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THE NONLOCAL PROBLEM FOR THE DIFFERENTIAL-OPERATOR EQUATION OF THE EVEN ORDER WITH THE INVOLUTION

In this paper, the problem with boundary non-self-adjoint conditions for differential-operator equations of the order $2n$ with involution is studied. Spectral properties of operator of the problem is investigated.

By analogy of separation of variables the nonlocal problem for the differential-operator equation of the even order is reduced to a sequence $\{L_k\}_{k=1}^{\infty}$ of operators of boundary value problems for ordinary differential equations of even order. It is established that each element L_k of this sequence is an isospectral perturbation of the self-adjoint operator $L_{0,k}$ of the boundary value problem for some linear differential equation of order $2n$.

We construct a commutative group of transformation operators whose elements reflect the system $V(L_{0,k})$ of the eigenfunctions of the operator $L_{0,k}$ in the system $V(L_k)$ of the eigenfunctions of the operators L_k. The eigenfunctions of the operator L of the boundary value problem for a differential equation with involution are obtained as the result of the action of some specially constructed operator on eigenfunctions of the sequence of operators $L_{0,k}$.

The conditions under which the system of eigenfunctions of the operator L of the studied problem is a Riesz basis is established.

Key words and phrases: operator of involution, differential-operator equation, eigenfunctions, Riesz basis.

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Introduction

The boundary value problems for linear differential-operator equations are used in the simulation of boundary value problems for differential equations with partial derivatives, in particular, in the study of nonlocal problems. Significant results concerning the theory of boundary value problems for differential-operator equations were obtained in the papers of Vishik M.I., Boehner M., Gorbachuk V.I., Gorbachuk M.L., Dezin O.O., Dubinsky Yu.V., Kochubei A.N., Lions J.L., Mamedov K.S., Romanko V.K., Shakhmurov Veli B., Triebel Kh., Yakubov S., Yurchuk N.Yu.

During recent years, the number of publications with the use of an involution operator in various sections of the theory of ordinary differential equations (see [4, 10, 12, 15, 16, 19]), of partial differential equations (see [3, 7, 9, 14, 16, 17, 20, 21]), of linear operators, T-invariant with respect to some group of homeomorphisms (see [8]), differential equations with operator coefficients (see [5–7]), PT-symmetric operators (see [1, 2]) increased significantly.

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1 STATEMENT OF PROBLEM

Let us make some notations. H is a separable Hilbert space; $A : D(A) \subset H \to H$ is the closed unbounded linear operator with the discrete spectrum $\sigma(A) \equiv \{z_k = \alpha k^\gamma, \alpha, \gamma > 0, k = 1,2,\ldots\}$; $V(A) \equiv \{v_k \in H : k = 1,2,\ldots\}$ is the system of the eigenfunctions which forms the Riesz basis in the space H; $H(A^s) \equiv \{h \in H : A^s h \in H\}$; $s \geq 0$; $W_1 \equiv L_2((0,1),H)$; $D_x : W_1 \to W_1$ is a strong derivative in the space W_1; $W_2 \equiv \{u \in W_1 : D_x^{2n}u \in W_1, A^{2n}u \in W_1\}$; $[H]$ is the algebra of the bounded linear operators $B : H \to H$; I is the operator of the involution in the space $L_2(0,1)$; $Iy(x) \equiv y(1-x)$; $p_j \equiv \frac{1}{2}(E + (-1)^j I)$ are the orthoprojectors of the space $L_2(0,1)$; $L_{2,j}(0,1) \equiv \{y \in L_2(0,1) : p_j y \equiv y\}$; $j = 0,1$; $W_2^{2n}(0,1) \equiv \left\{y \in L_2(0,1) : y^{(m)} \in C[0,1], m = 0,1,\ldots,2n-1, y^{(2n)} \in L_2(0,1)\right\}$; $W^*(0,1)$ is the space of continuous linear functionals over the space $W_2^{2n}(0,1)$; $W_j^*(0,1) \equiv \{l \in W^*(0,1) : ly = 0, y \in L_{2,1-j}(0,1) \bigcap W_2^{2n}(0,1)\}$; $j = 0,1$.

We consider the following boundary problem

$$\begin{aligned} Lw \equiv (-1)^n D_x^{2n} w(x) + A^{2n} w(x) \\ + \sum_{j=1}^{n} a_j \left(D_x^{2j-1} w(x) - D_x^{2j-1} w(1-x) \right) = f(x), \quad x \in (0,1), \end{aligned} \tag{1}$$

$$\ell_j w \equiv D_x^{m_j} w(0) + (-1)^{m_j} D_x^{m_j} w(1) = \varphi_j, \quad j = 1,2,\ldots,n, \tag{2}$$

$$\ell_{n+j} w \equiv D_x^{m_{n+j}} w(0) - (-1)^{m_{n+j}} D_x^{m_{n+j}} w(1) + l_j^1 w = \varphi_{n+j}, \quad j = 1,2,\ldots,n, \tag{3}$$

$$\ell_j^1 w \equiv \sum_{r=0}^{k_j} (b_{j,r,0} D_x^r w(0) + b_{j,r,1} D_x^r w(1)). \tag{4}$$

By solution of the problem (1)–(4) we mean a function that satisfies equalities

$$\begin{aligned} &\|Lw - f; W_1\| = 0, \ \|l_j w - \varphi_j; H\left(A^{\beta_j}\right)\| = 0, \\ &\beta_j = 2n - m_j - \frac{1}{2}, \ \beta_{n+j} = 2n - \max(m_{n+j}, k_j) - \frac{1}{2}, \\ &a_j, b_{j,r,s} \in \mathbb{R}, \ r = 0,1,\ldots,k_j, \ s = 0,1, \ j = 1,2,\ldots,n, \\ &m_n < m_{n-1} < \ldots < m_1, \ m_{2n} < m_{2n-1} < \ldots < m_{n+1}. \end{aligned}$$

2 AUXILIARY BOUNDARY VALUE PROBLEM

Consider the partial case of the problem (1)–(4), when $a_j = 0$, $b_{j,r,s} = 0$, $r = 0,1,\ldots,k_j$, $s = 0,1$, $j = 1,2,\ldots,n$,

$$(-1)^n D_x^{2n} u(x) + A^{2n} u(x) = f(x), \quad x \in (0,1), \tag{5}$$

$$\ell_{0,j} u \equiv D_x^{m_j} u(0) + (-1)^{m_j} D_x^{m_j} u(1) = 0, \tag{6}$$

$$\ell_{0,n+j} u \equiv D_x^{m_{n+j}} u(0) - (-1)^{m_{n+j}} D_x^{m_{n+j}} u(1) = 0, \quad j = 1,2,\ldots,n. \tag{7}$$

Remark 2.1. *The boundary conditions* (6), (7) *are numbered so that the following conditions are satisfied*

$$l_j \in W_0^*(0,1), \quad l_{n+j} \in W_1^*(0,1), \ j = 1,2,\ldots,n. \tag{8}$$

Let L_0 be the operator of the problem (5)–(7), $L_0u \equiv (-1)^n\ D_x^{2n}u + \ A^{2n}u,\ u \in D(L_0)$, $D(L_0) \equiv \{u \in W_2 : l_ju = 0, j = 1, 2, \ldots, 2n\}$. Consider the spectral problem for the operator L_0

$$(-1)^n\ D_x^{2n}u(x) + \ A^{2n}u(x) = \lambda u(x),\ \ l_ju = 0,\ \lambda \in \mathbb{C},\ j = 1, 2, \ldots, 2n. \tag{9}$$

The solution of the spectral problem (9) is defined as the product $u(x) = y(x)v_k$, $v_k \in V(A)$, $k = 1, 2, \ldots$. To determine the unknown function $y \in W_2^{2n}(0,1)$, we obtain the spectral problem

$$(-1)^n\ y^{(2n)}(x) + \ z_k^{2n}y(x) = \lambda y(x),\ \ \lambda \in \mathbb{C}, \tag{10}$$

$$\ell_{0,j}y \equiv\ y^{(m_j)}(0) + \ (-1)^{m_j}\, y^{(m_j)}(1) = 0, \quad j = 1, 2, \ldots, n, \tag{11}$$

$$\ell_{0,n+j}y \equiv\ y^{(m_{n+j})}y(0) - (-1)^{m_{n+j}}\, y^{(m_{n+j})}(1) = 0,\ \ j = 1, 2, \ldots, n. \tag{12}$$

Let $L_{0,k}$ be the operator of the problem (10)–(12), $L_{0,k}y \equiv (-1)^n\ y^{(2n)}(x) + \ z_k^{2n}y(x)$; $y \in D(L_{0,k})$; $D(L_{0,k}) \equiv \{y \in W_2^{2n}(0,1) : l_{0,j}y = 0, j = 1, 2, \ldots, 2n\}$.

Assumption B_1**.** The conditions (11), (12) are self-adjoint.

Assumption B_2**.** The boundary conditions (10), (11) are strongly regular according to Birkhoff (see [18]).

In what follows we assume that the assumptions B_1–B_2 are satisfied. The roots ρ_j of the characteristic equation $(-1)^n\rho^{2n} = \lambda - z_k^{2n}$, which corresponds to the differential equation

$$(-1)^n\ y^{(2n)}(x) + \ z_k^{2n}y(x) = \lambda y(x), \tag{13}$$

are determined by the relations $\rho_j = \omega_j\rho$, $\omega_1 = i$, $\omega_j = i \exp i\frac{\pi(j-1)}{2n}$, $\ j = 2, 3, \ldots, n$.

The fundamental system of the solutions of the differential equation (13) is defined by the formulas

$$y_j(x,\rho) \equiv \frac{1}{2}(\exp\omega_j\rho x + \exp\omega_j\rho(1-x)), \tag{14}$$

$$y_{n+j}(x,\rho) \equiv \frac{1}{2}(\exp\omega_j\rho x - \exp\omega_j\rho(1-x)),\ \ j = 1, 2, \ldots, n. \tag{15}$$

Substituting the general solution of the differential equation (13) $y(x,\rho) = \sum\limits_{s=1}^{2n} C_s y_s(x,\rho)$ into the boundary conditions (11), (12) we obtain an equation for determining the eigenvalues of the operator $L_{0,k}$

$$\Delta(\rho) = \det(l_r y_j)_{r,j=1}^{2n} = 0. \tag{16}$$

From the conditions (8) and from the properties of the functions (14), (15), we obtain

$$l_{0,r}y_{n+j} = 0,\ \ l_{0,n+r}y_j = 0,\ \ j, r = 1, 2, \ldots, n, \tag{17}$$

therefore,

$$\Delta(\rho) = \Delta_0(\rho)\Delta_1(\rho) = 0, \tag{18}$$

where $\Delta_s(\rho) = \det(l_{sn+r}y_{sn+j})_{r,j=1}^{n},\ \ s = 0, 1$.

The operator $L_{0,k}$ is self-adjoint, therefore the roots of the equation (18) lie on the semiaxis $Imz = 0$, $Rez \geq 0$. For any $s \in 0, 1$, we number the roots $\rho_{s,q}$ of the equation in ascending order $\rho_{s,1} < \rho_{s,2} < \ldots$.

Thus, the operator $L_{0,k}$ has the eigenvalues

$$\lambda_{s,q,k} = (\rho_{s,q})^{2n} + z_k^{2n},\ \ s \in 0, 1,\ \ q = 1, 2, \ldots . \tag{19}$$

Let $\beta_0 = m_1 + m_2 + \cdots + m_n,\ \ \beta_1 = m_{n+1} + m_{n+2} + \cdots + m_{2n}$. We define the eigenfunctions of the operator $L_{0,k}$, which are normalized in the space. Let $B(s, x, \rho)$ be a square matrix of the order n, the first row of which is determined by the functions $y_{sn+j}(x, \rho)$, and the r-th row is determined by the numbers $l_{sn+r}y_{sn+j},\ r = 2, 3, \ldots, n,\ s = 0, 1, j = 1, 2, \ldots, n$. Let

$$v_{s,q}(x, L_{0,k}) = (\rho_{s,q})^{-\beta_s}\theta_{s,q} \det B(s, x, \rho_{s,q}). \tag{20}$$

Then $\|v_{s,q}(x, L_{0,k}); L_2(0, 1)\| = 1,\ s = 0, 1,\ q = 1, 2, \ldots$.

Lemma 2.1. *Suppose that the assumptions B_1–B_2 hold. Then for each number $k \in \mathbb{N}$ the operator $L_{0,k}$ has the eigenvalues* (19)*, and it also has the system of the eigenfunctions* (20)*, which forms the orthogonal basis in the space $L_2(0, 1)$.*

Therefore, the operator L_0 has a system

$$V(L_0) \equiv \left\{v_{s,q,k}(x, L_0) \in W_1 : v_{s,q,k}(x, L_0) = v_{s,q}(x, L_{0,k})v_k,\ s = 0, 1,\ k, q = 1, 2, \ldots\right\}$$

of the eigenfunctions in the space W_1. The product of a system $V(A)$ and an orthonormal system $V(L_{0,k})$ is the Riesz basis in the space W_1. Thus, the following theorem is true.

Theorem 1. *Let the assumptions B_1–B_2 hold. The operator L_0 has the discrete spectrum*

$$\sigma(L_0) = \left\{\lambda_{s,q,k} = (\rho_{s,q})^{2n} + z_k^{2n},\ s = 0, 1,\ k, q = 1, 2, \ldots\right\}.$$

It also has the system of the eigenfunctions $V(L_0)$ which forms the Riesz basis in the space W_1.

We choose an arbitrary eigenvalue $\lambda_{0,q,k} \in \sigma(L_{0,k}),\ q \in \mathbb{N}$. Let

$$y_{n+j}(x, \rho_{0,q}) \equiv \frac{1}{2}(\exp \omega_j\rho_{0,q}x - \exp \omega_j\rho_{0,q}(1 - x)), j = 1, 2, \ldots, n,$$

$B_p(x, \rho_{0,q})$ is square matrix of the order n, the p-th row of which is defined by the functions $y_{n+j}(x, \rho_{0,q})$, and the r-th row is defined by the numbers $(\omega_j)^{m_{n+r+1}}(1 + (-1)^{m_{n+r+1}} \exp \omega_j\rho_{0,q})$, $r \neq p,\ j, r = 1, 2, \ldots, n$,

$$\begin{aligned} &y_{0,n+p}(x, \rho_{0,q}) \equiv \det B_p(x, \rho_{0,q}),\\ &\Delta_1(\rho_{0,q}) \equiv \det((\omega_j)^{m_{n+r+1}}(1 + (-1)^{m_{n+r+1}} \exp \omega_j\rho_{0,q}))_{r,j=1}^{n},\\ &y_{1,n+p}(x, \rho_{0,q}) \equiv (\Delta_1(\rho_{0,q}))^{-1}y_{0,n+p}(x, \rho_{0,q}). \end{aligned} \tag{21}$$

Substituting the expression (21) into the boundary conditions (11), (12), we see that

$$l_jy_{1,n+p}(x, \rho_{0,q}) = 0,\ j \neq n + p, j = 1, 2, \ldots, 2n, \tag{22}$$

$$l_{n+p}y_{1,n+p}(x, \rho_{0,q}) = (\rho_{0,q})^{m_{n+p}}. \tag{23}$$

3 NON SELF-ADJOINT BOUNDARY VALUE PROBLEM

For the differential-operator equation (5) we consider the following boundary value problem for arbitrary fixed $p \in \{1, 2, \ldots, n\}$, $b \in \mathbb{R}$,

$$\ell_{1,j} u \equiv D_x^{m_j} u(0) + (-1)^{m_j} D_x^{m_j} u(1) = 0, \quad j = 1, 2, \ldots, n, \tag{24}$$

$$\ell_{1,n+j} u \equiv D_x^{m_{n+j}} u(0) - (-1)^{m_{n+j}} D_x^{m_{n+j}} u(1) = 0, \quad j = 1, 2, \ldots, n, \ j \neq p, \tag{25}$$

$$\ell_{1,n+p} u \equiv D_x^{m_{n+p}} u(0) - (-1)^{m_{n+p}} D_x^{m_{n+p}} u(1) + l_p^2 u = 0, \tag{26}$$

$$\ell_p^2 u \equiv b(D_x^{m_{n+p}} u(0) + (-1)^{m_{n+p}} D_x^{m_{n+p}} u(1)) = 0. \tag{27}$$

Let L_1 be the operator of the problem (5), (24)–(27), $L_1 u \equiv (-1)^n D_x^{2n} u(x) + A^{2n} u(x)$, $u \in D(L_1)$, $D(L_1) \equiv \{u \in W_2 : l_{1,j} u = 0, \ j = 1, 2, \ldots, 2n\}$. The solution of the spectral problem (9), (24)–(27) is defined as the product $u(x) = y(x) v_k$, $v_k \in V(A)$, $k = 1, 2, \ldots$. To determine the unknown function $y \in W_2^{2n}(0, 1)$, we obtain the spectral problem

$$(-1)^n y^{(2n)}(x) + z_k^{2n} y(x) = \lambda y(x), \quad \lambda \in \mathbb{C}, \tag{28}$$

$$\ell_{1,j} y \equiv y^{(m_j)}(0) + (-1)^{m_j} y^{(m_j)}(1) = 0, \quad j = 1, 2, \ldots, n, \tag{29}$$

$$\ell_{1,n+j} y \equiv y^{(m_{n+j})}(0) - (-1)^{m_{n+j}} y^{(m_{n+j})}(1) = 0, \quad j = 1, 2, \ldots, n, j \neq p, \tag{30}$$

$$\ell_{1,n+p} y \equiv y^{(m_{n+p})}(0) - (-1)^{m_{n+p}} y^{m_{n+p}}(1) + b(y^{(m_{n+p})}(0) + (-1)^{m_{n+p}} y^{(m_{n+p})}(1)) = 0. \tag{31}$$

Let $L_{1,k}$ be the operator of the problem (28)–(31), $L_{1,k} y \equiv (-1)^n y^{(2n)}(x) + z_k^{2n} y(x)$; $y \in D(L_{1,k})$, $D(L_{1,k}) \equiv \{y \in W_2^{2n}(0, 1) : l_{1,j} y = 0, j = 1, 2, \ldots, 2n\}$.

Theorem 2. *Suppose that the assumptions B_1–B_2 hold. Then, for the any arbitrary fixed numbers $p \in \{1, 2, \ldots, n\}$, $b \in \mathbb{R}$,*

1) the eigenvalues of the operators $L_{0,k}$ and $L_{1,k}$ coincide;

2) the system $V(L_{1,k})$ of the eigenfunctions of the operator $L_{1,k}$ is the Riesz basis of the space $L_2(0, 1)$.

Proof. We show that the eigenvalues of the operators $L_{0,k}$ and $L_{1,k}$ coincide. We substitute the fundamental system (14), (15) of the solutions of the differential equation (28) into the boundary conditions (29)–(31).

$$\det(l_{1,j} y_r(x, \rho))_{j,r=1}^n = \det(l_{1,j} y_r(x, \rho))_{j,r=1}^n \det(l_{1,n+j} y_{n+r}(x, \rho))_{j,r=1}^n.$$

If $l_{p,b} y_{n+j}(x, \rho) = 0$, we obtain the same equations for determining the spectrum. Define the elements of the system $V(L_{1,k})$. Direct substitution shows that the functions $v_{1,q}(x, L_{0,k})$, $q = 1, 2, \ldots$, satisfy the conditions (29)–(31). Therefore, the eigenfunction of the operator $L_{1,k}$ that corresponds to the eigenvalue $\lambda_{1,q,k}$ is defined by

$$v_{1,q}(x, L_{1,k}) = v_{1,q}(x, L_{0,k}), \quad q = 1, 2, \ldots, \tag{32}$$

$$v_{0,q}(x, L_{1,k}) = v_{0,q}(x, L_{0,k}) - l_p^2(v_{0,q}(x, L_{0,k}))(l_{1,n+p} y_{1,n+p}(x, \rho_{0,q}))^{-1} y_{1,n+p}(x, \rho_{0,q}), \ q = 1, 2, \ldots.$$

Taking into account the formulas (31), (21), and the inequalities

$$|l_p^2(v_{0,q}(x, L_{0,k}))| \le K_1 (\rho_{0,q})^{m_{n+p}},$$

we obtain the estimates

$$|l_p^2(v_{0,q}(x, L_{0,k}))(l_{1,n+p}y_{1,n+p}(x, \rho_{0,q}))^{-1}| \le K_2|b|.$$

For the problem (29)–(31), there exists an adjoint problem whose system of the eigenfunctions $W(L_{1,k})$ form the biorthogonal system to the $V(L_{1,k})$. The boundary conditions (29)–(31) are strongly regular according to Birkhoff. Therefore, according to the Kesselman-Mikhailov's Theorem [18], the system $V(L_{1,k})$ is the Riesz basis of the space $L_2(0,1)$. □

4 TRANSFORMATION OPERATORS

For any fixed $k \in \mathbb{N}$, $p \in \{1, 2, \dots, n\}$, we consider the operator $B_p : L_2(0,1) \to L_2(0,1)$, the eigenvalues of which coincide with the eigenvalues of the operator $L_{0,k}$, and the eigenfunctions are defined by

$$v_{1,q}(x, B_p) \equiv v_{1,q}(x, L_{0,k}), v_{0,q}(x, B_p) \equiv v_{0,q}(x, L_{0,k}) + c_q(B_p) y_{1,n+p}(x, \rho_{0,q}), \quad (33)$$

$c_q(B_p) \in \mathbb{R}$, $q = 1, 2, \dots$.

The operator that maps the system $V(L_{0,k})$ into the system $V(B_p)$ of the eigenfunctions of the operator B_p is denoted by $R(B_p) \equiv E + S(B_p)$, $S(B_p) : L_{2,0}(0,1) \to L_{2,1}(0,1)$, $S(B_p) : L_{2,1}(0,1) \to 0$.

We consider the set $G_p(L_{0,k})$ of the operators $R(B_p)$ such that the eigenfunctions of the operator B_p are defined by the equalities (33).

Lemma 4.1. *Suppose that, the assumptions B_1–B_2 hold, $R(B_p) \in G_p(L_{0,k})$. Then the system of the functions $V(B_p)$ is complete and minimal in the space $L_2(0,1)$.*

Taking into account the uniqueness of the operator $R(B_p)^{-1} \equiv E - S(B_p)$, we obtain the statement of the lemma. Suppose that U is the set of systems of functions $(u_m)_{m=1}^{\infty} \subset L_2(0,1)$, that are complete and minimal in space $L_2(0,1)$, $Q(I)$ is a set of operators $R = E + S$, such that $S : L_{2,0}(0,1) \to L_{2,1}(0,1)$, $S : L_{2,1}(0,1) \to 0$, $Q_c(I) \equiv [L_2(0,1)] \bigcap Q(I)$.

Taking into account equality $S^2(B_p) = 0$, $R(B_p) \in G_p(L_{0,k}) \subset Q(I)$ on the set $Q(I)$, we can define the operation of multiplication

$$R_1 R_2 \equiv (E + S_1)(E + S_2) = E + S_1 + S_2, \quad R_1, R_2 \in Q(I).$$

In particular, $Q(I) = Q(I_0)$, $(E + S)(E - S) = E - S^2 = E, E + S = R \in Q(I)$. Therefore, for each operator $R = E + S \in Q(I)$ there exists a unique inverse operator $R^{-1} = E - S$.

