frac{1}{|q|}, \qquad l \le -2.$$

So, for all $j, k \in \{1, 2, ..., m\}$, and $l \ge 2$

$$\left| \frac{a_j}{b_k} q^l - 1 \right| \ge 1 - |q|,$$

and for $l \le -2$

$$\left| \frac{a_j}{b_k} q^l - 1 \right| \ge \frac{1}{|q|} - 1.$$

Let now $|l| < 2$. Choose

$$\varepsilon = \frac{1}{2} \min\{|a_j q^l - b_k| : j, k \in \{1, 2, ..., m\}, -1 \le l \le 1\}.$$

Then (10) implies

$$|a_j q^l - b_k| \le \varepsilon |b_k| \le \varepsilon\,.$$

That is

$$|a_j q^l - b_k| \le \frac{1}{2} \min\{|a_j q^l - b_k| : j, k \in \{1, 2, ..., m\}, -1 \le l \le 1\}$$

which gives a contradiction.

(iv) We remind that f has representation (7). It can be rewritten as follows

$$f(z) = c z^\nu \prod_{k=1}^{m} \frac{\prod\limits_{n=0}^{+\infty} \left(1 - \frac{q^n z}{a_k}\right) \prod\limits_{n=1}^{+\infty} \left(1 - \frac{q^n a_k}{z}\right)}{\prod\limits_{n=0}^{+\infty} \left(1 - \frac{q^n z}{b_k}\right) \prod\limits_{n=1}^{+\infty} \left(1 - \frac{q^n b_k}{z}\right)}, \qquad z \in \mathbb{C}^*. \tag{11}$$

Clearly, we can assume $c \ne 0$. Consider the integral means $I(r) = \frac{1}{2\pi} \int\limits_0^{2\pi} \log |f(re^{i\theta})|\, d\theta$, $r > 0$.

Let $z = re^{i\theta}$. We have for $r > 1$ [4, p. 8]

$$\frac{1}{2\pi}\int_0^{2\pi} \log\left|1 - \frac{z}{a_j}\right| d\theta = \log^+ \frac{r}{|a_j|},$$

and, if $|a_j| \le 1$

$$\frac{1}{2\pi}\int_0^{2\pi} \log\left|1 - \frac{a_j}{z}\right| d\theta = 0.$$

The same is true for b_j.

Since for every $k \in \{1, 2, ..., m\}$ we have $|c_k q^{-n}| > 1$ for $n \in \mathbb{N}$, and $|c_k q^n| \le 1$ for $n \in \mathbb{N} \cup \{0\}$, where c_k is a zero or pole of f, then (11) implies

$$I(r) = \nu \log r + \sum_{|a_j|>1} \log^+ \frac{r}{|a_j|} - \sum_{|b_j|>1} \log^+ \frac{r}{|b_j|} + \log|c|, \quad r > 1.$$

Similarly, for $0 < r \le 1$ we obtain

$$I(r) = \nu \log r + \sum_{|a_j|\le 1} \log^+ \frac{|a_j|}{r} - \sum_{|b_j|\le 1} \log^+ \frac{|b_j|}{r} + \log|c|.$$

Hence,

$$\mathfrak{M}_\nu(r) = \frac{1}{|p|^{\mu(r)}}\frac{1}{|c|}\exp I(r) = \frac{1}{|c|}\exp\{I(r) - \mu(r)\log|p|\}, \quad r > 0.$$

Since $I(r)$ is convex with respect to $\log r$ and consequently continuous, $I(r)$ is bounded on $[|q|, 1]$. It follows from the definition of a p-loxodromic function of multiplicator q that

$$I(|q|^k r) = I(r) + k\log|p|$$

for every $k \in \mathbb{Z}$. On the other hand

$$\mu(|q|^k r) = \left[\frac{k\log|q| + \log r}{\log|q|}\right] - 1 = k, \qquad |q| \le r < 1.$$

That is

$$\mathfrak{M}_\nu(|q|^k r) = \mathfrak{M}_\nu(r), \qquad |q| \le r < 1$$

for all $k \in \mathbb{Z}$. Then we conclude that $\mathfrak{M}_\nu(r)$ remains bounded for all $r > 0$ which completes the proof.