According to the definition of the operator B_p and of the set $G_p(L_{0,k})$ we have the inclusions

$$G_p(L_{0,k}) \subset Q(I),\ G_{c,p}(L_{0,k}) \subset Q_c(I),\ p \in \{1, 2, \dots, n\}.$$

Thus, the set $Q(I)$ is an Abelian group which contains the Abelian subgroups $Q_c(I)$, $G_p(L_{0,k})$, $G_{c,p}(L_{0,k})$, $p \in \{1, 2, \dots, n\}$. Therefore, for any operators $R_j = E + S_j \in Q_0(I)$, $j = 1, 2 \dots, d$, $d \in \mathbb{N}$, the following equality is satisfied

$$\prod_{j=1}^{d} R_j \equiv \prod_{j=1}^{d} (E + S_j) = E + \sum_{j=1}^{d} S_j, \quad d \in \mathbb{N}.$$

Lemma 4.2. *Suppose that the assumptions B_1–B_2 hold, $R(B_p) \in G_p(L_{0,k})$. The system of the functions $V(B_p)$ is the Riesz basis of the space $L_2(0,1)$ if and only if the sequence $\{c_q(B_p)\}$ is bounded.*

The proof of the lemma is carried out analogously in [13]. Therefore, the operator L_1 has the system

$$V(L_1) \equiv \left\{v_{s,q,k}(x, L_1) \in W_1 : v_{s,q,k}(x, L_1) = v_{s,q}(x, L_{1,k})v_k,\ s = 0, 1,\ q, k = 1, 2, \dots\right\}$$

of the eigenfunctions in the space W_1. The product of the system $V(A)$ and the system $V(L_{1,k})$ is a Riesz basis in the space W_1. Thus, the following theorem is true.

Theorem 3. *Suppose that the assumptions B_1–B_2 hold. Then for arbitrary fixed numbers $p \in \{1, 2, \dots, n\}$, $b \in \mathbb{R}$, the system of the functions $V(L_1)$ is the Riesz basis of the space W_1.*

5 THE SPECTRAL PROBLEM FOR A DIFFERENTIAL-OPERATOR EQUATION

For the differential-operator equation (5) for arbitrary fixed $b_{p,r,s} \in \mathbb{R}$, $p \in \{1, 2, \dots, n\}$, $r = 0, 1, \dots, k_j, s = 0, 1, j = 1, 2, \dots, n$, we consider problem, generated by nonlocal conditions

$$\ell_{2,j}w \equiv D_x^{m_j}w(0) + (-1)^{m_j} D_x^{m_j}w(1) = 0, \quad j = 1, 2, \dots, n, \tag{34}$$

$$\ell_{2,n+j}w \equiv D_x^{m_{n+j}}w(0) - (-1)^{m_{n+j}} D_x^{m_{n+j}}w(1) = 0, j \neq p, \quad j = 1, 2, \dots, n, \tag{35}$$

$$\ell_{2,n+p}w \equiv D_x^{m_{n+p}}w(0) - (-1)^{m_{n+p}} D_x^{m_{n+p}}w(1) + l_p^1 w = 0, \quad j = 1, 2, \dots, n, \tag{36}$$

$$\ell_p^1 w \equiv \sum_{r=0}^{k_j}(b_{p,r,0}D_x^r w(0) + b_{p,r,1}D_x^r w(1)). \tag{37}$$

Assumption B_3**.** $b_{p,r,0} = (-1)^r b_{p,r,1}$, $r = 0, 1, \dots, k_j$, $p = 1, 2, \dots, n$.

Remark 5.1. *Assumption B_3 implies that $l_p^1 \in W_0^*$, $p = 1, 2, \dots, n$.*

In what follows we assume that the assumptions B_1–B_3 are satisfied. Let L_2 be the operator of the problem (5), (34)–(37),

$$L_2 u \equiv (-1)^n\ D_x^{2n}u(x) + A^{2n}u(x), \quad u \in D(L_1),$$

$$D(L_2) \equiv \{u \in W_2 : l_{2,j}u = 0,\ j = 1, 2, \dots, 2n\}.$$

The solution of the spectral problem (5), (34)–(37) is defined as the product $u(x) = y(x)v_k$, $v_k \in V(A)$, $k = 1, 2, \dots$. To determine the unknown function $y \in W_2^{2n}(0, 1)$, we obtain the spectral problem

$$(-1)^n\ y^{(2n)}(x) + z_k^{2n}y(x) = \lambda y(x), \quad \lambda \in \mathbb{C}, \tag{38}$$

$$\ell_{2,j}y \equiv y^{(m_j)}(0) + (-1)^{m_j} y^{(m_j)}(1) = 0, \quad j = 1, 2, \dots, n, \tag{39}$$

$$\ell_{2,n+j}y \equiv y^{(m_{n+j})}(0) - (-1)^{m_{n+j}} y^{(m_{n+j})}(1) = 0, \quad j = 1, 2, \dots, n, \quad j \neq p, \tag{40}$$

$$\ell_{2,n+p}y \equiv y^{(m_{n+p})}(0) - (-1)^{m_{n+p}} y^{m_{n+p}}(1) + l_p^1 y = 0, \tag{41}$$

$$\ell_p^1 y \equiv \sum_{r=0}^{k_j}(b_{p,r,0}D_x^r y(0) + b_{p,r,1}D_x^r y(1)). \tag{42}$$

Let $L_{2,k}$ be the operator of the problem (38)–(42), $L_{2,k}y \equiv (-1)^n\ y^{(2n)}(x) + z_k^{2n}y(x)$; $y \in D(L_{2,k})$, $D(L_{2,k}) \equiv \{y \in W_2^{2n}(0, 1) : l_{2,j}y = 0, j = 1, 2, \dots, 2n\}$.

Theorem 4. *Suppose that the assumptions B_1–B_3 hold. Then for arbitrary fixed numbers $b_{p,r,s} \in \mathbb{R}$, $x_s \in (0,1)$, $s = 0,1$, $r = 0,1,\dots,k_p$, $p \in \{1,2,\dots,n\}$,*

1) the eigenvalues of the operators $L_{0,k}$ and $L_{2,k}$ coincide;

2) the system $V(L_{2,k})$ of the eigenfunctions of the operator $L_{2,k}$ is complete and minimal in the space $L_2(0,1)$;

3) if $k_p \leq m_p$ then the system $V(L_{2,k})$ is the Riesz basis of the space $L_2(0,1)$.

Proof. The proof of part 1 of the theorem is carried out analogously in Theorem 2. Define the elements of the system $V(L_{2,k})$. Direct substitution shows that the functions $v_{2,q}(x, L_{0,k}), q = 1,2,\dots$, satisfy the conditions (34)–(37).

Therefore, the eigenfunction of the operator $L_{2,k}$, that corresponds to the eigenvalue $\lambda_{q,k}$ is defined by

$$v_{1,q}(x, L_{2,k}) = v_{1,q}(x, L_{0,k}), \quad q = 1,2,\dots, \tag{43}$$

$$v_{0,q}(x, L_{2,k}) = v_{0,q}(x, L_{0,k}) - l_p^1(v_{0,q}(x, L_{0,k}))(l_{2,n+p}y_{1,n+p}(x, \rho_{0,q}))^{-1}y_{1,n+p}(x, \rho_{0,q}).$$

Consequently $L_{2,k} \in Q(I)$. Taking into account Lemma 4.1, we obtain the second statement of the theorem. Taking into account the formulas (31), (21), and the inequalities $|l_p^1 v_{0,q}| \leq K_1(\rho_{0,q})^{m_{n+p}}$, we obtain the estimates

$$|l_p^1(v_{0,q}(x, L_{0,k}))(l_{2,n+p}y_{1,n+p}(x, \rho_{0,q}))^{-1}| \leq K_2|b|. \tag{44}$$

Taking into account Lemma 4.2, we obtain the third statement of the theorem. □

6 THE SPECTRAL BOUNDARY VALUE PROBLEM FOR A DIFFERENTIAL-OPERATOR EQUATION WITH INVOLUTION

Consider the spectral problem

$$Lu \equiv (-1)^n D_x^{2n}u(x) + A^{2n}u(x) + \sum_{j=1}^{n} a_j \left(D_x^{2j-1}u(x) - D_x^{2j-1}u(1-x)\right) = \lambda u(x), \quad \lambda \in \mathbb{C}, \tag{45}$$

$$\ell_j u \equiv D_x^{m_j}u(0) + (-1)^{m_j} D_x^{m_j}u(1) = 0, \tag{46}$$

$$\ell_{n+j}u \equiv D_x^{m_{n+j}}u(0) - (-1)^{m_{n+j}} D_x^{m_{n+j}}u(1) + l_j^1 u = 0, \quad j = 1,2,\dots,n. \tag{47}$$

The solution of the spectral problem (45)–(47) is defined as the product $u(x) = y(x)v_k$, $v_k \in V(A)$, $k = 1,2,\dots$. To determine the unknown function $y \in W_2^{2n}(0,1)$ we obtain the spectral problem

$$(-1)^n y^{(2n)}(x) + z_k^{2n}y(x) + \sum_{j=1}^{n} a_j \left(y^{(2j-1)}y(x) - y^{(2j-1)}(1-x)\right) = \lambda y(x), \quad \lambda \in \mathbb{C}, \tag{48}$$

$$\ell_j y \equiv y^{(m_j)}(0) + (-1)^{m_j} y^{(m_j)}(1) = 0, \tag{49}$$

$$\ell_{n+j}y \equiv y^{(m_{n+j})}(0) - (-1)^{(m_{n+j})} y^{(m_{n+j})}(1) + l_j^1 y = 0, \quad j = 1,2,\dots,n. \tag{50}$$

Let $L_{3,k}$ be the operator of the problem (48)–(50);

$$L_{3,k}y \equiv (-1)^n y^{(2n)}y(x) + z_k^{2n}y(x) + \sum_{j=1}^{n} a_j \left(y^{(2j-1)}(x) + y^{(2j-1)}(1-x)\right), \quad y \in D(L_{3,k});$$

$$D(L_{3,k}) \equiv \{y \in W_2^{2n}(0,1) : l_j y = 0, j = 1,2,\dots,2n\};$$

$V(L_{3,k})$ is the system of the eigenfunctions of the operator $L_{3,k}$.

Theorem 5. *Suppose that $k_p \leq m_p$ and the assumptions B_1–B_3 hold. Then the system of the functions $V(L_{3,k})$ is the Riesz basis of the space $L_2(0,1)$.*

Proof. Define the elements of the system $V(L_{3,k})$. The functions $v_{1,q}(x, L_{0,k})$ satisfy the conditions (49)–(50), $q = 1, 2, \ldots$. Therefore, the eigenfunction of the operator $L_{3,k}$, that corresponds to the eigenvalue $\lambda_{1,q,k}$ is defined by

$$v_{1,q}(x, L_{3,k}) \equiv v_{1,q}(x, L_{0,k}), q = 1, 2, \ldots . \tag{51}$$

For convenience we consider the representation of an eigenfunction of the operator $L_{0,k}$ according to the formula

$$v_{0,q}(x, L_{0,k}) \equiv \theta_{0,q} \sum_{r=1}^{n} \Delta_0^{1,r}(\rho_{0,q}) y_r(x, \rho_{0,q}), \quad q = 1, 2, \ldots . \tag{52}$$

Let

$$y^1_{n+j}(x, \rho_{0,q}) \equiv (x - \frac{1}{2})(\exp \omega_j \rho_{0,q} x + \exp \omega_j \rho_{0,q}(1 - x)), \quad j = 1, 2, \ldots, n, \quad q = 1, 2, \ldots, \tag{53}$$

$$y^{1,1}(x, \rho_{0,q}) \equiv \sum_{r=1}^{n} h^{1,1}_{r,q} y^1_{n+r}(x, \rho_{0,q}) \in H_1, \quad q = 1, 2, \ldots, \tag{54}$$

be the linear combination of the functions (53) with the indeterminate coefficients $h^{1,1}_{r,q}$, and

$$y^{1,2}(x, \rho_{0,q}) \equiv \sum_{r=1}^{n} h^{1,2}_{r,q} y_{1,n+r}(x, \rho_{0,q}) \in H_1, \quad q = 1, 2, \ldots, \tag{55}$$

be the linear combination of the functions $y_{1,n+r}(x, \rho_{0,q})$ with the indeterminate coefficients $h^{1,2}_{r,q}$.

The eigenfunction $v_{0,q}(x, L_{3,k})$ of the operator $L_{3,k}$ is given by

$$v_{0,q}(x, L_{3,k}) \equiv v_{0,q}(x, L_{2,k}) + y^{1,1}(x, \rho_{0,q}) + y^{1,2}(x, \rho_{0,q}), \quad q = 1, 2, \ldots, \tag{56}$$

where

$$h^{1,1}_{r,q} = -\frac{1}{2n} \theta_{0,q} \sum_{j=1}^{n} a_j (\rho_{0,q})^{2j-2n} (\omega_r)^{2j-2} \Delta_0^{1,r}(\rho_{0,q}), \quad q = 1, 2, \ldots, \tag{57}$$

$$h^{1,2}_{r,q} = -(\rho_{0,q})^{-m_{n+r}} (\Delta_0(\rho_{0,q}))^{-1} \Delta_0^{1,r}(\rho_{0,q}) l_{n+r} y^{1,1}(x, \rho_{0,q}), \quad q = 1, 2, \ldots . \tag{58}$$

Let $\Delta_0^{1,r} = \lim\limits_{q \to \infty} \Delta_0^{1,r}(\rho_{0,k})$, $k = 1, 2, \ldots$; V_k be the system, whose elements are the functions

$$v_{1,q}(x) \equiv v_{1,q}(x, L_{0,k}),\ v_{0,q}(x) \equiv v_{0,q}(x, L_{0,k}) + \Delta_0^{1,r} y_{1,n+1}(x, \rho_{0,q}), \quad q = 1, 2, \ldots .$$

Using inequality $|\Delta_0^{1,r}| < K_3 < \infty$, and Lemma 4.2, we obtain the statement: V_k is Riesz basis of the space $L_2(0,1)$. Taking into account the quadratic proximity of the system V_k and complete the system $V(L_{2,k})$ in the space $L_2(0,1)$ and according to N.K.Bari's Theorem [11], we prove the Theorem. □

Therefore, the operator L has a system of the eigenfunctions

$$V(L) \equiv \{v_{s,q,k}(x,L) \in W_1 : v_{s,q,k}(x,L) \equiv v_{s,q}(x,L_{3,k})v_k, s = 0,1,k,q = 1,2,\dots\},$$

$\lambda_{s,q,k} = (\rho_{s,q})^{2n} + z_k^{2n}$ are the eigenvalues of the operator L, $s = 0,1, q = 1,2,\dots$.

Taking into account the formulas (56)–(58), we obtain the following statement: the sequence of the operators $\{R(L_{3,k}), k = 1,2,\dots\}$ is uniformly bounded by the norm $[L_2(0,1)]$. Thus, the following theorem is true.

Theorem 6. *Suppose that $k_p \le m_p$ and the assumptions B_1–B_3 hold. Then the system of the functions $V(L)$ is the Riesz basis of the space W_1.*

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Баранецький Я.О., Каленюк П.І., Коляса Л.І., Копач М.І. *Нелокальна задача для диференціально-операторного рівняння парного порядку з інволюцією* // Карпатські матем. публ. — 2017. — Т.9, №2. — С. 109–119.

У роботі досліджується задача з крайовими несамоспряженими умовами диференціально-операторних рівнянь порядку $2n$ з інволюцією. Досліджено спектральні властивості оператора задачі.

Аналогічно методу відокремлення змінних, крайова задача для диференціально-операторного рівняння парного порядку, зведена до послідовності операторів $\{L_k\}_{k=1}^{\infty}$ крайових задач для звичайних диференціальних рівнянь парного порядку. Встановлено, що кожен елемент L_k цієї послідовності є ізоспектральним збуренням оператора $L_{0,k}$ самоспряженої крайової задачі для деякого лінійного звичайного диференціального рівняння порядку $2n$.

Побудовано комутативну групу операторів перетворення, елементи якої відображають систему $V(L_{0,k})$ власних функцій оператора $L_{0,k}$ у систему $V(L_k)$ власних функцій операторів L_k. Власні функції оператора крайової задачі для диференціально - операторного рівняння з інволюцією отримано, як результат дії деякого спеціально побудованого оператора на власні функції послідовності операторів $\{L_k\}_{k=1}^{\infty}$.

Встановлено достатні умови, при яких система власних функцій оператора задачі є базисом Рісса.

Ключові слова і фрази: оператор інволюції, диференціально-операторне рівняння, власні функції, базис Рісса.

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DMYTRYSHYN R.I.

ON THE CONVERGENCE CRITERION FOR BRANCHED CONTINUED FRACTIONS WITH INDEPENDENT VARIABLES

In this paper, we consider the problem of convergence of an important type of multidimensional generalization of continued fractions, the branched continued fractions with independent variables. These fractions are an efficient apparatus for the approximation of multivariable functions, which are represented by multiple power series. We have established the effective criterion of absolute convergence of branched continued fractions of the special form in the case when the partial numerators are complex numbers and partial denominators are equal to one. This result is a multidimensional analog of the Worpitzky's criterion for continued fractions. We have investigated the polycircular domain of uniform convergence for multidimensional C-fractions with independent variables in the case of nonnegative coefficients of this fraction.

Key words and phrases: convergence, absolute convergence, uniform convergence, branched continued fraction with independent variables, multidimensional C-fraction with independent variables.

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INTRODUCTION

The problem of convergence of continued fractions whether their multidimensional generalizations, branched continued fractions, in particular, branched continued fractions with independent variables, is that on the basis of information about coefficients fraction to conclude its convergence or divergence. This class fractions was proposed by D.I. Bodnar [6], in the study of the convergence of branched continued fractions with positive elements for establishing a analog of the Seidel convergence criteria for continued fractions. In the thesis by Kh.Yo. Kuchminska [7] established the estimate of approximation of function by such fractions under the conditions of the type of Śleszyński-Pringsheim in the case of the two branches of branching. Further study of the convergence of branched continued fractions with independent variables, in particular, branched continued fraction of the special form

$$1+\sum_{i_1=1}^{N}\frac{c_{i(1)}}{1}+\sum_{i_2=1}^{i_1}\frac{c_{i(2)}}{1}+\sum_{i_3=1}^{i_2}\frac{c_{i(3)}}{1}+\cdots, \tag{1}$$

where N is fixed natural number, $c_{i(k)}$, $i(k)\in\mathcal{I}_k$, $k\geq 1$, are complex numbers,

$$\mathcal{I}_k=\left\{i(k):\ i(k)=(i_1,i_2,\ldots,i_k),\ 1\leq i_p\leq i_{p-1},\ 1\leq p\leq k,\ i_0=N\right\}$$

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denotes set of multiindices, $k \geq 1$, and branched continued fraction of the special form which is reciprocal to it

$$\frac{1}{1} + \sum_{i_1=1}^{N} \frac{c_{i(1)}}{1} + \sum_{i_2=1}^{i_1} \frac{c_{i(2)}}{1} + \sum_{i_3=1}^{i_2} \frac{c_{i(3)}}{1} + \cdots \tag{2}$$

received a continuation in the papers by O.E. Baran [3], where proved that (1) converges absolutely, if there exists the real numbers $0 \leq q_{i(k)} < 1$ or $0 < q_{i(k)} \leq 1$, $i(k) \in \mathcal{I}_k$, $k \geq 1$, such that

$$|c_{i(k)}| \leq i_{k-1}^{-1} g_{i(k)}(1 - g_{i(k-1)}), \quad g_{i(0)} = 0, \; i(k) \in \mathcal{I}_k, \; k \geq 1, \tag{3}$$

and by O.E. Baran [2], where investigated a convergence of (2) for

$$|c_{i(k)}| \leq i_{k-1}^{-1} \rho(1 - \rho), \quad 0 < \rho \leq 2^{-1}, \quad i(k) \in \mathcal{I}_k, \; k \geq 1. \tag{4}$$

The next stage of the study of convergence of branched continued fractions with independent variables associated with the paper by T.M. Antonova and D.I. Bodnar [1], where proved that (1) converges absolutely for

$$|c_{i(k)}| \leq t_{i(k)} \left(1 - \sum_{i_{k+1}=1}^{i_k} t_{i(k+1)} \right), \quad t_{i(k)} \geq 0, \; \sum_{i_{k+1}=1}^{i_k} t_{i(k+1)} < 1, \quad i(k) \in \mathcal{I}_k, \; k \geq 1. \tag{5}$$

In addition, we note the paper by Kh.Yo. Kuchminska [8], where was proved a convergence of (2) with the elements that satisfy (4) in a slightly more general form than it was done in [2], and the paper by D.I. Bodnar and M.M. Bubnyak [5], where was investigated a convergence of one-periodic branched continued fractions of a special form with the elements that lie in disks whose radius form a geometric sequences with common ratio 4^{-1}.

We remark that the convergence criteria of branched continued fractions of the special form (1) and (2) established in the above mentioned works are multidimensional analogs of the Worpitzky's criterion for continued fractions [9].

Our research continues to establish the convergence criteria for the branched continued fractions with independent variables.

1 BRANCHED CONTINUED FRACTIONS OF THE SPECIAL FORM

Let

$$f_n = 1 + \sum_{i_1=1}^{N} \frac{c_{i(1)}}{1} + \sum_{i_2=1}^{i_1} \frac{c_{i(2)}}{1} + \cdots + \sum_{i_n=1}^{i_{n-1}} \frac{c_{i(n)}}{1}$$

be the nth approximant of (1), $n \geq 1$.

We shall prove the following result.

Theorem 1. *Let for the elements $c_{i(k)}$, $i(k) \in \mathcal{I}_k$, $k \geq 1$, of branched continued fraction of the special form (1) hold the following conditions*

$$|c_{i(k)}| \leq q_{i(k)}^{i_k} q_{i(k-1)}^{i_k - 1} (1 - q_{i(k-1)}), \quad i(k) \in \mathcal{I}_k, \; k \geq 1, \tag{6}$$

where $q_{i(0)}$ and $q_{i(k)}$, $i(k) \in \mathcal{I}_k$, $k \geq 1$, are constants which satisfy one or the other of the conditions

$$0 \leq q_{i(0)} < 1, \quad 0 \leq q_{i(k)} < 1, \quad i(k) \in \mathcal{I}_k, \; k \geq 1, \tag{7}$$

or

$$0 < q_{i(0)} \leq 1, \quad 0 < q_{i(k)} \leq 1, \quad i(k) \in \mathcal{I}_k,\ k \geq 1. \tag{8}$$

Then

(A) the branched continued fraction of special form (1) converges absolutely;

(B) the values of branched continued fraction of the special form (1) and of its approximants are in the disk

$$|z-1| \leq 1 - q_{i(0)}^{N}; \tag{9}$$

(C) the disk (9) is the "best" set of values of branched continued fraction of the special form (1) and of its approximants for $q_{i(k)} = 2^{-1}$, $i(k) \in \mathcal{I}_k,\ k \geq 1$.

Proof. We show that branched continued fraction of the special form

$$1 - \sum_{i_1=1}^{N} \frac{q_{i(1)}^{i_1} q_{i(0)}^{i_1-1}(1-q_{i(0)})}{1} - \sum_{i_2=1}^{i_1} \frac{q_{i(2)}^{i_2} q_{i(1)}^{i_2-1}(1-q_{i(1)})}{1} - \sum_{i_3=1}^{i_2} \frac{q_{i(3)}^{i_3} q_{i(2)}^{i_3-1}(1-q_{i(2)})}{1} - \cdots \tag{10}$$

is a majorant of (1).

For the tails of (1) we introduce the following notation:

$$Q_{i(s)}^{(s)} = 1, \quad i(s) \in \mathcal{I}_s,\ s \geq 1,$$

$$Q_{i(k)}^{(s)} = 1 + \sum_{i_{k+1}=1}^{i_k} \frac{c_{i(k+1)}}{1} + \sum_{i_{k+2}=1}^{i_{k+1}} \frac{c_{i(k+2)}}{1} + \cdots + \sum_{i_s=1}^{i_{s-1}} \frac{c_{i(s)}}{1}, \quad i(k) \in \mathcal{I}_k,\ 1 \leq k \leq s-1,\ s \geq 2.$$

It is clear that the following recurrence relations hold

$$Q_{i(k)}^{(s)} = 1 + \sum_{i_{k+1}=1}^{i_k} \frac{c_{i(k+1)}}{Q_{i(k+1)}^{(s)}}, \quad i(k) \in \mathcal{I}_k,\ 1 \leq k \leq s-1,\ s \geq 2. \tag{11}$$

Let s be arbitrary integer number, moreover $s \geq 0$. Using relations (11), by induction on k for arbitrary of multiindex $i(k) \in \mathcal{I}_k$ we show that the following inequalities are valid

$$|Q_{i(k)}^{(s)}| \geq \tilde{Q}_{i(k)}^{(s)}, \quad i(k) \in \mathcal{I}_k,\ 1 \leq k \leq s, \tag{12}$$

where $\tilde{Q}_{i(k)}^{(s)}$, $i(k) \in \mathcal{I}_k$, $1 \leq k \leq s$, denote the tails of (10), and

$$\tilde{Q}_{i(k)}^{(s)} > q_{i(k)}^{i_k}, \quad i(k) \in \mathcal{I}_k,\ 1 \leq k \leq s, \tag{13}$$

if the conditions (7) hold,

$$\tilde{Q}_{i(k)}^{(s)} \geq q_{i(k)}^{i_k}, \quad i(k) \in \mathcal{I}_k,\ 1 \leq k \leq s, \tag{14}$$

if the conditions (8) hold.

It is clear that for $k = s$, $i(s) \in \mathcal{I}_s$, relations (12)–(14) hold. By induction hypothesis that (12)–(14) hold for $k = p+1$, $p+1 \leq s$, $i(p+1) \in \mathcal{I}_{p+1}$, we prove (12)–(14) for $k = p$, $i(p) \in \mathcal{I}_p$.

Indeed, use of relations (11) for arbitrary of multiindex $i(p) \in \mathcal{I}_p$ lead to

$$|Q_{i(p)}^{(s)}| \geq 1 - \sum_{i_{p+1}=1}^{i_p} \frac{|c_{i(p+1)}|}{|Q_{i(p+1)}^{(s)}|} \geq 1 - \sum_{i_{p+1}=1}^{i_p} \frac{q_{i(p+1)}^{i_{p+1}} q_{i(p)}^{i_{p+1}-1}(1-q_{i(p)})}{\tilde{Q}_{i(p+1)}^{(s)}} = \tilde{Q}_{i(p)}^{(s)}.$$

From (13) and (14) it follows that $\tilde{Q}_{i(p+1)}^{(s)} \neq 0$. Therefore, replacing $q_{i(p+1)}^{i_{p+1}}$ by $\tilde{Q}_{i(p+1)}^{(s)}$, inequalities (13) and (14) are obtained for $k = p$, $i(p) \in \mathcal{I}_p$.

Now, from (12)–(14) it follows that $Q_{i(k)}^{(s)} \neq 0$ and $\tilde{Q}_{i(k)}^{(s)} \neq 0$ for all indices. Applying the method suggested in [4, p. 28] and recurrence relations (11), for $m > n \geq 1$ we obtain

$$|f_m - f_n| \leq \sum_{i_1=1}^{N} \sum_{i_2=1}^{i_1} \cdots \sum_{i_{n+1}=1}^{i_n} \frac{\prod_{k=1}^{n+1} |c_{i(k)}|}{\prod_{k=1}^{n+1} |Q_{i(k)}^{(m)}| \prod_{k=1}^{n} |Q_{i(k)}^{(n)}|}$$

$$\leq (-1)^{n+1} \sum_{i_1=1}^{N} \sum_{i_2=1}^{i_1} \cdots \sum_{i_{n+1}=1}^{i_n} \frac{(-1)^{n+1} \prod_{k=1}^{n+1} q_{i(k)}^{i_k} q_{i(k-1)}^{i_k-1}(1-q_{i(k-1)})}{\prod_{k=1}^{n+1} \tilde{Q}_{i(k)}^{(m)} \prod_{k=1}^{n} \tilde{Q}_{i(k)}^{(n)}} = -(\tilde{f}_m - \tilde{f}_n),$$

where $\tilde{f}_k$, $k \geq 1$, denote the approximants of (10).

Hence,

$$|f_m - f_n| \leq \tilde{f}_n - \tilde{f}_m, \quad m > n \geq 1,$$

and

$$\sum_{r=1}^{k} |f_{r+1} - f_r| \leq \sum_{r=1}^{k} (\tilde{f}_r - \tilde{f}_{r+1}) = -\sum_{i_1=1}^{N} q_{i(1)}^{i_1} q_{i(0)}^{i_1-1}(1-q_{i(0)}) - \tilde{f}_{k+1}, \quad k \geq 1. \tag{15}$$

From this it follows that the sequence $\{\tilde{f}_k\}$ is a monotonically decreases. Furthermore, from (13) and (14) for arbitrary $k \geq 1$ we have

$$\tilde{f}_k = 1 - \sum_{i_1=1}^{N} \frac{q_{i(1)}^{i_1} q_{i(0)}^{i_1-1}(1-q_{i(0)})}{\tilde{Q}_{i(1)}^{(k)}} \geq q_{i(0)}^{N},$$

i.e. the sequence $\{\tilde{f}_k\}$ is bounded below. Therefore, the limit

$$\tilde{f} = \lim_{k \to \infty} \tilde{f}_k$$

exists and is a finite. Now, from (15) for $k \to \infty$ it follows that (1) converges absolutely. This proves part (A).

Next we prove part (B) and (C). Using (6), (13) and (14) for arbitrary $k \geq 1$ we have

$$|f_k - 1| \leq \sum_{i_1=1}^{N} \frac{|c_{i(1)}|}{|Q_{i(1)}^{(k)}|} \leq \sum_{i_1=1}^{N} \frac{q_{i(1)}^{i_1} q_{i(0)}^{i_1-1}(1-q_{i(0)})}{\tilde{Q}_{i(1)}^{(k)}} \leq 1 - q_{i(0)}^{N}.$$

Therefore, the disk (9) includes the set of value of (1). We show that it coincides with (9) for $q_{i(k)} = 2^{-1}$, $i(k) \in \mathcal{I}_k$, $k \geq 1$.

Let c be an arbitrary complex number such that $|c| < 1 - q_{i(0)}^N$. Then for the approximant f_1 of (1), where the $c_{i(1)} = c(1 - q_{i(0)}^N)^{-1} q_{i(0)}^{i_1 - 1}(1 - q_{i(0)})$, $1 \leq i_1 \leq N$, and the $c_{i(k)}$, $i(k) \in \mathcal{I}_k$, $k \geq 2$, are arbitrary complex numbers that satisfy (6), we obtain $f_1 = 1 + c$. If $|c| = 1 - q_{i(0)}^N$ then (1), where the $c_{i(1)} = 2^{-i_1} c(1 - q_{i(0)}^N)^{-1} q_{i(0)}^{i_1 - 1}(1 - q_{i(0)})$, $1 \leq i_1 \leq N$, and the $c_{i(k)} = -4^{-i_k}$, $i(k) \in \mathcal{I}_k$, $k \geq 2$, satisfy (6) for $q_{i(k)} = 2^{-1}$, $i(k) \in \mathcal{I}_k$, $k \geq 1$, get value $1 + c$. We show it.

Indeed, in the above mentioned values of elements of (1) by equivalent transformation $\rho_{i(k)} = 2^{i_k - 1}$, $i(k) \in \mathcal{I}_k$, $k \geq 1$, [4, pp. 29–33] we can write it in the form

$$1 + \sum_{i_1=1}^{N} \frac{2^{-1} c (1 - q_{i(0)}^N)^{-1} q_{i(0)}^{i_1 - 1} (1 - q_{i(0)})}{2^{i_1 - 1}} \underset{-}{} \sum_{i_2=1}^{i_1} \frac{2^{i_1 - i_2 - 2}}{2^{i_2 - 1}} \underset{-}{} \sum_{i_3=1}^{i_2} \frac{2^{i_2 - i_3 - 2}}{2^{i_3 - 1}} \underset{-}{} \cdots . \tag{16}$$

To prove that the value of (16) is equal to $1 + c$ it is sufficient to prove the following relations

$$f^{(k)} = 2^{k-1} - \sum_{i_2=1}^{k} \frac{2^{k - i_2 - 2}}{2^{i_2 - 1}} \underset{-}{} \sum_{i_3=1}^{i_2} \frac{2^{i_2 - i_3 - 2}}{2^{i_3 - 1}} \underset{-}{} \sum_{i_4=1}^{i_3} \frac{2^{i_3 - i_4 - 2}}{2^{i_4 - 1}} \underset{-}{} \cdots = 2^{-1}, \quad 1 \leq k \leq N. \tag{17}$$

By induction on k we show that the relations (17) are valid.

It is easy to shown that for $k = 1$ relation (17) holds. By induction hypothesis that (17) hold for $k = n - 1$, $n \geq 2$, we prove (17) for $k = n$. We have

$$f^{(n)} = 2^{n-1} - \frac{2^{n-3}}{f^{(1)}} - \frac{2^{n-4}}{f^{(2)}} - \ldots - \frac{1}{f^{(n-2)}} - \frac{2^{-1}}{f^{(n-1)}} - \frac{2^{-2}}{f^{(n)}}. \tag{18}$$

Since $f^{(k)} = 2^{-1}$, $1 \leq k \leq n - 1$, $n \geq 2$, and

$$2^{n-1} - 2^{n-2} - 2^{n-3} - \ldots - 2^0 = 2^{n-1} - 2^{n-2} \frac{2^{1-n} - 1}{2^{-1} - 1} = 1,$$

than from (18) we obtain $f^{(n)} = 2^{-1}$. From this it follows that the value of (16) is equal to $1 + c$. Finally, it follows from concept of equivalent transformation [4, pp. 29–33] that the value of (1) is also equal to $1 + c$. □

It is now a simple matter to prove the following theorem.

Theorem 2. *Let for the elements $c_{i(k)}$, $i(k) \in \mathcal{I}_k$, $k \geq 1$, of branched continued fraction of the special form (2) hold the conditions (6), where $q_{i(0)}$ and $q_{i(k)}$, $i(k) \in \mathcal{I}_k$, $k \geq 1$, are constants which satisfy one or the other of the conditions*

$$0 < q_{i(0)} \leq 1, \quad 0 \leq q_{i(k)} < 1, \quad i(k) \in \mathcal{I}_k, \ k \geq 1, \tag{19}$$

or

$$0 < q_{i(0)} \leq 1, \quad 0 < q_{i(k)} \leq 1, \quad i(k) \in \mathcal{I}_k, \ k \geq 1. \tag{20}$$

Then

(A) the branched continued fraction of special form (2) converges absolutely;

(B) the values of the branched continued fraction of the special form (2) and of its approximants are in the disk

$$\left| z - \frac{1}{q_{i(0)}^N (2 - q_{i(0)}^N)} \right| \le \frac{1 - q_{i(0)}^N}{q_{i(0)}^N (2 - q_{i(0)}^N)}; \tag{21}$$

(C) the disk (21) is the "best" set of values of branched continued fraction of the special form (2) and of its approximants for $q_{i(k)} = 2^{-1}$, $i(k) \in \mathcal{I}_k$, $k \ge 1$.

Proof. By analogous considerations as in the proof of Theorem 1, it is easy to shown that a majorant of (2) is the following branched continued fraction of the special form

$$\frac{1}{1} - \sum_{i_1=1}^{N} \frac{q_{i(1)}^{i_1} q_{i(0)}^{i_1-1}(1-q_{i(0)})}{1} - \sum_{i_2=1}^{i_1} \frac{q_{i(2)}^{i_2} q_{i(1)}^{i_2-1}(1-q_{i(1)})}{1} - \sum_{i_3=1}^{i_2} \frac{q_{i(3)}^{i_3} q_{i(2)}^{i_3-1}(1-q_{i(2)})}{1} - \cdots . \tag{22}$$

From the fact that the approximants of (22) form the sequence, which is a monotonically increasing and bounded above, it follows that (2) converges absolutely.

We write the kth approximant of (2) in the form

$$z = \left(1 + \sum_{i_1=1}^{N} \frac{c_{i(1)}}{Q_{i(1)}^{(k-1)}} \right)^{-1} = \frac{1}{1+w}.$$

Using relations (19), (20) and conditions (6), we have

$$|w| \le \sum_{i_1=1}^{N} \frac{|c_{i(1)}|}{|Q_{i(1)}^{(k-1)}|} \le \sum_{i_1=1}^{N} \frac{q_{i(1)}^{i_1} q_{i(0)}^{i_1-1}(1-q_{i(0)})}{\tilde{Q}_{i(1)}^{(k-1)}} \le 1 - q_{i(0)}^N.$$

Therefore,

$$\left| \frac{1-z}{z} \right| = |w| \le 1 - q_{i(0)}^N,$$

from where we obtain (21).

Since $0 < q_{i(0)} \le 1$ then (21) contains the point 1. In view of proof part (C) of Theorem 1, to show that (21) is the "best" set, it suffices to note that values of the particular branched continued fraction of special form

$$z = \frac{1}{1} + \sum_{i_1=1}^{N} \frac{2^{-1}c(1-q_{i(0)}^N)^{-1} q_{i(0)}^{i_1-1}(1-q_{i(0)})}{2^{i_1-1}} - \sum_{i_2=1}^{i_1} \frac{2^{i_1-i_2-2}}{2^{i_2-1}} - \sum_{i_3=1}^{i_2} \frac{2^{i_2-i_3-2}}{2^{i_3-1}} - \cdots = \frac{1}{1+c}.$$

fill the disk (21) as c ranges over the set $|c| \le 1 - q_{i(0)}^N$. □

2 MULTIDIMENSIONAL C-FRACTIONS WITH INDEPENDENT VARIABLES

In this section we have two convergence criteria for the multidimensional C-fractions with independent variables. Their proof is a simple application of Theorems 1 and 2 respectively.

Corollary 2.1. *Let $a_{i(k)}$, $i(k) \in \mathcal{I}_k$, $k \geq 1$, be nonnegative numbers such that*

$$a_{i(k)} \leq q_{i(k)}^{i_k} q_{i(k-1)}^{i_k - 1}(1 - q_{i(k-1)}), \quad i(k) \in \mathcal{I}_k,\ k \geq 1, \tag{23}$$

where $q_{i(0)}$ and $q_{i(k)}$, $i(k) \in \mathcal{I}_k$, $k \geq 1$, are constants which satisfy one or the other of the conditions (7) or (8). Then the multidimensional C-fraction with independent variables

$$1 + \sum_{i_1=1}^{N} \frac{a_{i(1)} z_{i_1}}{1} + \sum_{i_2=1}^{i_1} \frac{a_{i(2)} z_{i_2}}{1} + \sum_{i_3=1}^{i_2} \frac{a_{i(3)} z_{i_3}}{1} + \cdots$$

converges uniformly in the domain

$$G = \left\{ \mathbf{z} = (z_1, z_2, \ldots, z_N) \in \mathbb{C}^N : \ |z_k| < 1,\ 1 \leq k \leq N \right\}. \tag{24}$$

Corollary 2.2. *Let $a_{i(k)}$, $i(k) \in \mathcal{I}_k$, $k \geq 1$, be nonnegative numbers such that satisfy the inequalities (23), where $q_{i(0)}$ and $q_{i(k)}$, $i(k) \in \mathcal{I}_k$, $k \geq 1$, are constants which satisfy one or the other of the conditions (19) or (20). Then the multidimensional C-fraction with independent variables*

$$\frac{1}{1} + \sum_{i_1=1}^{N} \frac{a_{i(1)} z_{i_1}}{1} + \sum_{i_2=1}^{i_1} \frac{a_{i(2)} z_{i_2}}{1} + \sum_{i_3=1}^{i_2} \frac{a_{i(3)} z_{i_3}}{1} + \cdots$$

converges uniformly in the domain (24).

CONCLUSION

The convergence criteria (6), as well as (3) and (5), is an effective criterion for investigating the convergence of branched continued fractions with independent variables.

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Досліджується питання збіжності багатовимірних узагальнень неперервних дробів — гіллястих ланцюгових дробів з нерівнозначними змінними. Ці дроби є ефективним апаратом при наближенні функцій, заданих кратними степеневими рядами. Встановлено ефективні умови абсолютної збіжності гіллястих ланцюгових дробів з нерівнозначними змінними у випадку коли частинні чисельники комплексні числа, а частинні знаменники дорівнюють одиниці. Отриманий результат є багатовимірним аналогом критерію Ворпітського для неперервних дробів. Досліджено полікругову область рівномірної збіжності для багатовимірних C-дробів з нерівнозначними змінними у випадку невід'ємних коефіцієнтів дробу.

Ключові слова і фрази: збіжність, абсолютна збіжність, рівномірна збіжність, гіллястий ланцюговий дріб з нерівнозначними змінними, багатовимірний C-дріб з нерівнозначними змінними.

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SOME CLASSES OF DISPERSIBLE DCSL-GRAPHS

A distance compatible set labeling (dcsl) of a connected graph G is an injective set assignment $f : V(G) \to 2^X$, X being a non empty ground set, such that the corresponding induced function $f^{\oplus} : E(G) \to 2^X \setminus \{\varphi\}$ given by $f^{\oplus}(uv) = f(u) \oplus f(v)$ satisfies $|f^{\oplus}(uv)| = k^f_{(u,v)} d_G(u,v)$ for every pair of distinct vertices $u, v \in V(G)$, where $d_G(u,v)$ denotes the path distance between u and v and $k^f_{(u,v)}$ is a constant, not necessarily an integer, depending on the pair of vertices u, v chosen. G is distance compatible set labeled (dcsl) graph if it admits a dcsl. A dcsl f of a (p,q)-graph G is dispersive if the constants of proportionality $k^f_{(u,v)}$ with respect to $f, u \neq v, u, v \in V(G)$ are all distinct and G is dispersible if it admits a dispersive dcsl. In this paper, we prove that all paths and graphs with diameter less than or equal to 2 are dispersible.

Key words and phrases: set labeling of graphs, dcsl-graph, dispersible dcsl-graph.

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INTRODUCTION

Acharya B.D. [1] introduced the notion of vertex set valuation as a set analogue of number valuation. For a graph $G = (V, E)$ and a non empty set X, Acharya B.D. defined a set valuation of G as an injective set valued function $f : V(G) \to 2^X$, and he defined a set-indexer as a set valuation such that the function $f^{\oplus} : E(G) \to 2^X \setminus \{\varphi\}$ given by $f^{\oplus}(uv) = f(u) \oplus f(v)$ for every $uv \in E(G)$ is also injective, where 2^X is the set of all the subsets of X and $\oplus$ is the binary operation of taking the symmetric difference of subsets of X.

Acharya B.D. and Germina K.A., who has been studying topological set valuation, introduced the particular kind of set valuation for which a metric, especially the cardinality of the symmetric difference, is associated with each pair of vertices in proportion to the distance between them [2]. In otherwords, the question is whether one can determine those graphs $G = (V, E)$ that admit an injective function $f : V \to 2^X$, X being a non empty ground set such that the cardinality of the symmetric difference $f^{\oplus}(uv)$ is proportional to the usual path distance $d_G(u,v)$ between u and v in G, for each pair of distinct vertices u and v in G. They called f a *distance compatible set labeling* (dcsl) of G, and the ordered pair (G, f), a distance compatible set labeled (dcsl) graph.

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Definition 1 ([2]). *Let $G = (V, E)$ be any connected graph. A distance compatible set labeling (dcsl) of a graph G is an injective set assignment $f : V(G) \to 2^X$, X being a non empty ground set, such that the corresponding induced function $f^{\oplus} : E(G) \to 2^X \setminus \{\varphi\}$ given by $f^{\oplus}(uv) = f(u) \oplus f(v)$ satisfies $|f^{\oplus}(uv)| = k^f_{(u,v)} d_G(u,v)$ for every pair of distinct vertices $u, v \in V(G)$, where $d_G(u,v)$ denotes the path distance between u and v and $k^f_{(u,v)}$ is a constant, not necessarily an integer, depending on the pair of vertices u, v chosen.*

The following universal theorem has been established in [2].

Theorem 1 ([2]). *Every graph admits a dcsl.*

Definition 2 ([3]). *A dcsl f of a (p,q)-graph G is dispersive if the constants of proportionality $k^f_{(u,v)}$ with respect to f, $u \neq v$, $u, v \in V(G)$ are all distinct and G is dispersible if it admits a dispersive dcsl. A dispersive dcsl f of G is (k,r)-arithmetic, if the constants of proportionality with respect to f can be arranged in the arithmetic progression, $k, k+r, k+2r, \ldots, k+(q-1)r$ and if G admits such a dcsl then G is a (k,r)-arithmetic dcsl-graph.*

Theorem 2 ([3]). *K_n is dispersible for all $n \geq 1$.*

1 DISPERSIVE DCSL-GRAPH WITH DIAM(G) ≤ 2

Theorem 3. *The star graph $K_{1,n}$ is dispersible for any $n \geq 1$.*

Proof. Let $V(K_{1,n}) = \{v_0, v_1, v_2, \ldots, v_n\}$ with v_0 is the *central vertex*. Let $X = \{1, 2, \ldots, 2^{2n+1}\}$. Define $f : V(K_{1,n}) \to 2^X$ by $f(v_0) = \varphi$ and $f(v_i) = \{1, 2, 3, \ldots, 2^{2i+1}\}, 1 \leq i \leq n$. Clearly $f(v_i) \subset f(v_j)$ and $|f(v_i) \oplus f(v_j)| = 2^{2j+1} - 2^{2i+1}, |f(v_0) \oplus f(v_i)| = 2^{2i+1}$ for $i < j$ and $1 \leq i, j \leq n$. Now, we prove that the constant of proportionality $k^f_{(u,v)}$ are all distinct, for distinct $u, v \in V(K_{1,n})$.

Case 1. For $i \neq j$, if possible

$$\begin{aligned} k^f_{(v_0,v_i)} = k^f_{(v_0,v_j)} &\Rightarrow \frac{|f(v_0) \oplus f(v_i)|}{d(v_0,v_i)} = \frac{|f(v_0) \oplus f(v_j)|}{d(v_0,v_j)} \\ &\Rightarrow \frac{2^{2i+1} - 0}{1} = \frac{2^{2j+1} - 0}{1} \Rightarrow 2^{2i+1} = 2^{2j+1}, \text{ a contradiction.} \end{aligned}$$

Case 2. For i, j, k and $j > k$, if possible

$$\begin{aligned} k^f_{(v_0,v_i)} = k^f_{(v_j,v_k)} &\Rightarrow \frac{|f(v_0) \oplus f(v_i)|}{d(v_0,v_i)} = \frac{|f(v_j) \oplus f(v_k)|}{d(v_j,v_k)} \\ &\Rightarrow \frac{2^{2i+1} - 0}{1} = \frac{2^{2j+1} - 2^{2k+1}}{2} \Rightarrow 2^{2i+1} = 2^{2j} - 2^{2k} \\ &\Rightarrow 2^{2i+1} = 2^{2k}(2^{2j-2k} - 1) \Rightarrow 2^{2i+1-2k} = 2^{2j-2k} - 1 (\text{if } 2i+1 > 2k). \end{aligned}$$

Here the left hand side is even and right hand side is odd, a contradiction. Also $2i + 1 = 2k$ is not possible and for $2i + 1 < 2k$, a similar contradiction can be derived.