$\square$

4 JULIA EXCEPTIONALITY

Definition 3. *Let $f_n, n \in \mathbb{N}$, be meromorphic functions in a domain G. A sequence $\{f_n(z)\}$ is said to be uniformly convergent to $f(z)$ on G in the Carathéodory-Landau sense [1] if for any point $z_0 \in G$ there exists a disk $K(z_0)$ centered at this point such that $K(z_0) \subset G$ and*

$$(\forall \varepsilon > 0)(\exists n_0 \in \mathbb{N})(\forall n > n_0)(\forall z \in K(z_0)) : |f_n(z) - f(z)| < \varepsilon,$$

whenever $f(z_0) \ne \infty$, or

$$\left|\frac{1}{f_n(z)} - \frac{1}{f(z)}\right| < \varepsilon,$$

whenever $f(z_0) = \infty$.

Note that this convergence is equivalent to the convergence in the spherical metric.

Definition 4. *A family $\mathcal{F}$ of meromorphic in $\mathbb{C}^*$ functions is said to be normal if every sequence $\{f_n\} \subseteq \mathcal{F}$ contains a subsequence which converges uniformly in the Carathéodory-Landau sense.*

Definition 5. *A meromorphic in $\mathbb{C}^*$ function f is called Julia exceptional (see [7]) if for some q, $0 < |q| < 1$, the family $\{f_n(z)\}$, $n \in \mathbb{Z}$, where $f_n(z) = f(q^n z)$, is normal in $\mathbb{C}^*$.*

In $\mathbb{C}$ there are few simple examples of Julia exceptional functions. But in $\mathbb{C}^*$ we have the following.

Let $f \in \mathcal{L}_{qp}$. We have

$$f_n(z) = f(q^n z) = p^n f(z)$$

for every $z \in \mathbb{C}^*$.

If $|p| > 1$, then a limiting function of the family $\{f_n(z)\}$, $n \in \mathbb{Z}$, is ∞. Otherwise, if $|p| < 1$, then a limiting function is 0. If $|p| = 1$, that is $p = e^{i\alpha}$, we have $f_n(z) = e^{in\alpha} f(z)$. Hence, the set of limit functions depends on α. If $\alpha = \frac{\pi m}{k}$, where $m \in \mathbb{Z}$, $k \in \mathbb{N}$, the number of limiting functions is less than or equals to $2k$. Otherwise, if $\alpha = \pi r$, where $r \in \mathbb{R} \backslash \mathbb{Q}$, the number of limiting functions is infinite.

Example 2. *Let $f \in \mathcal{L}_q^\alpha$ with $\alpha = \frac{\pi}{4}$. Then*

$$f_n(z) = f(q^n z) = p^n f(z) = e^{in\frac{\pi}{4}} f(z).$$

Thus, we obtain eight limiting functions

$$\pm f, \ \pm i f, \ \left(\frac{\sqrt{2}}{2} \pm i \frac{\sqrt{2}}{2}\right) f, \ \left(-\frac{\sqrt{2}}{2} \pm i \frac{\sqrt{2}}{2}\right) f.$$

Hence, f is Julia exceptional in $\mathbb{C}^*$.

These results can be summarized as follows.

Theorem 4. *Each function $f \in \mathcal{L}_{qp}$ is Julia exceptional in $\mathbb{C}^*$.*

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Хорощак В.С., Христіянин А.Я., Луківська Д.В. *Один клас Жюліа виняткових функцій* // Карпатські матем. публ. — 2016. — Т.8, №1. — С. 172–180.

Досліджується клас p-локсодромних функцій (мероморфних функцій, що задовольняють умову $f(qz) = pf(z)$ при деяких $q \in \mathbb{C}\setminus\{0\}$ для всіх $z \in \mathbb{C} \setminus \{0\}$). Доведено, що кожна p-локсодромна функція є Жюліа винятковою. Подано зображення таких функцій та описано розподіл їх нулів та полюсів.

Ключові слова і фрази: p-локсодромна функція, первинна функція Шотткі-Кляйна, Жюліа винятковість.