Case 3. Let $v_i, v_j, v_k, v_l, 1 \leq i, j, k, l \leq n$ are four vertices of $K_{1,n}$ with all the four vertices are distinct. We also assume with out loss of generality that $i < j, l < k$ and $i < l$.

$$\begin{aligned} k^f_{(v_i,v_j)} = k^f_{(v_k,v_l)} &\Rightarrow \frac{|f(v_i) \oplus f(v_j)|}{d(v_i,v_j)} = \frac{|f(v_k) \oplus f(v_l)|}{d(v_k,v_l)} \\ &\Rightarrow \frac{2^{2j+1} - 2^{2i+1}}{2} = \frac{2^{2k+1} - 2^{2l+1}}{2} \Rightarrow 2^{2j} - 2^{2i} = 2^{2k} - 2^{2l} \\ &\Rightarrow 2^{2i}(2^{2j-2i} - 1) = 2^{2l}(2^{2k-2l} - 1) \Rightarrow (2^{2j-2i} - 1) = 2^{2l-2i}(2^{2k-2l} - 1), \end{aligned}$$

a contradiction that left hand side is odd and right hand side is even. Now if $k^f_{(v_i,v_j)} = k^f_{(v_k,v_l)}$ and any two vertices are same then it is easy to see that the other two vertices are also same. Hence, $k^f_{(u,v)}$ are all distinct for all distinct $u, v \in V(K_{1,n})$, so that $K_{1,n}$ is dispersible dcsl-graph. □

Remark 1. *For $K_{1,n}$, $max\{d(u,v) : u,v \in V(K_{1,n})\} = 2$. The diameter of a connected graph G is defined as $max\{d(u,v) : u,v \in V(G)\}$ and is denoted by $diam(G)$. It can be shown for a graph G with $diam(G) \leq 2$ that it is dispersible dcsl-graph. The result is proved in the following statement.*

Theorem 4. *Any graph G for which $diam(G) \leq 2$, is dispersible dcsl-graph.*

Proof. Let G be a graph with $diam(G) \leq 2$ and $|V(G)| = n$. Choose any 1-1 function $g : V(G) \longrightarrow \{1,3,5,7\ldots\}$. Consider the function $f : V(G) \longrightarrow 2^{\mathbb{N}}$, where $\mathbb{N} = \{1,2,3,\ldots\}$ given by $f(v) = \{1,2,3,4\ldots,2^{g(v)}\}$. We prove that f is a dispersive dcsl of G. Rename the vertices of G as $v \in V(G)$ changes to $v_{g(v)}$. We need to prove $k^f_{(v_i,v_j)} \neq k^f_{(v_k,v_l)}$ for all $v_i, v_j, v_k, v_l \in V(G)$. Assume the contrary that,

$$\frac{|f(v_i) \oplus f(v_j)|}{d(v_i,v_j)} = \frac{|f(v_k) \oplus f(v_l)|}{d(v_k,v_l)}.$$

Case 1. $d(v_i,v_j) = d(v_k,v_l) = 1$.

Subcase a. If $v_i = v_k$,

$$2^j - 2^i = 2^l - 2^k \Rightarrow 2^j - 2^i = 2^l - 2^i \Rightarrow 2^j = 2^l \Rightarrow j = l \Rightarrow v_j = v_l.$$

Similarly for $v_j = v_l \Rightarrow v_i = v_k$.

Subcase b. If $v_i = v_l$ and for $j > i > k$,

$$\begin{aligned} 2^j - 2^i = 2^l - 2^k &\Rightarrow 2^j + 2^k = 2^i + 2^l \Rightarrow 2^j + 2^k = 2^{i+1} \\ &\Rightarrow 2^k(2^{j-k} + 1) = 2^{i+1} \Rightarrow 2^{j-k} + 1 = 2^{i+1-k}. \end{aligned}$$

Left hand side is odd and right hand side is even, a contradiction.

Subcase c. If $v_j = v_k$ and for $l > j > i$,

$$2^j - 2^i = 2^l - 2^k \Rightarrow 2^j + 2^k = 2^i + 2^l \Rightarrow 2^{j+1} = 2^i + 2^l \Rightarrow 2^{j+1-i} = 2^{l-i} + 1.$$

Here left hand side is even and right hand side is odd, a contradiction.

Case 1 implies that if any two vertices are same, either the other two must be same or we arrive

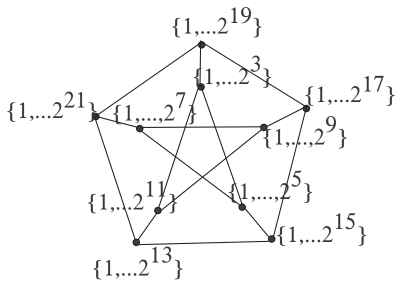

Figure 1: Dispersive dcsl of Peterson graph $[diam(P) = 2]$.

at a contradiction.
Case 2. $d(v_i, v_j) = d(v_k, v_l) = 2$.
Similar arguments of Case 1 implies that if any two vertices are same, either the other two must be same or we arrive at a contradiction.
Case 3. $d(v_i, v_j) = 2$ and $d(v_k, v_l) = 1$.
Subcase a. If $v_j = v_l$, then $v_i \neq v_k$ and for $j > i > k$,

$$2^{j-1} - 2^{i-1} = 2^l - 2^k \Rightarrow 2^{j-1} - 2^{i-1} = 2^j - 2^k \Rightarrow 2^k(2^{j-1-k} - 2^{i-1-k}) = 2^k(2^{j-k} - 1).$$

Left hand side is even and right hand side is odd, a contradiction. A similar contradiction can be obtained when $k > i$.
Subcase b. if $v_j = v_k$ and for $l > j > i$,

$$2^{j-1} - 2^{i-1} = 2^l - 2^k \Rightarrow 2^{j-1-(i-1)} - 1 = 2^{l-(i-1)} - 2^{j-(i-1)}.$$

A contradiction(left hand side is odd and right hand side is even).
Subcase c. If $v_i = v_l$ and for $j > i > k$,

$$2^{j-1} - 2^{i-1} = 2^l - 2^k \Rightarrow 2^{j-1} - 2^{i-1} = 2^i - 2^k \Rightarrow 2^k(2^{j-1-k} - 2^{i-1-k}) = 2^k(2^{i-k} - 1).$$

A contradiction(left hand side is even and right hand side is odd). Case 3 implies that if any two vertices are same, then we arrive at a contradiction.
Case 4. All the four vertices are distinct. if for any i, j, k, l distinct odd natural numbers,

$$2^j - 2^i \neq 2^l - 2^k, 2^{j-1} - 2^{i-1} \neq 2^{l-1} - 2^{k-1}$$

and $2^{j-1} - 2^{i-1} \neq 2^l - 2^k$. So in every case all the four vertices should be distinct, implies $k^f_{(u,v)}$ is distinct for every pair of vertices (u, v) of a connected graph G with $diam(G) \leq 2$. □

Corollary 1. *A graph G with a full degree vertex is dispersive dcsl-graph.*

Proof. Since G has a full degree vertex, $K_{1,n}$ is a spanning subgraph of G. So $diam(G) \leq 2$. □

Corollary 2. *$K_n, K_{m,n}, C_4, C_5$ and Peterson graph are dispersive dcsl-graphs.*

Corollary 3. *Join of two graphs is always dispersive dcsl-graphs.*

Proof. Since $diam(G_1 \vee G_2) \leq 2$ for any two graphs G_1 and G_2, by theorem 5 join of two graph is always dispersible. ☐

Corollary 4. *The Wheel graph $(K_1 \vee C_n)$ is dispersive dcsl-graph.*

Corollary 5. *A graph G with $\delta(G) > \frac{n}{2}$ is dispersible.*

Proof. Let $u, v \in V(G)$. Since degree of each vertex in G is greater than or equal to $\frac{n}{2}$, both u and v should have a common neighbor. Which in turn implies that $d(u, v) \leq 2$. This is true for any pair of vertices implies the $diam(G) \leq 2$. ☐

Remark 2. *It is proved in Theorem 4 that all the graphs with diameter less than or equal to two are dispersible. It does not imply that graphs with higher diameter are not dispersible. In fact for every n, we get a dispersible graph with $diam(G) = n$ as shown in the next Theorem 5.*

Theorem 5. *Paths are dispersible dcsl-graphs.*

Proof. Let $P_{n+1} = v_0v_1v_2 \dots v_{n-1}v_n$ be a path of length n with $n+1$ vertices. Label the vertices with sets which are mutually disjoint and of size in the following way.

$$|f(v_0)| = 0,$$
$$|f(v_1)| = n!,$$
$$|f(v_i)| = i[|f(v_{i-1})| + |f(v_{i-2})|] + n!, \ for \ 2 \leq i \leq n+1.$$

Here the constant $k^f_{(v_0,v_i)}$ is greater than all other constants upto v_{i-1}. Also

$$k^f_{(v_0,v_i)} < k^f_{(v_1,v_i)} < \dots < k^f_{(v_{i-1},v_i)}$$

for all $2 \leq i \leq n+1$. Since all the constants of proportionality are distinct, this dcsl is a dispersive dcsl. ☐

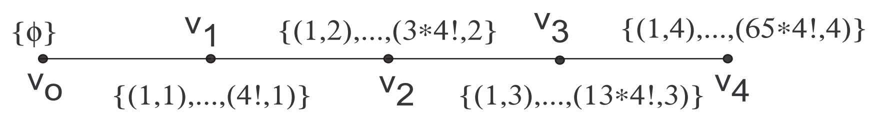

Figure 2: Dispersive dcsl of P_5.

2 CONCLUSION

Much work has been done when the constant of proportionality $k^f_{u,v}$ is a constant for every pair $(u, v) \in V(G) \times V(G)$ of a dcsl-graph G [2, 4, 5]. Here we proved that some classes of graphs are dispersible. But we did not get any graph which is not dispersible. Also dispersive dcsl is not unique for a dispersible graph. So some problems arise automatically.

1. What is the minimum cardinality of ground set X of dispersible graph G, denoted by $\nu(G)$?
2. Trees are dispersible?
3. Every graph admits a dispersive dcsl?
4. Any graph G with $diam(G) \leq 2$ is (k, r)-arithmetic?

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Джінто Дж., Ґерміна К.А., Шаіні П. *Деякі класи розсіюваних dcsl графів* // Карпатські матем. публ. — 2017. — Т.9, №2. — С. 128–133.

Сумісна з відстанями множина міток (dcsl) зв'язаного графа G є ін'єктивним відображенням множин $f : V(G) \to 2^X$, де X — непорожня базова множина така, що відповідна індукована функція $f^{\oplus} : E(G) \to 2^X \setminus \{\varphi\}$ задана як $f^{\oplus}(uv) = f(u) \oplus f(v)$ задовольняє умову $|f^{\oplus}(uv)| = k^f_{(u,v)} d_G(u,v)$ для кожної пари різних вершин $u, v \in V(G)$, де $d_G(u,v)$ позначає довжину шляху між u і v, та $k^f_{(u,v)}$ не обов'язково ціла константа, що залежить від пари обраних вершин u, v. G є графом з сумісною з відстанями множиною міток (dcsl-графом), якщо він дозволяє dcsl. Сумісна з відстанями множина міток f деякого (p,q)-графа G є дисперсною, якщо сталі пропорційності $k^f_{(u,v)}$ відносно $f, u \neq v, u, v \in V(G)$ є різними і G є дисперсним, якщо він доспускає дисперсну dcsl. У цій статті доведено, що всі шляхи і графи з діаметром не більшим 2 є дисперсними.

Ключові слова і фрази: множини міток графів, dcsl-граф, дисперсний dcsl-граф.

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FIRST REFORMULATED ZAGREB INDICES OF SOME CLASSES OF GRAPHS

A topological index of a graph is a parameter related to the graph; it does not depend on labeling or pictorial representation of the graph. Graph operations plays a vital role to analyze the structure and properties of a large graph which is derived from the smaller graphs. The Zagreb indices are the important topological indices found to have the applications in Quantitative Structure Property Relationship (QSPR) and Quantitative Structure Activity Relationship (QSAR) studies as well. There are various studies of different versions of Zagreb indices. One of the most important Zagreb indices is the reformulated Zagreb index which is used in QSPR study.

In this paper, we obtain the first reformulated Zagreb indices of some derived graphs such as double graph, extended double graph, thorn graph, subdivision vertex corona graph, subdivision graph and triangle parallel graph. In addition, we compute the first reformulated Zagreb indices of two important transformation graphs such as the generalized transformation graph and generalized Mycielskian graph.

Key words and phrases: Zagreb index, reformulated Zagreb index, derived graphs.

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INTRODUCTION

All the graphs considered in this paper are connected and simple. For vertex $u \in V(G)$, the degree of the vertex u in G, denoted by $d_G(u)$, is the number of edges incident to u in G. A *topological index* of a graph is a parameter related to the graph; it does not depend on labeling or pictorial representation of the graph. In theoretical chemistry, molecular structure descriptors (also called topological indices) are used for modeling physicochemical, pharmacologic, toxicologic, biological and other properties of chemical compounds. Several types of such indices exist, especially those based on vertex and edge distances. One of the most intensively studied topological indices is the Wiener index. Two of these topological indices are known under various names, the most commonly used one are the first and second Zagreb indices.

The Zagreb indices have been introduced more than thirty years ago by Gutman I. and Trinajstić N. [6]. They are defined as

$$M_1(G) = \sum_{u \in V(G)} d_G(u)^2, \qquad M_2(G) = \sum_{uv \in E(G)} d_G(u) d_G(v).$$

Note that the first Zagreb index may also be written as

$$M_1(G) = \sum_{uv \in E(G)} (d_G(u) + d_G(v)).$$

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2010 *Mathematics Subject Classification:* 05C12, 05C76.

The Zagreb indices are found to have appilications in QSPR and QSAR studies as well. For the survey on theory and application of Zagreb indices see [7]. Feng L. et al. [5] have given the sharp bounds for the Zagreb indices of graphs with a given matching number. Khalifeh M.H. et al. [12] have obtained the Zagreb indices of the Cartesian product, composition, join, disjunction and symmetric difference of graphs. The extremal values of Zagreb coindices over some special class of graphs determined by Ashrafi A.R. et al. [1].

Milićević A. et al. [15] in 2004 reformulated the Zagreb indices in terms of edge-degrees instead of vertex-degrees $EM_1(G) = \sum\limits_{e\in E(G)} d(e)^2$, where $d(e)$ denotes the degree of the edge e in G, which is defined by $d(e) = d(u) + d(v) - 2$ with $e = uv$. The use of these descriptors in QSPR study was also discussed in their report [15]. Reformulated Zagreb index, particularly its upper/lower bounds has attracted recently theat tention of many mathematicians and computer scientists, see [3, 4, 10, 11, 15, 17, 20]. The aim of this paper is to obtain, the first reformulated Zagreb indices of some derived graphs such as double, extended double, thorn graph, subdivision vertex corona of graphs, subdivision graph and triangle parallel graph. In addition, we compute the first reformulated Zagreb indices of two important transformation graphs such as the generalized transformations graphs and generalized Mycielskian graphs.

1 MAIN RESULTS

The hyper Zagreb index and its coindex are defined as

$$HM(G) = \sum_{uv\in E(G)} (d_G(u) + d_G(v))^2 \qquad \text{and} \qquad \overline{HM}(G) = \sum_{uv\notin E(G)} (d_G(u) + d_G(v))^2.$$

The *F-index* of a graph G is defined as $F = F(G) = \sum\limits_{u\in V(G)} d_G^3(u) = \sum\limits_{uv\in E(G)} (d_G^2(u) + d_G^2(v))$.

1.1 Double graph and extended double cover

Let us denote the double graph of a graph G by G^*, which is constructed from two copies of G in the following manner [9, 2]. Let the vertex set of G be $V(G) = \{v_1, v_2, \ldots, v_n\}$, and the vertices of G^* are given by the two sets $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$. Thus for each vertex $v_i \in V(G)$, there are two vertices x_i and y_i in $V(G^*)$. The *double graph* G^* includes the initial edge set of each copies of G, and for any edge $v_iv_j \in E(G)$, two more edges x_iy_j and x_jy_i are added, see Figure 1. Now we compute the first reformulated Zagreb index of the double of a given graph.

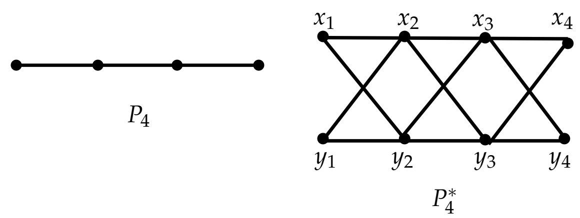

Figure 1: The double graph of P_4.

Theorem 1. *Let G be a connected graph with m edges. If G^* is a double graph of G, then $EM_1(G^*) = 16HM(G) - 32M_1(G) + 16m$.*

Proof. From the definition of a double graph it is clear that $d_{G^*}(x_i) = d_{G^*}(y_i) = 2d_G(v_i)$, where $v_i \in V(G)$ and $x_i, y_i \in V(G^*)$ are corresponding clone vertices of v_i. By the definition of EM_1,

$$\begin{aligned} EM_1(G^*) &= \sum_{uv \in E(G^*)} (d_{G^*}(u) + d_{G^*}(v) - 2)^2 \\ &= \sum_{x_i x_j \in E(G^*)} (d_{G^*}(x_i) + d_{G^*}(x_j) - 2)^2 + \sum_{y_i y_j \in E(G^*)} (d_{G^*}(y_i) + d_{G^*}(y_j) - 2)^2 \\ &+ \sum_{x_i y_j \in E(G^*)} (d_{G^*}(x_i) + d_{G^*}(y_j) - 2)^2 + \sum_{x_j y_i \in E(G^*)} (d_{G^*}(x_j) + d_{G^*}(y_i) - 2)^2 \\ &= 4 \sum_{v_i v_j \in E(G)} (2d_G(v_i) + 2d_G(v_j) - 2)^2 = 16 \sum_{v_i v_j \in E(G)} (d_G(v_i) + d_G(v_j) - 1)^2 \\ &= 16 \sum_{v_i v_j \in E(G)} \left[(d_G(v_i) + d_G(v_j))^2 - 2(d_G(v_i) + d_G(v_j)) + 1 \right] \\ &= 16HM(G) - 32M_1(G) + 16m. \end{aligned}$$

□

Let G be a simple connected graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$. The *extended double cover* of G, denoted by G^{**} is the bipartite graph with bipartition (X, Y) where $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ in which x_i and y_j are adjacent if and only if either v_i and v_j are adjacent in G or $i = j$, see Figure 2. This construction of the extended double cover was introduced by Alon N. [2] in 1986. Here we obtain the first reformulated Zagreb index of extended double cover of a given graph.

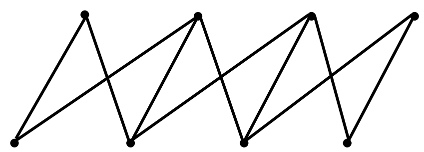

Figure 2: Extended double cover of P_4.

Theorem 2. *Let G be a graph and G^{**} its extended double cover. Then $EM_1(G^{**}) = 2HM(G)$.*

Proof. Let G be a graph with n vertices and m edges. The definition of the extended double cover implies that G^{**} consists of $2n$ vertices and $n + 2m$ edges. Moreover, $d_{G^{**}}(x_i) = d_{G^{**}}(y_i) = d_G(v_i) + 1$, for $i = \{1, 2, \ldots, n\}$. Here, $v_i \in V(G)$ and $x_i, y_i \in V(G^{**})$ are corresponding clone vertices of v_i. Hence

$$\begin{aligned} EM_1(G^{**}) &= \sum_{uv \in E(G^{**})} (d_{G^{**}}(u) + d_{G^{**}}(v) - 2)^2 \\ &= \sum_{x_i y_j \in E(G^{**})} (d_{G^{**}}(x_i) + d_{G^{**}}(y_j) - 2)^2 + \sum_{x_j y_i \in E(G^{**})} (d_{G^{**}}(x_j) + d_{G^{**}}(y_i) - 2)^2 \\ &+ \sum_{i=1}^{n} (d_{G^{**}}(x_i) + d_{G^{**}}(y_i) - 2)^2 = 2 \sum_{v_i v_j \in E(G)} (d_G(v_i) + 1 + d_G(v_j) + 1 - 2)^2 \\ &= 2 \sum_{v_i v_j \in E(G)} (d_G(v_i) + d_G(v_j))^2 = 2HM(G). \end{aligned}$$

□

1.2 Thorn Graph

An edge $e = uv$ of a graph G is called a *thorn* if either $d_G(u) = 1$ or $d_G(v) = 1$. The concept of thorn graph was introduced by Gutman I. [8] by joining a number of thorn to each vertex of any given graph G. Some of the topological indices of thorn graphs are studied in [13, 18, 19].

Let $V(G)$ and $V(G^T)$ be the vertex sets of G and its thorn graph G^T respectively. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $V^T(G) = V(G) \cup V_1 \cup V_2 \cup \ldots \cup V_n$, where V_i are the set of degree one vertices attached to the vertices v_i in G^T and $Vi \cup V_j = \varphi$, $i \neq j$. Let the vertices of the set V_i are denoted by v_{ij} for $j = 1, 2, \ldots, p_i$ and $i = 1, 2, \ldots, n$. Thus $\left|V(G^T)\right| = n + z$ where, $z = \sum\limits_{i=1}^{n} p_i$. Then the degree of the vertices v_i in G^T are given by $d_{G^T}(v_i) = d_G(v_i) + p_i$, for $i = 1, 2, \ldots, n$. Now we compute the first reformulated Zagreb index of thorn of a given graph.

Theorem 3. *Let G be a graph. Then*

$$EM_1(G^T) = HM(G) + \sum_{v_iv_j \in E(G)} (p_i + p_j - 2)^2 + 2 \sum_{v_iv_j \in E(G)} (d_G(v_i) + d_G(v_j))(p_i + p_j - 2)$$
$$+ \sum_{i=1}^{n} p_i \Big[d_G^2(v_i) + (p_i - 1)^2 + 2d_G(v_i)(p_i - 1)\Big].$$

Proof. From the definition of reformulated first Zagreb index,

$$\begin{aligned}
EM_1(G^T) &= \sum_{v_iv_j \in E(G^T)} (d_{G^T}(v_i) + d_{G^T}(v_j) - 2)^2 \\
&= \sum_{v_iv_j \in E(G^T)} (d_{G^T}(v_i) + d_{G^T}(v_j) - 2)^2 + \sum_{i=1}^{n} \sum_{j=1}^{p_i} (d_{G^T}(v_i) + d_{G^T}(v_j) - 2)^2 \\
&= \sum_{v_iv_j \in E(G)} (d_G(v_i) + p_i + d_G(v_j) + p_j - 2)^2 + \sum_{i=1}^{n} \sum_{j=1}^{p_i} (d_G(v_i) + p_i + 1 - 2)^2 \\
&= \sum_{v_iv_j \in E(G)} \Big[(d_G(v_i) + d_G(v_j))^2 + (p_i + p_j - 2)^2 \\
&+ 2(d_G(v_i) + d_G(v_j))(p_i + p_j - 2)\Big] + \sum_{i=1}^{n} p_i (d_G(v_i) + p_i - 1)^2 \\
&= HM(G) + \sum_{v_iv_j \in E(G)} (p_i + p_j - 2)^2 + 2 \sum_{v_iv_j \in E(G)} (d_G(v_i) + d_G(v_j))(p_i + p_j - 2) \\
&+ \sum_{i=1}^{n} p_i \Big[d_G^2(v_i) + (p_i - 1)^2 + 2d_G(v_i)(p_i - 1)\Big].
\end{aligned}$$

□

1.3 Subdivision Vertex Corona of Graphs

Let G_1 and G_2 be any two simple connected graphs with n_1 and n_2 number of vertices and m_1 and m_2 number of edges respectively. The *subdivision vertex corona* of G_1 and G_2 is denoted by $G_1 \circ G_2$ and was introduced by Lu P. and Miao Y. [14]. The graph $G_1 \circ G_2$ is obtained from the subdivision graph $S(G_1)$ and n_1 copies of G_2, by joining the i-th vertex of $V(G_1)$ to every vertex in the i-th copy of G_2. Let $V(G_1) = \{v_1, v_2, \ldots, v_{n_1}\}$, $I(G_1) = \{v_1^e, v_2^e, \ldots, v_{m_1}^e\}$

and $V(G_2) = \{u_1, u_2, \ldots, u_{n_2}\}$, so that $V(S(G)) = V(G) \cup I(G)$. Let $u_1^i, u_2^i, \ldots, u_{n_2}^i$ denote the vertices of the i-th copy of $G_{2,i}$, $i = 1, 2, \ldots, n_1$, so that

$$V(G_1 \circ G_2) = V(G_1) \cup I(G_1) \cup [V(G_{2,1}) \cup V(G_{2,2}) \cup \ldots \cup V(G_{2,n_1})].$$

Here we compute the first reformulated Zagreb index of Subdivision vertex corona of graphs.

Theorem 4. *Let G_1 and G_2 be two graphs with n_1, n_2 and m_1, m_2 edges, respectively. Then*

$$EM_1(G_1 \circ G_2) = n_1 HM(G_2) + F(G_1) + 3n_2 M_1(G_1) + n_1 M_1(G_2) + n_2\Big[2m_1n_2 + n_1(n_2-1)^2\Big]$$
$$+ 8m_1m_2 + 4\Big[m_1(n_2-1) + m_2(n_1-1)\Big].$$

Proof. The degree of the vertices of $G_1 \circ G_2$ is given by $d_{G_1 \circ G_2}(v_i) = d_{G_1}(v_i) + n_2$ for $i = 1, 2, \ldots, n_1$, $d_{G_1 \circ G_2}(e_i) = 2$ for $i = 1, 2, \ldots, m_1$, $d_{G_1 \circ G_2}(u_j^i) = d_{G_2}(u_j) + 1$ for $i = 1, 2, \ldots, n_1$ and $j = 1, 2, \ldots, n_2$. Let the vertex set of $G_1 \circ G_2$ can be partitioned into three subsets $E_1 = \{xy \in E(G_1 \circ G_2) | x, y \in V(G_{2,i}), i = 1, 2, \ldots, n_1\}$, $E_2 = \{xy \in E(G_1 \circ G_2) | x \in V(G_1), y \in I(G_1)\}$, and $E_3 = \{xy \in E(G_1 \circ G_2) | x \in V(G_1), y \in V(G_{2,i}),\ i = 1, 2, \ldots, n_1\}$. The contribution of the edges in E_1 to the first reformulated Zagreb index of $G_1 \odot G_2$ is given by

$$\begin{aligned} EM_1(G_1 \odot G_2) &= \sum_{xy \in E_1} (d_{G_1 \odot G_2}(x) + d_{G_1 \odot G_2}(y) - 2)^2 \\ &= \sum_{i=1}^{n_1} \sum_{u_iu_j \in E(G_2)} (d_{G_2}(u_i) + 1 + d_{G_2}(u_j) + 1 - 2)^2 \\ &= \sum_{i=1}^{n_1} \sum_{u_iu_j \in E(G_2)} (d_{G_2}(u_i) + d_{G_2}(u_j))^2 = n_1 HM(G_2). \end{aligned}$$

Similarly, the contribution of the edges in E_2 to the first reformulated Zagreb index of $G_1 \odot G_2$ is given by

$$\begin{aligned} EM_1(G_1 \odot G_2) &= \sum_{xy \in E_2} (d_{G_1 \odot G_2}(x) + d_{G_1 \odot G_2}(y) - 2)^2 = \sum_{i=1}^{n} (d_{G_1}(v_i) + n_2 + 2 - 2)^2 d_{G_1}(v_i) \\ &= \sum_{i=1}^{n} \Big[d_{G_1}^2(v_i) + n_2^2 + 2d_{G_1}(v_i)n_2\Big] d_{G_1}(v_i) \\ &= F(G_1) + 2n_2M_1(G_1) + 2m_1n_2^2. \end{aligned}$$

The contribution of the edges in E_3 to the first reformulated Zagreb index of $G_1 \odot G_2$ is given by

$$\begin{aligned} EM_1(G_1 \odot G_2) &= \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} (d_{G_1}(v_i) + n_2 + d_{G_2}(u_j) + 1 - 2)^2 \\ &= \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} (d_{G_1}(v_i) + d_{G_2}(u_j) + (n_2 - 1))^2 \\ &= \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \Big[d_{G_1}^2(v_i) + d_{G_2}^2(u_j) + (n_2-1)^2 + 2d_{G_1}(v_i)d_{G_2}(u_j) \\ &\quad + 2d_{G_2}(u_j)(n_2-1) + 2d_{G_1}(v_i)(n_2-1)\Big] = n_2M_1(G_1) + n_1M_1(G_2) \\ &\quad + n_1n_2(n_2-1)^2 + 8m_1m_2 + 4m_2(n_2-1) + 4m_1(n_2-1). \end{aligned}$$

The desired expression for the first reformulated Zagreb index of $G_1 \odot G_2$ is obtained by summing the above three expressions. ◻

1.4 Some derived graphs

The *subdivision graph*$S(G)$ is the graph obtained from G by replacing each edge of G by a path of length two. The *triangle parallel graph* of a graph G is denoted by $R(G)$ and is obtained from G by replacing each edge of G by a triangle. Now we compute the first reformulated Zagreb index of $S(G)$ and $R(G)$ for a given graph G.

Theorem 5. *Let G be a graph. Then $EM_1(S(G)) = F(G)$.*

Proof. Observe that $V(S(G)) = \Big(V(S(G)) \cap V(G)\Big) \cup \Big(V(S(G)) \setminus V(G)\Big)$, that is $|V(S(G))| = p + q$ and $|E(S(G))| = 2q$. Note that for $x \in V(S(G)) \cap V(G)$, $d_{S(G)}(x) = d_G(x)$ and for $x \in V(S(G)) \setminus V(G)$, $d_{S(G)}(x) = 2$. The first reformulated Zagreb index is given by

$$\begin{aligned} EM_1(S(G)) &= \sum_{uv \in E(S(G))} (d_{S(G)}(u) + d_{S(G)}(v) - 2)^2 = \sum_{u \in V(S(G))} (d_{S(G)}(u) + 2 - 2)^2 \\ &= \sum_{u \in V(G)} d_G(u)(d_G(u))^2 = F(G). \end{aligned}$$

□

Theorem 6. *Let G be a graph on m edges. Then*

$$EM_1(R(G)) = 4HM(G) - 8M_1(G) + 4F(G) + 4m.$$

Proof. From the definition of $R(G)$, we have

$$\begin{aligned} RM_1(R(G)) &= \sum_{uv \in E(R(G))} (d_{R(G)}(u) + d_{R(G)}(v) - 2)^2 \\ &= \sum_{u,v \in V(G),\, uv \in E(R(G))} (d_{R(G)}(u) + d_{R(G)}(v) - 2)^2 \\ &+ \sum_{x \in V(G),\, y \in V(R(G))/V(G),\, xy \in E(R(G))} (d_{R(G)}(x) + d_{R(G)}(y) - 2)^2 \\ &= \sum_{uv \in E(G)} (2d_G(u) + 2d_G(v) - 2)^2 + \sum_{p \in V(G)} (2d_G(p) + 2 - 2)^2 d_G(p) \\ &= \sum_{uv \in E(G)} [(2d_G(u) + 2d_G(v))^2 + 4 - 4(2d_G(u) + 2d_G(v))] + \sum_{p \in V(G)} 4d_G^3(p) \\ &= 4HM(G) - 8M_1(G) + 4F(G) + 4m. \end{aligned}$$

□

1.5 Generalized transformation graphs

Sampathkumar E. and Chikkodimath S.B. [16] defined the *semitotal-point graph* of given graph. Based on this definition, Gutman introduced some new graphical transformations. These generalize the concept of a semitotal-point graph.

Let $G = (V, E)$ be a graph, and let α, β be two elements of $V(G) \cup E(G)$. We say that the associativity of α and β is $+$ if they are adjacent or incident in G, otherwise is $-$. Let ab be a $2-$permutation of the set $\{+, -\}$. We say that α and β correspond to the first term a of ab if both α and β are in $V(G)$, whereas α and β correspond to the second term b of ab if one of α and β is in $V(G)$ and the other is in $E(G)$. The generalized transformation graph G^{ab} of G is defined on the vertex set $V(G) \cup E(G)$. Two vertices α and β of G^{ab} are joined by an edge if and only if their associativity in G is consistent with the corresponding term of ab.

In view of above, one can obtain four graphical transformations of graphs, since there are four distinct 2−permutations of $\{+-\}$. Note that G^{++} is just the semitotal-point graph $T_2(G)$ of G, whereas the other generalized transformation graphs are G^{+-}, G^{-+} and G^{--}. In other words, the generalized transformation graph G^{ab} is a graph whose vertex set is $V(G) \cup E(G)$, and $\alpha, \beta \in V(G^{ab})$. α and β are adjacent in G^{ab} if and only if either (i) and (ii) holds:

(i) for any $\alpha, \beta \in V(G), \alpha, \beta$ are adjacent in G if $a = +$ and; α, β are not adjacent in G if $a = -$;

(ii) for any $\alpha \in V(G)$ and $\beta \in E(G), \alpha, \beta$ are incident in G if $b = +$ and; α, β are not incident in G if $b = -$.

The vertex v_i of G^{ab} corresponding to a vertex v_i of G is referred to as a point vertex. The vertex e_i of G^{ab} corresponding to an edge e_i of G is referred to as a line vertex.

Theorem 7. *Let G be a connected graph on n vertices and m edges. Then $EM_1(G^{++}) = 4HM(G) - 8M_1(G) + 4F(G) + 4m$.*

Proof. One can observe that the number of vertices and edges of G^{++} are $n + m$ and $2m$, respectively. $d_{G^{++}}(v_i) = 2d_G(v_i)$ and $d_{G^{++}}(e_i) = 2$.

$$\begin{aligned}
EM_1(G^{++}) &= \sum_{uv \in E(G^{++})} (d_{G^{++}}(u) + d_{G^{++}}(v) - 2)^2 \\
&= \sum_{uv \in E(G^{++}) \cap E(G)} (d_{G^{++}}(u) + d_{G^{++}}(v) - 2)^2 \\
&+ \sum_{uv \in E(G^{++}) - E(G)} (d_{G^{++}}(u) + d_{G^{++}}(v) - 2)^2 \\
&= \sum_{uv \in E(G)} (2d_G(u) + 2d_G(v) - 2)^2 + \sum_{uv \in E(G^{++}) - E(G)} (2 + 2d_G(v) - 2)^2 \\
&= 4HM(G) - 8M_1(G) + 4 \sum_{v \in V(G)} d_G^3(v) + 4m \\
&= 4HM(G) - 8M_1(G) + 4F(G) + 4m.
\end{aligned}$$

□

Theorem 8. *Let G be a connected graph on n vertices and m edges. Then*

$$EM_1(G^{+-}) = 4m(m-1)^2 + (nm - 2m)(n + m - 4)^2.$$

Proof. Note that $|V(G^{+-})| = n + m$ and $|E(G^{+-})| = m(n-1)$. Moreover, $d_{G^{++}}(v_i) = m$ and $d_{G^{++}}(e_i) = n - 2$.

$$\begin{aligned}
EM_1(G^{+-}) &= \sum_{uv \in E(G^{+-})} (d_{G^{+-}}(u) + d_{G^{+-}}(v) - 2)^2 \\
&= \sum_{uv \in E(G^{+-}) \cap E(G)} (d_{G^{+-}}(u) + d_{G^{+-}}(v) - 2)^2 \\
&+ \sum_{uv \in E(G^{+-}) - E(G)} (d_{G^{+-}}(u) + d_{G^{+-}}(v) - 2)^2 \\
&= \sum_{uv \in E(G)} (2m - 2)^2 + \sum_{uv \in E(G^{+-}) - E(G)} (m + (n-2) - 2)^2 \\
&= m(2m-2)^2 + (m(n-1) - m)(n + m - 4)^2.
\end{aligned}$$

□

Theorem 9. *Let G be a connected graph on n vertices and m edges. Then*

$$EM_1(G^{-+}) = 2n^2(n(n-1)-2m) + 2m(n-1)^2.$$

Proof. Note that $|V(G^{-+})| = n+m$ and $|E(G^{-+})| = m + \frac{n(n-1)}{2}$. Moreover, $d_{G^{-+}}(v_i) = n-1$ and $d_{G^{++}}(e_i) = 2$.

$$\begin{aligned}
EM_1(G^{-+}) &= \sum_{uv\in E(G^{-+})} (d_{G^{-+}}(u) + d_{G^{-+}}(v) - 2)^2 \\
&= \sum_{uv\in E(G^{-+})\cap E(\overline{G})} (d_{G^{-+}}(u) + d_{G^{-+}}(v) - 2)^2 \\
&\quad + \sum_{uv\in E(G^{-+})-E(\overline{G})} (d_{G^{-+}}(u) + d_{G^{-+}}(v) - 2)^2 \\
&= [\frac{n(n-1)}{2} - m](2n)^2 + (n-1)^2(\frac{n(n-1)}{2} + m - \frac{n(n-1)}{2} + m) \\
&= 4n^2(\frac{n(n-1)}{2} - m) + 2m(n-1)^2.
\end{aligned}$$

□

Theorem 10. *Let G be a connected graph on n vertices and m edges. Then*

$$\begin{aligned}
EM_1(G^{--}) &= 4\overline{HM}(G) - 8(n+m-2)\overline{M}_1(G) + 2(n+m-2)^2(n^2-n-m) \\
&\quad + (2n+m-5)^2 m(n-2) + 4\sum_{uv\in E(G^{--})-E(\overline{G})} \Big(d_G^2(v) - (2n+m-5)d_G(v)\Big).
\end{aligned}$$

Proof. Note that $|V(G^{--})| = p+q$ and $|E(G^{--})| = \frac{p(p-1)}{2} + q(p-3)$. Moreover, $d_{G^{--}}(v_i) = p+q-1-2d_G(v_i)$ and $d_{G^{--}}(e_i) = p-2$.

$$\begin{aligned}
EM_1(G^{--}) &= \sum_{uv\in E(G^{--})} (d_{G^{--}}(u) + d_{G^{--}}(v) - 2)^2 = \sum_{uv\in E(G^{--})\cap E(\overline{G})} (d_{G^{--}}(u) + d_{G^{--}}(v) - 2)^2 \\
&+ \sum_{uv\in E(G^{--})-E(\overline{G})} (d_{G^{--}}(u) + d_{G^{--}}(v) - 2)^2 \\
&= \sum_{uv\in E(\overline{G})} ((n+m-1) - 2d_G(u) + (n+m-1) - 2d_G(v) - 2)^2 \\
&+ \sum_{uv\in E(G^{--})-E(\overline{G})} (n-2+n+m-1-2d_G(v)-2)^2 \\
&= \sum_{uv\in E(\overline{G})} (2(n+m-1) - 2(d_G(u)+d_G(v)) - 2)^2 \\
&+ \sum_{uv\in E(G^{--})-E(\overline{G})} (2n+m-5-2d_G(v))^2 \\
&= \sum_{uv\in E(\overline{G})} ((2(n+m-1)-2)^2 + 4(d_G(u)+d_G(v))^2 \\
&- 4(2(n+m-1)-2)(d_G(u)+d_G(v))) \\
&+ \sum_{uv\in E(G^{--})-E(\overline{G})} ((2n+m-5)^2 + 4d_G^2(v) - 4(2n+m-5)d_G(v)) \\
&= 4\overline{HM}(G) - 8(n+m-2)\overline{M}_1(G) + 2(n+m-2)^2(n^2-n-m) \\
&+ (2n+m-5)^2 m(n-2) + 4\sum_{uv\in E(G^{--})-E(\overline{G})} \Big(d_G^2(v) - (2n+m-5)d_G(v)\Big).
\end{aligned}$$

□

1.6 Generalized Mycielskian graphs

Let G be a simple connected graph with n vertices and m edges, $V(G) = \{v_1, v_2, \ldots, v_n\}$. For a graph $G = (V, E)$, the *Mycielskian* of G is the graph $\mu(G)$ with the vertex set consisting of the disjoint union $V \cup V' \cup \{u\}$, where $V' = \{x' | x \in V\}$ and edge set $E \cup \{x'y, xy' | xy \in E\} \cup \{x'u | x' \in V'\}$.

For a graph $G = (V, E)$, the *generalized Mycielskian*, denoted by $\mu_k(G)$, of G is the graph whose vertex set is the disjoint union $V \cup (\bigcup\limits_{i=1}^{k} V^i) \cup \{u\}$, where $V^i = \{x^i | x \in V\}$ is an independent set, $1 \leq i \leq k$, and edge set $E(\mu_k(G)) = E \cup \{\bigcup\limits_{i=1}^{k} \{y^{i-1}x^i; x^{i-1}y^i | xy \in E\}\} \bigcup \{x^k u | x^k \in V^k\}$, where $x^0 = x$ and $y^0 = y$.

The proof of the following lemma easily follows from the definition of the generalized Mycielskian of G.

Lemma 1. *Let G be a connected graph. Then*
(i) $|V(\mu_k(G))| = (k+1)n + 1$;
(ii) $|E(\mu_k(G))| = (2k+1)m + n$;
(iii) *If $u^0v^0 \in E(G)$, then $u^0v^0, u^iv^{i+1}, u^{i+1}v^i \in E(\mu_k(G))$ for $0 \leq i \leq k-1$;*
(iv) $d_{\mu_k(G)}(v^i) = 2d_G(v), 0 \leq i \leq k-1$;
(v) $d_{\mu_k(G)}(v^k) = d_G(v) + 1$ *for all* $v \in V(G)$;
(vi) $d_{\mu_k(G)}(u) = n$.

Here we obtain the first reformulated Zagreb index of $\mu_k(G)$.

Theorem 11. *Let G be a connected graph with n vertices and m edges. Then*
$$EM_1(\mu_k(G)) = 2(4k-1)HM(G) + 6F(G) - (16k-1)M_1(G) + 4M_2(G) + n(n-1)^2 + 2m(4k+2n-3).$$

Proof. By the definition of EM_1, we have
$$EM_1(\mu_k(G)) = \sum_{uv \in E(\mu_k(G))} \Big(d_{\mu_k(G)}(u) + d_{\mu_k(G)}(v) - 2\Big)^2.$$

By Lemma 1, we get
$$\begin{aligned}
EM_1(\mu_k(G)) &= \sum_{uv \in E(G)} \Big(2d_G(u) + 2d_G(v) - 2\Big)^2 + 2(k-1) \sum_{uv \in E(G)} \Big(2d_G(u) + 2d_G(v) - 2\Big)^2 \\
&\quad + \sum_{uv \in E(G)} \Big(2d_G(u) + (d_G(v)+1) - 2\Big)^2 + \sum_{uv \in E(G)} \Big(2d_G(v) + (d_G(u)+1) - 2\Big)^2 \\
&\quad + \sum_{v \in V(G)} \Big((d_G(v)+1) + n - 2\Big)^2 \\
&= (2k-1) \sum_{uv \in E(G)} \Big(2d_G(u) + 2d_G(v) - 2\Big)^2 \\
&\quad + \sum_{uv \in E(G)} \Big(2d_G(u) + d_G(v) - 1\Big)^2 + \sum_{uv \in E(G)} \Big(2d_G(v) + d_G(u) - 1\Big)^2 \\
&\quad + \sum_{v \in V(G)} \Big(d_G(v) + n - 1\Big)^2 \\
&= S_1 + S_2 + S_3 + S_4,
\end{aligned}$$

where

$$\begin{aligned} S_1 &= (2k-1)\sum_{uv\in E(G)}\Big(2d_G(u)+2d_G(v)-2\Big)^2 \\ &= 4(2k-1)\sum_{uv\in E(G)}\Big((d_G(u)+d_G(v))^2-2(d_G(u)+d_G(v))+1\Big) \\ &= (2k-1)\Big(4HM(G)-8M_1(G)+4m\Big), \end{aligned}$$

$$\begin{aligned} S_2 &= \sum_{uv\in E(G)}\Big(2d_G(u)+d_G(v)-1\Big)^2 \\ &= \sum_{uv\in E(G)}\Big((d_G(u)+d_G(v))^2+(d_G(u))^2+2(d_G(u)+d_G(v))d_G(u) \\ &\quad -2(d_G(u)+d_G(v))-2d_G(u)+1\Big) \\ &= HM(G)+3\sum_{v\in V(G)}d_G(v)(d_G(v))^2+2M_2(G)-2M_1(G)-2\sum_{v\in V(G)}(d_G(v))^2+m \\ &= HM(G)+3F(G)+2M_2(G)-4M_1(G)+m. \end{aligned}$$

Similarly,

$$S_3=\sum_{uv\in E(G)}\Big(2d_G(v)+d_G(u)-1\Big)^2=HM(G)+3F(G)+2M_2(G)-4M_1(G)+m,$$

$$\begin{aligned} S_4=\sum_{v\in V(G)}\Big(d_G(v)+n-1\Big)^2 &= \sum_{v\in V(G)}\Big((d_G(v))^2+2(n_1)d_G(v)+(n-1)^2\Big)^2 \\ &= M_1(G)+n(n-1)^2+4(n-1)m. \end{aligned}$$

The desired expression for the first reformulated Zagreb index of $\mu_k(G)$ is obtained by summing S_1 to S_4. □

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Топологічний індекс графа — це параметр, пов'язаний з графом; він не залежить від маркування або наочного зображення графа. Операції з графами відіграють важливу роль для аналізу структури і властивостей великого графа, що породжений від менших графів. Індекси Загреба є важливими топологічними показниками, які знайшли застосування в вивченні кількісної структури відносин властності (QSPR) та кількісної структури відносин активності (QSAR). Є різні дослідження окремих видів індексів Загреба. Один з найважливіших індексів Загреба — це переформульований індекс Загреба, який використовується в дослідженні QSPR.

У статті ми отримуємо значення перших переформульованих індексів Загреба деяких похідних графів, таких як подвійний граф, подовжений подвійний граф, шиповий граф, напівподілений вершиний коронний граф, напівподілений граф та паралельний трикутний граф. Крім того обчислено перші переформульовані індекси Загреба для двох важливих перетворень графів таких як граф узагальненого перетворення та узагальнений граф Міцельскіяна.

Ключові слова і фрази: індекс Загреба, перевизначений індекс Загреба, похідні графи.

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LITOVCHENKO V.A.

PARABOLIC BY SHILOV SYSTEMS WITH VARIABLE COEFFICIENTS

Because of the parabolic instability of the Shilov systems to change their coefficients, the definition parabolicity of Shilov for systems with time-dependent t coefficients, unlike the definition parabolicity of Petrovsky, is formulated by imposing conditions on the matricant of corresponding dual by Fourier system. For parabolic systems by Petrovsky with time-dependent coefficients, these conditions are the property of a matricant, which follows directly from the definition of parabolicity. In connection with this, the question of the wealth of the class Shilov systems with time-dependent coefficients is important.

A new class of linear parabolic systems with partial derivatives to the first order by the time t with time-dependent coefficients is considered in this work. It covers the class by Petrovsky systems with time-dependent younger coefficients. A main part of differential expression of each such system is parabolic (by Shilov) expression with constant coefficients. The fundamental solution of the Cauchy problem for systems of this class is constructed by the Fourier transform method. Also proved their parabolicity by Shilov. Only the structure of the system and the conditions on the eigenvalues of the matrix symbol were used. First of all, this class characterizes the wealth by Shilov class of systems with time-dependents coefficients.

Also it is given a general method for investigating a fundamental solution of the Cauchy problem for Shilov parabolic systems with positive genus, which is the development of the well-known method of Y.I. Zhitomirskii.

Key words and phrases: parabolic by Shilov system, fundamental solution, Cauchy problem.

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INTRODUCTION

In [1] G.E. Shilov formulated a new definition of parabolicity of systems of partial differential equations which generalizes the notion of parabolicity by I.G. Petrovsky [2] and leads to a significant expansion of the Petrovsky class of systems appearance

$$\partial_t u(t;x) = P(t;i\partial_x)u(t;x), \quad (t;x) \in \Pi_{(\tau;T]} := (\tau;T] \times \mathbb{R}^n, \tau \in [0;T). \tag{1}$$

Here i is imaginary unit, u is unknown vector function of m dimension, $P(t;i\partial_x)$ is matrix differential expression of $p \in \mathbb{N}$ order with t time-dependent coefficients.

If coefficients of system (1) are constants and $P(t;i\partial_x) \equiv P(i\partial_x)$, parabolicity by Shilov is defined like parabolicity by Petrovsky: by imposing conditions on the real part of the characteristic numbers $\lambda_j(\cdot)$ of matrix symbol $P(\sigma)$, $\sigma \in \mathbb{C}^n$, of differential expression of system (1): exists $h > 0$, exists $\delta_0 > 0$ and exists $\delta_1 \geq 0$ for all $\xi \in \mathbb{R}^n$ such that

$$\max_{j \in \mathbb{N}_m} \mathrm{Re}\lambda_j(\xi) \leq -\delta_0 \|\xi\|^h + \delta_1. \tag{2}$$

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Here h is index of parabolicity of system (1), $0 < h \leq p$; $\mathbb{N}_m := \{1; 2; \ldots; m\}$, $\|\cdot\| := (\cdot,\cdot)^{1/2}$, $(\cdot,\cdot)$ is scalar product in $\mathbb{R}^n$.

If the coefficients of system (1) depend on t (continuously), it has, unlike parabolicity by Petrovsky, parabolicity by Shilov for this system with the index of parabolicity h means performance for the matricant $\Theta^t_\tau(\cdot)$, $0 \leq \tau < t \leq T$, of corresponding dual by Fourier system following estimate [3]:

$$|\Theta^t_\tau(\xi)| \leq c(1 + \|\xi\|^\gamma) e^{-\delta(t-\tau)\|\xi\|^h}, \quad (t;\xi) \in \Pi_{(\tau;T]} \tag{3}$$

(here $\gamma := (p-h)(m-1)$). Let's note that for parabolic by Petrovsky systems (1) condition (3) is characteristic property which is a direct consequence of the relevant condition of parabolicity of type (2). For parabolic systems (1) with dependent on t coefficients in the case of $p \neq h$ it is not possible to confirm this fact by means of classical theory of parabolic systems, generally speaking, due to the parabolic instability of such systems to changing of their coefficients [4]. So the information about the richness of Shilov class of systems with coefficients dependent on t, in particular about the examples of such systems which are not parabolic by Petrovsky is important.

In this paper a new class of systems of partial differential equations whose coefficients depend on t is defined; it is substantiated their parabolicity by Shilov and examples are given. This class of systems characterizes the richness of Shilov class of systems with depend on t coefficients. In addition, estimates of the derivatives of the fundamental solution of the Cauchy problem (FSCP) are established for parabolic by Shilov systems with coefficients dependent on t the genus of which is positive.

The study FSCP for Shilov-type parabolic systems with coefficients independent of t was carried out in the papers [3, 5–7] and scalar parabolic equations by Shilov, whose coefficients can depend on t was carried out in the papers [8–11].

1 PRELIMINARIES

Let $\mathbb{R}^n$ and $\mathbb{C}^n$ are respectively real and complex space of n dimension, $\mathbb{R} := \mathbb{R}^1$, $\mathbb{Z}^n_+$ is the set of all n -dimensional multi indices; $|x + iy| := (x^2 + y^2)^{1/2}$, $\{x, y\} \subset \mathbb{R}$, $|(a_{lj})^m_{l,j=1}| := \max\limits_{\{l,j\}\subset\mathbb{N}_m} |a_{lj}|$; $z^l := z_1^{l_1} \ldots z_n^{l_n}$, $|z|^h_+ := |z_1|^h + \ldots + |z_n|^h$, $|z|_+ := |z|^1_+$, if $z := (z_1; \ldots; z_n) \in \mathbb{C}^n$, $l := (l_1; \ldots; l_n) \in \mathbb{Z}^n_+$ and $h > 0$.

We shall consider the system (1) with matrix differential expression

$$P(t; i\partial_x) = \left(\sum_{|k|_+ \leq p} a^{lj}_k(t) i^{|k|_+} \partial^k_x \right)^m_{l,j=1}$$

of p order coefficients $a^{lj}_k(\cdot)$ of which are continuous complex-valued functions on $[0; T]$. We shall suppose that this system is parabolic by Shilov on the set $\Pi_{(\tau;T]}$ with the index of parabolicity h, $0 < h \leq p$, consolidated order p_0 and genus μ [3].

Let's remind now that matricant $\Theta^t_\tau(\cdot)$ of appropriate dual by Fourier to (1) system has the structure

$$\Theta^t_\tau(\xi) = E + \sum_{r=1}^{\infty} \int_\tau^t \int_\tau^{t_1} \ldots \int_\tau^{t_{r-1}} \left(\prod_{j=1}^{r} P(t_j; \xi) \right) dt_r \ldots dt_2 dt_1. \tag{4}$$

Here E is the identity matrix of m order. Hence, it abides bound

$$|P(t;\sigma)| \le c(1+\|\sigma\|^p),\ 0 \le t \le T, \sigma \in \mathbb{C}^n,$$

we obtain that

$$|\Theta_\tau^t(\sigma)| \le c_0 e^{\delta_0 (t-\tau)\|\sigma\|^p},\ 0 \le t \le T, \sigma \in \mathbb{C}^n \tag{5}$$

(here c_0 and δ_0 are positive constants which are not dependent on τ, t and σ).

The exact order of exponential increase of matricant $\Theta_\tau^t(\cdot)$ in complex space $\mathbb{C}^n$ is called the consolidated order p_0 of the system (1). Always $p \ge p_0 > 1$ for parabolic systems [3].

The genus of parabolic by Shilov system we shall call the maximum rate μ such that in the domain

$$\mathbb{K}_\mu = \{\xi + i\eta \in \mathbb{C}^n : \|\eta\| \le K(1+\|\xi\|)^\mu\}$$

with some $K > 0$ for matricant the following estimate holds

$$|\Theta_\tau^t(\xi+i\eta)| \le c(1+\|\xi\|^\gamma) e^{-\delta(t-\tau)\|\xi\|^h}, 0 \le \tau < t \le T. \tag{6}$$

In [3] it is established that $1-(p_0-h) \le \mu \le 1$.

2 One class of parabolic systems

Let's consider the system of equations

$$\partial_t u(t;x) = \{P_0(i\partial_x) + P_1(t;i\partial_x)\} u(t;x), \quad (t;x) \in \Pi_{(\tau;T]},\ \tau \in [0;T), \tag{7}$$

with $p \in \mathbb{N}$ order in which $u := \mathrm{col}(u_1, \dots, u_m)$,

$$P_0(i\partial_x) := \left(\sum_{|k|_+ \le p} a_k^{lj} i^{|k|_+} \partial_x^k \right)_{l,j=1}^m, \quad P_1(t;i\partial_x) := \left(\sum_{|k|_+ \le p_1} a_k^{lj}(t) i^{|k|_+} \partial_x^k \right)_{l,j=1}^m.$$

We shall assume that corresponding system

$$\partial_t u(t;x) = P_0(i\partial_x) u(t;x), \quad (t;x) \in \Pi_{(\tau;T]}, \tag{8}$$

on the set $\Pi_{(\tau;T]}$ is parabolic by Shilov with constant coefficients and index of parabolicity h and coefficients of differential expression $P_1(t;i\partial_x)$ are continuous complex-valued functions defined on [0;T] with the values p, p_1 and h satisfying condition

$$0 \le p_1 + (p-h)(m-1) < h. \tag{A}$$

Examples of system (7) with condition (A).

I. Each parabolic by Petrovsky system (1) of $p = 2b$ order, $b \in \mathbb{N}$, with constant coefficients of group of senior members and dependent continuously on t coefficients of group of younger members is a system of kind (7) with condition (A). Because in this case $p = h = 2b$, $p_1 = 2b-1$ and respectively

$$0 < p_1 + (p-h)(m-1) = 2b-1 < 2b = h.$$

II. Let $n = 1, m = 2,\ a > 0$ i $c_j(\cdot),\ j \in \mathbb{N}_5$, are some continuous on [0;T] complex-valued functions. Then the system

$$\begin{cases} \partial_t u_1 &= \{-a\partial_x^4 + c_1(t)\partial_x^2\}u_1 + \{\partial_x^5 - \partial_x^3 + c_2(t)\partial_x\}u_2, \\ \partial_t u_2 &= \{(c_3(t)-1)\partial_x^3\}u_1 - \{a\partial_x^4 - c_4(t)\partial_x^3 - c_5(t)\}u_2, \end{cases}$$

is the system of kind (7) with condition (A). Indeed, putting

$$P_0(i\partial_x) = \begin{pmatrix} -a\partial_x^4 & \partial_x^5 - \partial_x^3 \\ -\partial_x^3 & -a\partial_x^4 \end{pmatrix},$$

$$P_1(t; i\partial_x) = \begin{pmatrix} c_1(t)\partial_x^2 & c_2(t)\partial_x \\ c_3(t)\partial_x^3 & c_4(t)\partial_x^3 + c_5(t) \end{pmatrix}$$

and solving the appropriate equation

$$\det(P_0(\sigma) - \lambda E) = 0, \quad \sigma \in \mathbb{C}^n,$$

we obtain that $\lambda_{1,2}(\sigma) = -a\sigma^4 \pm i\sqrt{\sigma^8 + \sigma^6}$, $p = 5$, $p_1 = 3$ i $h = 4$. For these values p, p_1 and h, obviously the condition (A) holds.

Theorem 1. *Let* (7) *is system with continuous coefficients for which the condition (A) holds. Then for matricant* $\Theta_\tau^t(\cdot)$ *of appropriate dual by Fourier system on the set* $\Pi_{(\tau;T]}$, $\tau \in [0;T)$, *the estimate* (3) *holds.*

Proof. Let's write down appropriate dual by Fourier system to (7):

$$\partial_t v(t;\xi) = \{P_0(\xi) + P_1(t;\xi)\}v(t;\xi), \quad (t;\xi) \in \Pi_{(\tau;T]}. \tag{9}$$

With the continuity of the coefficients matricant $\Theta_\tau^t(\cdot)$ is the only solution of the Cauchy problem for system (9) with the initial condition

$$v(t;\cdot)\ |_{t=\tau} = E. \tag{10}$$

Then the following equality holds

$$\partial_t \Theta_\tau^t(\xi) = P_0(\xi)\Theta_\tau^t(\xi) + Q(\tau, t; \xi). \tag{11}$$

Here $Q(\tau, t; \xi) := P_1(t;\xi)\Theta_\tau^t(\xi)$. Solving the Cauchy problem (11), (10), we obtain such representation:

$$\Theta_\tau^t(\xi) = e^{(t-\tau)P_0(\xi)} + \int_\tau^t e^{(t-\beta)P_0(\xi)} Q(\tau, \beta; \xi)d\beta,\ (t;\xi) \in \Pi_{(\tau;T]},\ \tau \in [0;T).$$

Hence, it abides performance of estimate (3) for $e^{(t-\tau)P_0(\cdot)}$ because $e^{(t-\tau)P_0(\cdot)}$ is matricant dual by Fourier system to (8) and inequality

$$|Q(\tau, t; \xi)| \le c_0(1 + \|\xi\|^{p_1})|\Theta_\tau^t(\xi)|,\ (t;\xi) \in \Pi_{(\tau;T]},\ \tau \in [0;T)$$

(here positive constant c_0 does not depend on τ, t i ξ), we get the estimate

$$|\Theta_\tau^t(\xi)| \le c(1 + \|\xi\|^\gamma)e^{-\delta(t-\tau)\|\xi\|^h} + c_1(1 + \|\xi\|^\gamma)(1 + \|\xi\|^{p_1}) \int_\tau^t e^{-\delta(t-\beta)\|\xi\|^h} |\Theta_\tau^\beta(\xi)|d\beta,$$

from which we come to correlation

$$\frac{|\Theta_\tau^t(\xi)|e^{\delta(t-\tau)\|\xi\|^h}}{(1+\|\xi\|^\gamma)} \le c + c_1(1+\|\xi\|^\gamma)(1+\|\xi\|^{p_1})\int_\tau^t \frac{|\Theta_\tau^\beta(\xi)|e^{\delta(\beta-\tau)\|\xi\|^h}}{(1+\|\xi\|^\gamma)}d\beta.$$

Using now Lemma of Gronwall [12] we obtain

$$|\Theta_\tau^t(\xi)| \le c(1+\|\xi\|^\gamma)e^{-(t-\tau)\left(\delta\|\xi\|^h - c_1(1+\|\xi\|^\gamma)(1+\|\xi\|^{p_1})\right)},\ (t;\xi)\in\Pi_{(\tau;T]},\ \tau\in[0;T).$$

From here considering the condition (A) we come to existence of positive constants c and δ with which for all $(t;\xi)\in\Pi_{(\tau;T]}$, $\tau\in[0;T)$, bound (3) holds. Theorem is proved. □

Corollary 1. *System (7) with condition (A) is parabolic by Shilov system with coefficients dependent on t and index of parabolicity h.*

3 PROPERTIES OF FSCP

Let (1) is parabolic by Shilov system with continuous on $[0;T]$ coefficients. Solving this system by Fourier transform we obtain a representation of the fundamental solution of its Cauchy problem:

$$G(\tau,t;\cdot) = F^{-1}[\Theta_\tau^t(\xi)](\tau,t;\cdot),\quad 0\le\tau<t\le T$$

(here $F^{-1}[\cdot]$ is inverse Fourier transform and $\Theta_\tau^t(\cdot)$ is appropriate matricant (4)).

The following statement holds.

Theorem 2. *Let the system (1) is parabolic by Shilov with dependent continuously on t coefficients and positive genus μ. Then its FSCP on the set $\mathbb{R}^n$ for spatial variable is infinitely differentiable function such that exists $\{c,B,\delta\}\subset(0;+\infty)$ for all $k\in\mathbb{Z}_+^n$, $\tau\in[0;T)$, $t\in(\tau;T]$, $x\in\mathbb{R}^n$ such that*

$$|\partial_x^k G(\tau,t;x)| \le c(t-\tau)^{-\frac{n+\gamma+|k|_+}{h}}B^{|k|_+}k^{k\frac{1}{h}}e^{-\delta\left(\frac{\|x\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}},$$

here $\alpha := \mu/p_0$.

Proof. Let's consider the matrix function

$$\varphi_{\tau,t}^k(x) := (t-\tau)^{\frac{\gamma+|k|_+}{h}}x^k\Theta_\tau^t(x),\ k\in\mathbb{Z}_+^n,\ x\in\mathbb{R}^n,\ 0\le\tau<t\le T,$$

which obviously continues in a complex space $\mathbb{C}^n$ to an entire analytic function at each fixed k,t and τ.

Directly to the condition (3) we obtain that

$$\begin{aligned}|\varphi_{\tau,t}^k(x)| &\le c(t-\tau)^{\frac{\gamma+|k|_+}{h}}\|x\|^{|k|_+}(1+\|x\|^\gamma)e^{-\delta(t-\tau)\|x\|^h}\\ &= c\left(((t-\tau)\|x\|^h)^{\frac{|k|_+}{h}}(t-\tau)^{\frac{\gamma}{h}} + ((t-\tau)\|x\|^h)^{\frac{\gamma+|k|_+}{h}}\right)e^{-\delta(t-\tau)\|x\|^h}\\ &\le cT_0^{\frac{\gamma}{h}}\left(\sup_{\xi\ge0}\{\xi^{\frac{|k|_+}{h}}e^{-\frac{\delta}{2}\xi}\} + \sup_{\xi\ge0}\{\xi^{\frac{\gamma+|k|_+}{h}}e^{-\frac{\delta}{2}\xi}\}\right)e^{-\frac{\delta}{2}(t-\tau)\|x\|^h},\ \text{where } T_0=\max\{1,T\}.\end{aligned}$$

Hence, taking the equality

$$\sup_{\xi\geq 0}\{\xi^\beta e^{-\delta\xi}\} = \left(\frac{\beta}{e\delta}\right)^\beta, \ \beta > 0, \ \delta > 0, \tag{12}$$

into account we come to existence of positive constants c_1, B_1 and δ_1 such that for all $x \in \mathbb{R}^n$, $k \in \mathbb{Z}_+^n$, $\tau \in [0; T)$ and $t \in (\tau; T]$ inequality

$$|\varphi_{\tau,t}^k(x)| \leq c_1 B_1^{|k|+} k^{k\frac{1}{h}} e^{-\delta_1(t-\tau)\|x\|^h}$$

holds. Similarly way due to the definition of genus μ of parabolic system (1) we come to such an bound of matrix function $\varphi_{\tau,t}^k(\cdot)$ in the relevant domain $\mathbb{K}_\mu \subset \mathbb{C}^n$:

$$|\varphi_{\tau,t}^k(x+iy)| \leq c_2 B_2^{|k|+} k^{k\frac{1}{h}} e^{-\delta_2(t-\tau)\|x\|^h}, \ k \in \mathbb{Z}_+^n, \ 0 \leq \tau < t \leq T \tag{13}$$

(here positive constants c_2, B_2 and δ_2 do not depend on k, x, y, τ and t). In addition, using the estimate (5) and the equality (12) we obtain inequality

$$|\varphi_{\tau,t}^k(z)| \leq c_3 B_3^{|k|+} k^{k\frac{1}{h}} e^{\delta_3(t-\tau)\|z\|^{p_0}}, \ k \in \mathbb{Z}_+^n, \ z \in \mathbb{C}^n, \ 0 \leq \tau < t \leq T, \tag{14}$$

with an estimated constant not dependent on k, z, τ and t.

Note that when $\mu > 0$ estimate (14) can be specified. Indeed, let $z = x + iy \in \mathbb{C}^n \setminus \mathbb{K}_\mu$, then inequality $\|y\|/K > \|x\|^\mu$ holds. From here, the estimates $\|z\|^{p_0} \leq c(\|x\|^{p_0} + \|y\|^{p_0})$, $z \in \mathbb{C}^n$ and (14) taking into account that $\mu \leq 1$ for all $z \in \mathbb{C}^n \setminus \mathbb{K}_\mu$, $\tau \in [0; T)$ and $t \in (\tau; T]$ we obtain

$$\begin{aligned}|\varphi_{\tau,t}^k(z)| &\leq c_3 B_3^{|k|+} k^{k\frac{1}{h}} e^{-\delta_2(t-\tau)\|x\|^h} e^{(t-\tau)(\delta_3\|z\|^{p_0}+\delta_2\|x\|^h)} \\ &\leq c_4 B_3^{|k|+} k^{k\frac{1}{h}} e^{(t-\tau)\left(\delta_0\|y\|^{\frac{p_0}{\mu}} - \delta_2\|x\|^h\right)}\end{aligned}$$

(here estimated constants also do not depend on k, z, τ and t). If we now consider estimate (13) then we come to this statement: exists $\{c, B, \delta_1, \delta_2\} \subset (0; +\infty)$ for all $z = x + iy \in \mathbb{C}^n$, $k \in \mathbb{Z}_+^n$, $\tau \in [0; T)$ $t \in (\tau; T]$ such that

$$|\varphi_{\tau,t}^k(z)| \leq c B^{|k|+} k^{k\frac{1}{h}} e^{(t-\tau)\left(\delta_1\|y\|^{\frac{p_0}{\mu}} - \delta_2\|x\|^h\right)}. \tag{15}$$

Further, according to Cauchy integral formula we have

$$\partial_x^q \varphi_{\tau,t}^k(x) = \prod_{j=1}^n \frac{q_j!}{2\pi i} \int\limits_{\Gamma_{R_j}} \frac{\varphi_{\tau,t}^k(\sigma) d\sigma_j}{(\sigma_j - x_j)^{q_j+1}}, \ \{k, q\} \subset \mathbb{Z}_+^n, \, x \in \mathbb{R}^n, \, 0 \leq \tau < t \leq T, \tag{16}$$

here Γ_{R_j} is circle of radius R_j with center in the point x_j.

Let $\Gamma_R := \Gamma_{R_1} \times \ldots \times \Gamma_{R_n}$. Let us denote $\sigma^* = \xi^* + i\eta^*$ is the point from Γ_R such that

$$|\varphi_{\tau,t}^k(\sigma^*)| = \max_{\sigma\in\Gamma_R} |\varphi_{\tau,t}^k(\sigma)|.$$

Since the coordinates σ_j^* of the point σ^* are in Γ_{R_j} then the equality

$$(\xi_j^* - x_j)^2 + \eta_j^{*2} = R_j^2, \ j \in \mathbb{N}_n,$$

holds and implies such correlations:

$$|\zeta_j^* - x_j| \le R_j, \ |\eta_j^*| \le R_j, \ \ j \in \mathbb{N}_n. \tag{17}$$

Taking into consideration all above mentioned, estimations (15), inequality

$$n^{-1}|x|_+^r \le \|x\|^r \le n^{r/2}|x|_+^r, \ \ r > 0, x \in \mathbb{R}^n,$$

and (16) for $\mu > 0$ we obtain for all $R_j > 0$, $j \in \mathbb{N}_n$

$$|\partial_x^q \varphi_{\tau,t}^k(x)| \le cB^{|k|_+} k^{k\frac{1}{h}} \prod_{j=1}^n \frac{q_j!}{R_j^{q_j}} e^{(t-\tau)(\hat{\delta}_1 R_j^{p_0/\mu} - \hat{\delta}_2|\zeta_j^*|^h)} \tag{18}$$

(here $\hat{\delta}_1 =: \delta_1 n^{\frac{p_0}{2\mu}}$ and $\hat{\delta}_2 := \delta_2/n$).

Let us take radiuses R_j such that the ratio $e^{(t-\tau)\hat{\delta}_1 R_j^{p_0/\mu}}/R_j^{q_j}$ reaches a minimum. Then we put

$$R_j = \left(\frac{q_j \mu}{(t-\tau)\hat{\delta}_1 p_0}\right)^{\mu/p_0}, \ \ j \in \mathbb{N}_n.$$

Then bound (18) is reduced to

$$|\partial_x^q \varphi_{\tau,t}^k(x)| \le cB^{|k|_+} \big((t-\tau)ep_0\hat{\delta}_1/\mu\big)^{\mu|q|_+/p_0} k^{k\frac{1}{h}} q^{q(1-\frac{\mu}{p_0})} e^{-(t-\tau)\hat{\delta}_2|\zeta^*|_+^h}. \tag{19}$$

Next, let's estimate the exponent $e^{-(t-\tau)\hat{\delta}_2|\zeta_j^*|^h}$, $j \in \mathbb{N}_n$.

If $2|\zeta_j^*| \ge |x_j|$ then we have

$$e^{-(t-\tau)\hat{\delta}_2|\zeta_j^*|^h} \le e^{-(t-\tau)\hat{\delta}_2(|x_j|/2)^h}.$$

If $|x_j| > 2|\zeta_j^*|$ then according to (17) the following inequalities hold:

$$\begin{aligned} R_j^h &\ge |x_j - \zeta_j^*|^h \ge ||x_j| - |\zeta_j^*||^h = (|x_j|^h - |\zeta_j^*|^h)\frac{||x_j| - |\zeta_j^*||^h}{|x_j|^h - |\zeta_j^*|^h} \\ &\ge (|x_j|^h - |\zeta_j^*|^h)\big|1 - |\zeta_j^*|/|x_j|\big|^h \ge (|x_j|^h - |\zeta_j^*|^h)/2^h, \end{aligned}$$

and

$$|x_j|^h - |\zeta_j^*|^h \le (2R_j)^h.$$

Then $-|\zeta_j^*|^h = -|x_j|^h + (|x_j|^h - |\zeta_j^*|^h) \le -|x_j|^h + (2R_j)^h$ and

$$e^{-(t-\tau)\hat{\delta}_2|\zeta_j^*|^h} \le e^{-(t-\tau)\hat{\delta}_2|x_j|^h + \hat{\delta}_0(t-\tau)R_j^h}, \ \ \hat{\delta}_0 := \hat{\delta}_2 2^h.$$

From here and the astimate (19), abides by that

$$\begin{aligned} (t-\tau)R_j^h &= (t-\tau)^{1-\mu h/p_0}\left(\frac{q_j\mu}{\hat{\delta}_1 p_0}\right)^{\mu h/p_0} \le T_0^{1-\mu h/p_0}\left(\frac{q_j\mu}{\hat{\delta}_1 p_0}\right)^{\mu h/p_0} \\ &\equiv c q_j^{\mu h/p_0} \le c q_j, \ \ j \in \mathbb{N}_n. \end{aligned}$$

If $\mu > 0$ we get the next statement: exists $\{c, A, B, \delta\} \subset (0; +\infty)$ for all $\{k, q\} \subset \mathbb{Z}_+^n$, $\tau \in [0; T)$, $t \in (\tau; T]$, $x \in \mathbb{R}^n$ such that

$$|\partial_x^q \varphi_{\tau,t}^k(x)| \leq c((t-\tau)^\alpha A)^{|q|_+} B^{|k|_+} k^{k\frac{1}{h}} q^{q(1-\alpha)} e^{-(t-\tau)\delta|x|_+^h}. \tag{20}$$

Directly from the estimate (20) and the definition of matrix function $\varphi_{\tau,t}^k(\cdot)$ and with the equality

$$(ix)^q \partial_x^k G(\tau, t; x) = (-i)^{|k|_+} (2\pi)^{-n} (t-\tau)^{-\frac{\gamma+|k|_+}{h}} \int_{\mathbb{R}^n} \partial_\xi^q \varphi_{\tau,t}^k(\xi) e^{-i(x,\xi)} d\xi,$$

we obtain that

$$|\partial_x^k G(\tau, t; x)| \leq c_0 (t-\tau)^{-\frac{n+\gamma+|k|_+}{h}} B^{|k|_+} k^{k\frac{1}{h}} \times \left(\prod_{j=1}^n \inf_{q_j} \left\{ ((t-\tau)^\alpha A)^{q_j} q_j^{q_j(1-\alpha)} |x_j|^{-q_j} \right\} \right)$$
$$\leq c(t-\tau)^{-\frac{n+\gamma+|k|_+}{h}} B^{|k|_+} k^{k\frac{1}{h}} e^{-\delta\left(\frac{\|x\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}},$$

for all $k \in \mathbb{Z}_+^n$, $x \in \mathbb{R}^n$ and $0 \leq \tau < t \leq T$, while estimated constants c, B and δ do not depend on t, τ, k and x. Theorem is proved. $\square$

4 CONCLUSIONS

Parabolic systems of Shilov type are parabolically unstable systems to a change in their coefficients, in contrast to Petrovsky's parabolic systems. In this respect, information is important about parabolic systems with variable coefficients that significantly extend the Petrovsky class in the Shilov class and allow us to use the means of the classical theory of the Cauchy problem for their investigation. The class of systems defined in this article is such. The presence of this class, in particular, convinces that the class of Shilov vector equations with variable coefficients is not exhausted by the class of Petrine systems with time-dependent coefficients, but is much wider.

The obtained here estimates of the fundamental solution of the Cauchy problem for Shilov parabolic systems with coefficients that depend on t important to establish the correct solvability of the Cauchy problem in various functional spaces and, in the study of properties of solutions to this problem.

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Літовченко В.А. *Параболічні за Шиловим системи із змінними коефіцієнтами* // Карпатські матем. публ. — 2017. — Т.9, №2. — С. 145–153.

Через параболічну нестійкість систем Шилова до зміни своїх коефіцієнтів, означення параболічності за Шиловим для систем із залежними від часу t коефіцієнтами, на відміну від параболічності за Петровським, формулюється шляхом накладання умов на матрицант відповідної двоїстої за Фур'є системи. Для параболічних за Петровським систем із залежними від часу коефіцієнтами ці умови є характерною властивістю матриціанта, які випливають безпосередньо із означення параболічності. У зв'язку з цим, набуває актуальності питання про багатство класу Шилова систем із змінними коефіцієнтами.

У даній роботі наведено новий клас лінійних параболічних систем рівнянь із частинними похідними першого порядку за t із залежними від часу коефіцієнтами, який охоплює клас Петровського систем із молодшими коефіцієнтами, залежними від t. Головна частина диференціального виразу кожної такої системи є параболічним за Шиловим виразом із сталими коефіцієнтами. Методом перетворення Фур'є побудовано фундаментальний розв'язок задачі Коші для систем цього класу та обгрунтовано їх параболічність за Шиловим. При цьому використано лише структуру системи та умови на власні числа її головного матричного символу. Цей клас, перед усім, характеризує багатство класу Шилова систем із змінними коефіцієнтами та невичерпність його системами Петровського.

Також наведено загальний метод дослідження фундаментального розв'язку задачі Коші для параболічних за Шиловим систем, який є розвиненням відомого методу Я.І. Житомирського.

Ключові слова і фрази: параболічна за Шиловим система, фундаментальний розв'язок, задача Коші.

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MULYAVA O.[1], TRUKHAN YU.[2]

ON MEROMORPHICALLY STARLIKE FUNCTIONS OF ORDER α AND TYPE β, WHICH SATISFY SHAH'S DIFFERENTIAL EQUATION

According to M.L. Mogra, T.R. Reddy and O.P. Juneja an analytic in $\mathbb{D}_0 = \{z:\ 0<|z|<1\}$ function $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} f_n z^n$ is said to be meromorphically starlike of order $\alpha \in [0, 1)$ and type $\beta \in (0, 1]$ if $|zf'(z) + f(z)| < \beta|zf'(z) + (2\alpha - 1)f(z)|$, $z \in \mathbb{D}_0$. Here we investigate conditions on complex parameters β_0, β_1, γ_0, γ_1, γ_2, under which the differential equation of S. Shah $z^2w'' + (\beta_0 z^2 + \beta_1 z)w' + (\gamma_0 z^2 + \gamma_1 z + \gamma_2)w = 0$ has meromorphically starlike solutions of order $\alpha \in [0, 1)$ and type $\beta \in (0, 1]$. Beside the main case $n + \gamma_2 \neq 0$, $n \geq 1$, cases $\gamma_2 = -1$ and $\gamma_2 = -2$ are considered. Also the possibility of the existence of the solutions of the form $f(z) = \frac{1}{z} + \sum_{n=1}^{m} f_n z^n$, $m \geq 2$, is studied. In addition we call an analytic in $\mathbb{D}_0$ function $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} f_n z^n$ meromorphically convex of order $\alpha \in [0,1)$ and type $\beta \in (0,1]$ if $|zf''(z) + 2f'(z)| < \beta|zf''(z) + 2\alpha f'(z)|$, $z \in \mathbb{D}_0$ and investigate sufficient conditions on parameters β_0, β_1, γ_0, γ_1, γ_2 under which the differential equation of S. Shah has meromorphically convex solutions of order $\alpha \in [0, 1)$ and type $\beta \in (0, 1]$. The same cases as for the meromorphically starlike solutions are considered.

Key words and phrases: meromorphically starlike function of order α and type β, meromorphically convex function of order α and type β, Shah's differential equation.

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INTRODUCTION AND PRELIMINARY LEMMAS

An analytic univalent in $\mathbb{D} = \{z:\ |z|<1\}$ function

$$f(z) = \sum_{n=0}^{\infty} f_n z^n \tag{1}$$

is said to be convex if $f(\mathbb{D})$ is a convex domain. It is well known [2, p. 203] that the condition $\mathrm{Re}\,\{1 + zf''(z)/f'(z)\} > 0$ $(z \in \mathbb{D})$ is necessary and sufficient for the convexity of f. By W. Kaplan [4] a function f is said to be close-to-convex in $\mathbb{D}$ (see also [2, p. 583]) if there exists a convex in $\mathbb{D}$ function Φ such that $\mathrm{Re}\,(f'(z)/\Phi'(z)) > 0$ $(z \in \mathbb{D})$. A close-to-convex function f has the characteristic property that the complement G to the domain $f(\mathbb{D})$ can be filled with rays L which go from ∂G and lie in G. Every close-to-convex in $\mathbb{D}$ function f is univalent in $\mathbb{D}$ and, therefore, $f'(0) \neq 0$. Hence it follows that a function f is close-to-convex in $\mathbb{D}$ if and only if the function

$$g(z) = z + \sum_{n=2}^{\infty} g_n z^n \tag{2}$$

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is close-to-convex in $\mathbb{D}$, where $g_n = f_n/f_1$. We remark also, that a function defined by (2) is said to be starlike in $\mathbb{D}$, if $g(\mathbb{D})$ is a starlike domain with respect to the origin. It is clear that every starlike function is close-to-convex.

S.M. Shah [8] indicated conditions on real parameters β_0, β_1, γ_0, γ_1, γ_2 of the differential equation

$$z^2w'' + (\beta_0 z^2 + \beta_1 z)w' + (\gamma_0 z^2 + \gamma_1 z + \gamma_2)w = 0, \tag{3}$$

under which there exists an entire transcendental solution given by (1) such that f and all its derivatives are close-to-convex in $\mathbb{D}$. In particular he obtained the following result: if $\beta_1 + \gamma_2 = 0$, $-1 \le \beta_0 < 0$, $\beta_1 > 0$, $\gamma_0 = 0$ and $-\beta_1/2 < \gamma_1 \le 0$, then equation (3) has an entire solution (2) such that all $g^{(n)}$ $(n \ge 0)$ are close-to-convex in $\mathbb{D}$ and $\ln M_g(r) = (1 + o(1))|\beta_0|r$ as $r \to +\infty$, where $M_g(r) = \max\{|g(z)| : |z| = r\}$.

The investigations are continued in papers [9–14]. In particular in the case of complex parameters β_0, β_1, γ_0, γ_1, γ_2 in [13] it is proved that if $\gamma_0 = 0$, $\beta_1 + \gamma_2 = 0$, $\beta_0 \ne 0$, $|\beta_1| < 2$ and $\dfrac{2(|\beta_1| + |\gamma_1|)}{2 - |\beta_1|} < \ln 2$ then equation (3) has an entire solution (2) such that all $g^{(n)}$ $(n \ge 0)$ are starlike and, thus, close-to-convex in $\mathbb{D}$ and $\ln M_g(r) = (1 + o(1))|\beta_0|r$ as $r \to +\infty$. An analog of this assertion for convex functions is obtained in [14], where it is proved that if $\gamma_0 = 0$, $\beta_1 + \gamma_2 = 0$, $\beta_0 \ne 0$, $|\beta_1| < 2$ and $\dfrac{2(|\beta_1| + |\gamma_1|)}{2 - |\beta_1|} < \dfrac{\ln 2}{2}$ then equation (3) has an entire solution (2) such that all $g^{(n)}$ $(n \ge 0)$ are convex in $\mathbb{D}$.

Let Σ be the class of functions defined by

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} f_n z^n, \tag{4}$$

analytic in $\mathbb{D}_0 = \{z : 0 < |z| < 1\}$. A function $f \in \Sigma$ is said ([3,5]) to be meromorphically starlike of order $\alpha \in [0, 1)$ if $\mathrm{Re}\,\{-zf'(z)/f(z)\} > \alpha$ $(z \in \mathbb{D}_0)$, and is said to be meromorphically convex of order $\alpha \in [0, 1)$ if $\mathrm{Re}\{-(1 + zf''(z))/f'(z)\} > \alpha$ $(z \in \mathbb{D}_0)$.

Conditions on complex parameters β_0, β_1, γ_0, γ_1, γ_2 under which Shah's differential equation has meromorphically starlike and meromorphically convex solutions of order $\alpha \in [0, 1)$ are investigated in [1] . It is known ([1,7]) that if

$$|zf'(z) + f(z)| < |zf'(z) + (2\alpha - 1)f(z)| \tag{5}$$

for all $z \in \mathbb{D}_0$ then the function f is meromorphically starlike of order $\alpha \in [0, 1)$.

By B. Uralegaddi [15] a function $f \in \Sigma$ is meromorphically starlike of order $\beta \in (0, 1]$ if

$$|zf'(z) + f(z)| < \beta|zf'(z) - f(z)|, \quad z \in \mathbb{D}_0. \tag{6}$$

Finally, combining (5) and (6), M.L. Mogra, T.R. Reddy and O.P. Juneja [6] called a function $f \in \Sigma$ meromorphically starlike of order $\alpha \in [0, 1)$ and type $\beta \in (0, 1]$ if

$$|zf'(z) + f(z)| < \beta|zf'(z) + (2\alpha - 1)f(z)|, \quad z \in \mathbb{D}_0,$$

and proved the following lemma.

Lemma 1. *If*

$$\sum_{n=1}^{\infty} ((1 + \beta)n + \beta(2\alpha - 1) + 1)|f_n| \le 2\beta(1 - \alpha), \tag{7}$$

then the function defined by (4) *is meromorphically starlike of order* $\alpha \in [0, 1)$ *and type* $\beta \in (0, 1]$.

Here we investigate conditions on complex parameters β_0, β_1, γ_0, γ_1, γ_2 such that equation (3) has meromorphically starlike solutions of order $\alpha \in [0, 1)$ and type $\beta \in (0, 1]$.

We need also the following lemma [1].

Lemma 2. *A function defined by* (4) *is a solution of equation* (3) *if and only if*

$$2-\beta_1+\gamma_2=0, \quad -\beta_0+\gamma_1=0, \gamma_0+2(1+\gamma_2)f_1=0, \quad 3(2+\gamma_2)f_2+2\gamma_1 f_1=0 \tag{8}$$

and for $n \geq 3$

$$(n+1)(n+\gamma_2)f_n+n\gamma_1 f_{n-1}+\gamma_0 f_{n-2}=0. \tag{9}$$

1 MEROMORPHICALLY STARLIKE SOLUTIONS

We assume that

$$n+\gamma_2 \neq 0, \quad n \geq 1. \tag{10}$$

Then equalities (8) and (9) yields that if $\gamma_0 = 0$ then all $f_n = 0$, that is, the condition (7) is equivalent to the condition $0 \leq \beta(1-\alpha)$. Therefore, the following statement is true ([1]).

Proposition 1. *If* $\beta_1 = 2+\gamma_2$, $\beta_0 = \gamma_1$, $\gamma_0 = 0$ *and condition* (10) *holds then differential equation* (3) *has the solution* $f(z) = 1/z$, *which is meromorphically starlike of order* α *and type* β *for each* $\alpha \in [0, 1)$ *and* $\beta \in (0, 1]$.

Now we assume that $\gamma_0 \neq 0$. Then $f_1 = -\dfrac{\gamma_0}{2(1+\gamma_2)}$, $f_2 = -\dfrac{2\gamma_1}{3(2+\gamma_2)}f_1$ and $f_n = -\dfrac{n\gamma_1}{(n+1)(n+\gamma_2)}f_{n-1} - \dfrac{\gamma_0}{(n+1)(n+\gamma_2)}f_{n-2}$. Using these equalities and Lemma 1 we prove the following theorem.

Theorem 1. *Let* $\alpha \in [0, 1)$ *and* $\beta \in (0, 1]$. *If* $\beta_1 = 2+\gamma_2$, $|\gamma_2| < 1$, $\beta_0 = \gamma_1$ *then differential equation* (3) *has a solution given by* (4), *which by the condition*

$$\frac{(1+\beta\alpha)|\gamma_0|}{1-|\gamma_2|} \leq 2\beta(1-\alpha)\left(1-\frac{(3+(1+2\alpha)\beta)|\gamma_1|}{3(1+\alpha\beta)(2-|\gamma_2|)}-\frac{(2+\beta(1+\alpha))|\gamma_0|}{4(1+\alpha\beta)(3-|\gamma_2|)}\right) \tag{11}$$

is meromorphically starlike of order α *and type* β.

Proof. Since $|\gamma_2| < 1$ then (10) holds and from the indicated above equalities for f_j we obtain

$$\begin{aligned}
&\sum_{n=1}^{\infty}((1+\beta)n+\beta(2\alpha-1)+1)|f_n| = 2(1+\beta\alpha)|f_1| + (3+\beta(1+2\alpha))|f_2| \\
&\quad + \sum_{n=3}^{\infty}((1+\beta)n+\beta(2\alpha-1)+1)\left|\frac{n\gamma_1}{(n+1)(n+\gamma_2)}f_{n-1} + \frac{\gamma_0}{(n+1)(n+\gamma_2)}f_{n-2}\right| \\
&\quad \leq 2(1+\beta\alpha)|f_1| + (3+\beta(1+2\alpha))|f_2| \\
&\quad + \sum_{n=3}^{\infty}\frac{n((1+\beta)n+\beta(2\alpha-1)+1)|\gamma_1|}{(n+1)(n-|\gamma_2|)}|f_{n-1}| + \sum_{n=3}^{\infty}\frac{((1+\beta)n+\beta(2\alpha-1)+1)|\gamma_0|}{(n+1)(n-|\gamma_2|)}|f_{n-2}| \\
&\quad = 2(1+\beta\alpha)|f_1| + (3+\beta(1+2\alpha))|f_2| \\
&\quad + \sum_{n=2}^{\infty}\frac{(n+1)((1+\beta)(n+1)+\beta(2\alpha-1)+1)|\gamma_1|}{(n+2)(n+1-|\gamma_2|)}|f_n| \\
&\quad + \sum_{n=1}^{\infty}\frac{((1+\beta)(n+2)+\beta(2\alpha-1)+1)|\gamma_0|}{(n+3)(n+2-|\gamma_2|)}|f_n| \\
&\quad = 2(1+\beta\alpha)|f_1| + (3+\beta(1+2\alpha))|f_2| - \frac{2(3+\beta(1+2\alpha))|\gamma_1|}{3(2-|\gamma_2|)}|f_1| \\
&\quad + \sum_{n=1}^{\infty}\frac{(n+1)(n+2+(n+2\alpha)\beta)|\gamma_1|}{(n+2)(n+1-|\gamma_2|)}|f_n| + \sum_{n=1}^{\infty}\frac{(n+3+\beta(n+1+2\alpha))|\gamma_0|}{(n+3)(n+2-|\gamma_2|)}|f_n|,
\end{aligned}$$

whence

$$\begin{aligned}
&\sum_{n=1}^{\infty}\left(1 - \frac{(n+1)(n+2+(n+2\alpha)\beta)|\gamma_1|}{(n+1+\beta(n-1+2\alpha))(n+2)(n+1-|\gamma_2|)}\right. \\
&\quad \left. - \frac{(n+3+\beta(n+1+2\alpha))|\gamma_0|}{(n+3)(n+1+\beta(n-1+2\alpha))(n+2-|\gamma_2|)}\right)((1+\beta)n+\beta(2\alpha-1)+1)|f_n| \\
&\quad \leq 2(1+\beta\alpha)|f_1| + \frac{2(3+\beta(1+2\alpha))|\gamma_1|}{3(2-|\gamma_2|)}|f_1| - \frac{2(3+\beta(1+2\alpha))|\gamma_1|}{3(2-|\gamma_2|)}|f_1| \\
&\quad = 2(1+\beta\alpha)|f_1| \leq \frac{(1+\beta\alpha)|\gamma_0|}{1-|\gamma_2|}.
\end{aligned} \tag{12}$$

Since the sequences $\left(\dfrac{(n+2+(n+2\alpha)\beta)}{n+1+\beta(n-1+2\alpha)}\right)$ and $\left(\dfrac{n+1}{(n+2)(n+1-|\gamma_2|)}\right)$ are decreasing then

$$\frac{(n+1)(n+2+(n+2\alpha)\beta)|\gamma_1|}{(n+1+\beta(n-1+2\alpha))(n+2)(n+1-|\gamma_2|)} \leq \frac{(3+(1+2\alpha)\beta)|\gamma_1|}{3(1+\alpha\beta)(2-|\gamma_2|)}, \tag{13}$$

and by analogy

$$\frac{(n+3+\beta(n+1+2\alpha))|\gamma_0|}{(n+3)(n+1+\beta(n-1+2\alpha))(n+2-|\gamma_2|)} \leq \frac{(2+\beta(1+\alpha))|\gamma_0|}{4(1+\alpha\beta))(3-|\gamma_2|)}. \tag{14}$$

Condition (11) implies the inequality

$$\frac{(3+(1+2\alpha)\beta)|\gamma_1|}{3(1+\alpha\beta)(2-|\gamma_2|)} + \frac{(2+\beta(1+\alpha))|\gamma_0|}{4(1+\alpha\beta))(3-|\gamma_2|)} < 1.$$

Therefore, from (12) in view of (13) and (14) we have

$$\begin{aligned}
\left(1 - \frac{(3+(1+2\alpha)\beta)|\gamma_1|}{3(1+\alpha\beta)(2-|\gamma_2|)} - \frac{(2+\beta(1+\alpha))|\gamma_0|}{4(1+\alpha\beta))(3-|\gamma_2|)}\right)&\sum_{n=1}^{\infty}((1+\beta)n+\beta(2\alpha-1)+1)|f_n| \\
&\leq \frac{(1+\beta\alpha)|\gamma_0|}{1-|\gamma_2|},
\end{aligned}$$

whence in view of (11) we obtain inequality (7). By Lemma 1 function (4) is meromorphically starlike of order α and type β. □

Now we consider the cases where the condition (10) does not hold. At first, we assume that $1+\gamma_2 = 0$. Then in view of (8) $\gamma_0 = 0$ and we can choose $f_1 \neq 0$, because if $f_1 = 0$ then in view of (8) $f_2 = 0$, and in view of (9) all $f_n = 0$ and we come to the case $f(z) = 1/z$, which we considered above.

We assume that $f_1 = a^2 \neq 0$ and $\gamma_1 = 0$. Since $2+\gamma_2 \neq 0$, we have $f_2 = 0$ and in view of the equality $\gamma_0 = 0$, all $f_n = 0$ for $n \geq 2$. Thus, the solution has the form $f(z) = 1/z + a^2 z = a(1/(az) + az) = 2aJ(az)$, where J is the function of Joukowsky. Therefore, using Lemma 1, we get the following statement.

Proposition 2. *If $\beta_1 = 1$, $\gamma_2 = -1$ and $\beta_0 = \gamma_1 = \gamma_0 = 0$ then differential equation* (3) *has the solution $f(z) = J(az)$, which by the condition $(1+\beta\alpha)|a|^2 \leq \beta(1-\alpha)$ is meromorphically starlike of order α and type β.*

If $\gamma_1 \neq 0$ then in view of the equality $\gamma_2 = -1$ from (8) we have $f_2 = -2\gamma_1 f_1/3$ and since $\gamma_0 = 0$, we obtain $f_n = -\dfrac{n\gamma_1}{n^2-1} f_{n-1}$ for $n \geq 3$. Using the recurrent formula we prove the following theorem.

Theorem 2. *If $\beta_1 = 1$, $\gamma_2 = -1$, $\gamma_0 = 0$, $\beta_0 = \gamma_1 \neq 0$ then there exists a solution given by* (4) *of differential equation* (3)*, which by the condition*

$$\frac{3+\beta+2\alpha\beta}{3(1+\alpha\beta)}|\gamma_1| < 1 \tag{15}$$

is meromorphically starlike of order α and type β.

Proof. Since, as above,

$$\begin{aligned}
&\sum_{n=1}^{\infty}((1+\beta)n+\beta(2\alpha-1)+1)|f_n| = 2(1+\beta\alpha)|f_1| \\
&\quad + \sum_{n=1}^{\infty}\frac{((1+\beta)(n+1)+\beta(2\alpha-1)+1)}{(1+\beta)n+\beta(2\alpha-1)+1}\frac{(n+1)|\gamma_1|}{((n+1)^2-1)}((1+\beta)n+\beta(2\alpha-1)+1)|f_n| \\
&\quad \leq 2(1+\beta\alpha)|f_1| + \sum_{n=1}^{\infty}\frac{3+\beta+2\alpha\beta}{2(1+\alpha\beta)}\frac{2|\gamma_1|}{3}((1+\beta)n+\beta(2\alpha-1)+1)|f_n|,
\end{aligned}$$

then by the condition (15) we have

$$\left(1-\frac{3+\beta+2\alpha\beta}{3(1+\alpha\beta)}|\gamma_1|\right)\sum_{n=1}^{\infty}((1+\beta)n+\beta(2\alpha-1)+1)|f_n| \leq 2(1+\beta\alpha)|f_1|.$$

Therefore, if

$$2(1+\beta\alpha)|f_1| \leq 2\beta(1-\alpha)\left(1-\frac{3+\beta+2\alpha\beta}{3(1+\alpha\beta)}|\gamma_1|\right), \tag{16}$$

then by Lemma 1 the function given by (4) is meromorphically starlike of order α and type β. In view of the arbitrariness of f_1 and the condition (15) we can choose f_1 such that the condition (16) holds. □

Now, let $2+\gamma_2=0$. Then $\beta_1=0$ and from (8) and (9) we obtain $f_1=\gamma_0/2$, $\gamma_1 f_1=0$ and $f_n=-\dfrac{n\gamma_1}{(n+1)(n-2)}f_{n-1}-\dfrac{\gamma_0}{(n+1)(n-2)}f_{n-2}$ for $n\geq 3$. Hence it follows that either $f_1=0$ or $\gamma_1=0$, and f_2 may be arbitrary number.

At first we suppose that $f_1=0$. Then $\gamma_0=0$ and for $n\geq 3$

$$|f_n|=\frac{n|\gamma_1|}{(n+1)(n-2)}|f_{n-1}|\leq\frac{|\gamma_1|}{n-2}|f_{n-1}|\leq\frac{|\gamma_1|^2}{(n-2)(n-3)}|f_{n-2}|\leq\cdots\leq\frac{|\gamma_1|^{n-2}}{(n-2)!}|f_2|.$$

Hence it follows that

$$\sum_{n=1}^{\infty}((1+\beta)n+\beta(2\alpha-1)+1)|f_n|\leq(3+\beta+2\alpha\beta)|f_2|$$
$$+\sum_{n=3}^{\infty}((1+\beta)n+\beta(2\alpha-1)+1)\frac{|\gamma_1|^{n-2}}{(n-2)!}|f_2|=K_1(\alpha,\beta,|\gamma_1|)|f_2|$$

where $K(\alpha,\beta,|\gamma_1|)=\text{const}>0$. Since f_2 may be arbitrary the following proposition holds.

Proposition 3. *If $\gamma_2=-2$, $\beta_1=\gamma_0=0$, $\beta_0=\gamma_1\neq 0$ then for each $\alpha\in[0,1)$ and $\beta\in(0,1]$ there exists a solution given by* (4) *of differential equation* (3)*, which is meromorphically starlike of order α and type β.*

Now, we assume that $\gamma_1=0$. Then $f_1=\gamma_0/2$, f_2 may be arbitrary and

$$|f_n|=\frac{|\gamma_0|}{(n+1)(n-2)}|f_{n-2}|\qquad\text{for}\qquad n\geq 3.$$

Using these relations, we prove the following theorem.

Theorem 3. *Let $\alpha\in[0,1)$ and $\beta\in(0,1]$. If $\gamma_2=-2,\beta_1=\beta_0=\gamma_1=0$ then there exists a solution given by* (4) *of differential equation* (3)*, which by the condition*

$$(1+\beta\alpha)|\gamma_0|\leq 2\beta(1-\alpha)\left(1-\frac{(2+\beta(1+\alpha))|\gamma_0|}{4(1+\beta\alpha)}\right),\tag{17}$$

is meromorphically starlike of order α and type β.

Proof. Since f_2 may be arbitrary, we set $f_2=0$. Then

$$\sum_{n=1}^{\infty}((1+\beta)n+\beta(2\alpha-1)+1)|f_n|=2(1+\alpha\beta)|f_1|+\sum_{n=3}^{\infty}\frac{((1+\beta)n+\beta(2\alpha-1)+1)|\gamma_0|}{(n+1)(n-2)}|f_{n-2}|$$
$$=2(1+\alpha\beta)|f_1|+\sum_{n=1}^{\infty}\frac{((1+\beta)(n+2)+\beta(2\alpha-1)+1)|\gamma_0|}{n(n+3)}|f_n|$$

and, thus,

$$\sum_{n=1}^{\infty}\left(1-\frac{((1+\beta)(n+2)+\beta(2\alpha-1)+1)|\gamma_0|}{((1+\beta)n+\beta(2\alpha-1)+1)n(n+3)}\right)((1+\beta)n+\beta(2\alpha-1)+1)|f_n|$$
$$\leq 2(1+\alpha\beta)|f_1|.$$

But

$$\frac{((1+\beta)(n+2)+\beta(2\alpha-1)+1)|\gamma_0|}{((1+\beta)n+\beta(2\alpha-1)+1)n(n+3)}\leq\frac{(2+\beta(1+\alpha))|\gamma_0|}{4(1+\alpha\beta)}.$$

Therefore,

$$\left(1-\frac{(2+\beta(1+\alpha))|\gamma_0|}{4(1+\alpha\beta)}\right)\sum_{n=1}^{\infty}((1+\beta)n+\beta(2\alpha-1)+1)|f_n|\leq(1+\alpha\beta)|\gamma_0|,$$

whence in view of (17) we obtain (7), and by Lemma 1 function (4) is meromorphically starlike of order α and type β. □

Finally, we consider the case, where equation (3) has a solution of the form

$$f(z)=\frac{1}{z}+\sum_{n=1}^{m}f_nz^n,\quad m\geq 2,\tag{18}$$

where $f_m\neq 0$. Equality (9) yields that

$$(m+2)(m+1+\gamma_2)f_{m+1}+m\gamma_1f_m+\gamma_0f_{m-1}=0\quad\text{and}$$
$$(m+3)(m+2+\gamma_2)f_{m+2}+(m+1)\gamma_1f_{m+1}+\gamma_0f_m=0.$$

Since $f_{m+2}=f_{m+1}=0$ and $f_m\neq 0$, the second equality implies the equality $\gamma_0=0$ and consequently the first equality implies the equality $\gamma_1=0$. Therefore, in view of (8) and (9) $(n+1)(n+\gamma_2)f_n=0$ for all $n\geq 1$. Since $f_m\neq 0$, so $m+\gamma_2=0$. Thus $n+\gamma_2\neq 0$ for all $n\neq m$ and, therefore, $f_n=0$, except f_m, which may be arbitrary. Hence it follows that the solution given by (18) is possible only if $m+\gamma_2=0$ and is of the form

$$f(z)=\frac{1}{z}+f_mz^m,\tag{19}$$

where f_m is an arbitrary number. (It is easy to verify directly that the function (19) is a solution of equation (3) if and only if $\beta_0=\gamma_0=\gamma_1=2-\beta_1+\gamma_2=m+\gamma_2=0$.)

For each $\alpha\in[0,1)$ and $\beta\in(0,1]$ we choose f_m such that $((1+\beta)m+\beta(2\alpha-1)+1)|f_m|\leq 2\beta(1-\alpha)$. Then the function (19) is meromorphically starlike of order α and type β.

2 MEROMORPHICALLY CONVEX SOLUTIONS

We call a function $f\in\Sigma$ meromorphically convex of order $\alpha\in[0,1)$ and type $\beta\in(0,1]$ if

$$|zf''(z)+2f'(z)|<\beta|zf''(z)+2\alpha f'(z)|,\quad z\in\mathbb{D}_0.$$

Clearly, f is meromorphically convex of order α and type β if and only if $\varphi(z)=-zf'(z)$ is meromorphically starlike of order α and type β. Since $\varphi(z)=\frac{1}{z}-\sum\limits_{n=1}^{\infty}nf_nz^n$, by Lemma 1 the condition

$$\sum_{n=1}^{\infty}((1+\beta)n+\beta(2\alpha-1)+1)n|f_n|\leq 2\beta(1-\alpha),\tag{20}$$

is sufficient in order that f is meromorphically convex of order $\alpha\in[0,1)$ and type $\beta\in(0,1]$. Therefore, using Lemma 2 we can prove analogues of Theorems 1 - 3.

Theorem 4. *Let $\alpha\in[0,1)$ and $\beta\in(0,1]$. If $\beta_1=2+\gamma_2$, $|\gamma_2|<1$, $\beta_0=\gamma_1$ then differential equation (3) has a solution given by (4), which by the condition*

$$\frac{(1+\beta\alpha)|\gamma_0|}{1-|\gamma_2|}\leq 2\beta(1-\alpha)\left(1-\frac{2(3+(1+2\alpha)\beta)|\gamma_1|}{3(1+\alpha\beta))(2-|\gamma_2|)}-\frac{3(2+\beta(1+\alpha))|\gamma_0|}{4(1+\alpha\beta))(3-|\gamma_2|)}\right)\tag{21}$$

is meromorphically convex of order α and type β.

Proof. As in the proof of Theorem 1 we have

$$\sum_{n=1}^{\infty}((1+\beta)n+\beta(2\alpha-1)+1)n|f_n| = 2(1+\beta\alpha)|f_1| + (3+\beta(1+2\alpha))2|f_2|$$
$$+\sum_{n=3}^{\infty}((1+\beta)n+\beta(2\alpha-1)+1)\left|\frac{n^2\gamma_1(n-1)f_{n-1}}{(n-1)(n+1)(n+\gamma_2)}+\frac{n\gamma_0(n-2)f_{n-2}}{(n-2)(n+1)(n+\gamma_2)}\right|$$
$$\le 2(1+\beta\alpha)|f_1| + 2(3+\beta(1+2\alpha))|f_2|$$
$$+\sum_{n=2}^{\infty}\frac{(n+1)^2((1+\beta)(n+1)+\beta(2\alpha-1)+1)|\gamma_1|}{n(n+2)(n+1-|\gamma_2|)}n|f_n|$$
$$+\sum_{n=1}^{\infty}\frac{(n+2)((1+\beta)(n+2)+\beta(2\alpha-1)+1)|\gamma_0|}{n(n+3)(n+2-|\gamma_2|)}n|f_n|$$
$$= 2(1+\beta\alpha)|f_1| + 2(3+\beta(1+2\alpha))|f_2| - \frac{4(3+\beta(1+2\alpha))|\gamma_1|}{3(2-|\gamma_2|)}|f_1|$$
$$+\sum_{n=1}^{\infty}\frac{(n+1)^2(n+2+(n+2\alpha)\beta)|\gamma_1|}{n(n+2)(n+1-|\gamma_2|)}n|f_n| + \sum_{n=1}^{\infty}\frac{(n+2)(n+3+\beta(n+1+2\alpha))|\gamma_0|}{n(n+3)(n+2-|\gamma_2|)}n|f_n|,$$

whence as above

$$\left(1-\frac{2(3+(1+2\alpha)\beta)|\gamma_1|}{3(1+\alpha\beta)(2-|\gamma_2|)}-\frac{3(2+\beta(1+\alpha))|\gamma_0|}{4(1+\alpha\beta)(3-|\gamma_2|)}\right)\sum_{n=1}^{\infty}((1+\beta)n+\beta(2\alpha-1)+1)n|f_n|$$
$$\le \frac{(1+\beta\alpha)|\gamma_0|}{1-|\gamma_2|}$$

and in view of (21) we obtain (20). Therefore, the function defined by (4) is meromorphically convex of order α and type β. □

The following theorems can be proved by analogy.

Theorem 5. *If* $\beta_1 = 1$, $\gamma_2 = -1$, $\gamma_0 = 0$, $\beta_0 = \gamma_1 \neq 0$ *then there exists a solution given by* (4) *of differential equation* (3)*, which by the condition*

$$\frac{2(3+\beta+2\alpha\beta)}{3(1+\alpha\beta)}|\gamma_1| < 1$$

is meromorphically convex of order α *and type* β.

Theorem 6. *Let* $\alpha \in [0,\,1)$ *and* $\beta \in (0,\,1]$. *If* $\gamma_2 = -2$ *and* $\beta_1 = \beta_0 = \gamma_1 = 0$ *then there exists a solution given by* (4) *of differential equation* (3)*, which by the condition*

$$(1+\beta\alpha)|\gamma_0| \le 2\beta(1-\alpha)\left(1-\frac{3(2+\beta(1+\alpha))|\gamma_0|}{4(1+\beta\alpha)}\right)$$

is meromorphically convex of order α *and type* β.

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Мулява О., Трухан Ю. *Про мероморфно зіркові функції порядку α і типу β, що задовольняють диференційне рівняння Шаха* // Карпатські матем. публ. — 2017. — Т.9, №2. — С. 154–162.

Згідно з М.Л. Могра, Т.Р. Редді та О.П. Жюнея аналітична в $\mathbb{D}_0 = \{z : 0 < |z| < 1\}$ функція $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} f_n z^n$ називається мероморфно зірковою порядку $\alpha \in [0, 1)$ і типу $\beta \in (0, 1]$, якщо $|zf'(z) + f(z)| < \beta|zf'(z) + (2\alpha - 1)f(z)|$, $z \in \mathbb{D}_0$. Тут досліджено умови на комплексні параметри β_0, β_1, γ_0, γ_1, γ_2, за яких диференційне рівняння С. Шаха $z^2w'' + (\beta_0 z^2 + \beta_1 z)w' + (\gamma_0 z^2 + \gamma_1 z + \gamma_2)w = 0$ має мероморфно зіркові розв'язки порядку $\alpha \in [0, 1)$ і типу $\beta \in (0, 1]$. Окрім основного випадку $n + \gamma_2 \neq 0$, $n \geq 1$, розглядаються випадки $\gamma_2 = -1$ і $\gamma_2 = -2$. Також вивчено можливість існування розв'язків вигляду $f(z) = \frac{1}{z} + \sum_{n=1}^{m} f_n z^n$, $m \geq 2$. Крім того, ми називаємо аналітичну в $\mathbb{D}_0$ функцію $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} f_n z^n$ мероморфно опуклою порядку $\alpha \in [0,1)$ і типу $\beta \in (0,1]$, якщо $|zf''(z) + 2f'(z)| < \beta|zf''(z) + 2\alpha f'(z)|$, $z \in \mathbb{D}_0$, і досліджуємо достатні умови на параметри β_0, β_1, γ_0, γ_1, γ_2, за яких диференційне рівняння С. Шаха має мероморфно опуклі розв'язки порядку $\alpha \in [0, 1)$ і типу $\beta \in (0,1]$. Розглядаються ті ж випадки, що і для мероморфно зіркових розв'язків.

Ключові слова і фрази: мероморфно зіркова функція порядку α та типу β, мероморфно опукла функція порядку α та типу β, диференційне рівняння Шаха.

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PRAJISHA E., SHAINI P.

FG-COUPLED FIXED POINT THEOREMS IN CONE METRIC SPACES

The concept of FG-coupled fixed point introduced recently is a generalization of coupled fixed point introduced by Guo and Lakshmikantham. A point $(x, y) \in X \times X$ is said to be a coupled fixed point of the mapping $F : X \times X \to X$ if $F(x, y) = x$ and $F(y, x) = y$, where X is a non empty set. In this paper, we introduce FG-coupled fixed point in cone metric spaces for the mappings $F : X \times Y \to X$ and $G : Y \times X \to Y$ and establish some FG-coupled fixed point theorems for various mappings such as contraction type mappings, Kannan type mappings and Chatterjea type mappings. All the theorems assure the uniqueness of FG-coupled fixed point. Our results generalize several results in literature, mainly the coupled fixed point theorems established by Sabetghadam et al. for various contraction type mappings. An example is provided to substantiate the main theorem.

Key words and phrases: FG-coupled fixed point, cone metric space, contraction type mappings.

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1 INTRODUCTION

The classical Banach contraction theorem is proved to be one of the most fruitful and durable results in metric fixed point theory. Due to its enormous applications, several authors have studied and made very many generalizations of Banach contraction principle. In 2004 A.C.M. Ran and M.C.B. Reurings [1] proved an analogue of Banach contraction principle in partially ordered metric spaces and used the theorem to solve matrix equations. Following this, J.J. Nieto and R.R. Lopez [5, 6] established several fixed point theorems in partially ordered metric spaces and obtained applications to periodic boundary value problems. As an extension of fixed point, a new concept called coupled fixed point is introduced by D. Guo and V. Lakshmikantham [2]. They investigated some coupled fixed point theorems of mixed monotone operator, and applied their results to solve initial value problem of ordinary differential equations with discontinuous right hand sides. Using the notion of coupled fixed points they explored the existence and uniqueness of fixed point of non-monotone operator. Later T.G. Bhaskar and V. Lakshmikantham [13] established existence and uniqueness theorems of coupled fixed point for mixed monotone mappings defined on partially ordered complete metric spaces satisfying contraction type condition and applied their result to solve periodic boundary value problems. After the work of Gnana Bhaskar and Lakshmikantham, in 2009 V. Lakshmikantham and L. Ciric [14] introduced a new mapping called mixed g-monotone mapping. Using this, they proved coupled coincidence and coupled common fixed point theorems

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which generalize the results of Gnana Bhaskar and Lakshmikantham. In 2007 L.G. Huang and X. Zhang [8] introduced a metric called cone metric by replacing the real line by a real Banach space equipped with a partial ordering with respect to the cone. They proved some fixed point theorems for contraction mappings defined on cone metric spaces. Following them several authors have proved various fixed point theorems in cone metric spaces [10–12]. Later in 2009 F. Sabetghadam et al. [4] introduced the concept of coupled fixed point in cone metric spaces, and proved several coupled fixed point theorems for different contraction type mappings. In 2011 M.O. Olatinwo [9] proved coupled fixed point theorems by considering two different cone metrics on the same ambient space. Recently E. Prajisha and P. Shaini [3] introduced a concept called FG-coupled fixed point in partially ordered metric spaces which is a generalization of coupled fixed point. They established some FG-coupled fixed point theorems, in which F and G satisfy different contraction type conditions. Subsequently, K. Deepa and P. Shaini [7] proved several FG-coupled fixed point theorems for various contractive and generalized quasi-contractive mappings.

In this paper we define FG-coupled fixed point in cone metric spaces and prove FG-coupled fixed point theorems for different contraction type mappings on complete cone metric spaces. Let us give some useful definitions.

Definition 1. *A cone P is a subset of real Banach space E such that:*

(i) P *is closed, nonempty and* $P \neq \{0\}$;

(ii) if a, b *are non-negative real numbers and* $x, y \in P$, *then* $ax + by \in P$;

(iii) $P \cap (-P) = \{0\}$.

For a given cone $P \subseteq E$, the partial ordering $\leq$ with respect to P is defined by $x \leq y$ if and only if $y - x \in P$. The notation $x \ll y$ stands for $y - x \in intP$ where $intP$ denotes the interior of P. Also we will use $x < y$ to indicate that $x \leq y$ and $x \neq y$. The cone P is called normal if there is a number $M > 0$ such that for all $x, y \in E$, $0 \leq x \leq y$ implies that $\| x \| \leq M \ \| y \|$. The least positive number satisfying the above is called the normal constant of P. The cone P is called regular if every increasing (decreasing) sequence which is bounded above (below) is convergent. It is known that every regular cone is normal.

Definition 2 ([8])**.** *Let X be a non empty set and let E be a real Banach space equipped with the partial ordering $\leq$ with respect to the cone $P \subseteq E$. Suppose that the mapping $d : X \times X \to E$ satisfies the following conditions:*

(i) $0 \leq d(x, y)$ *for all* $x, y \in X$ *and* $d(x, y) = 0$ *if and only if* $x = y$;

(ii) $d(x, y) = d(y, x)$ *for all* $x, y \in X$;

(iii) $d(x, y) \leq d(x, z) + d(z, y)$ *for all* $x, y, z \in X$.

Then d is called a cone metric on X and the pair (X, d) is called a cone metric space.

Definition 3 ([8])**.** *Let (X, d) be a cone metric space, $x \in X$ and $\{x_n\}$ a sequence in X. Then*

(i) $\{x_n\}$ *converges to* x *whenever for every* $c \in E$ *with* $0 \ll c$ *there is a natural number* N *such that* $d(x_n, x) \ll c$ *for all* $n \geq N$. *We denote this by* $\lim_{n\to\infty} x_n = x$ *or* $x_n \to x$;

(ii) $\{x_n\}$ *is a Cauchy sequence whenever for every* $c \in E$ *with* $0 \ll c$ *there is a natural number* N *such that* $d(x_n, x_m) \ll c$ *for all* $n, m \geq N$.

A cone metric space (X, d) is said to be complete if every Cauchy sequence is convergent.

Definition 4 ([4]). *Let* (X, d) *be a cone metric space and* $F : X \times X \to X$ *be a mapping. An element* $(x, y) \in X \times X$ *is said to be coupled fixed point of* F *if* $F(x, y) = x$ *and* $F(y, x) = y$.

Definition 5 ([3]). *Let* $F : X \times Y \to X$ *and* $G : Y \times X \to Y$ *be two mappings, then for* $n \geq 1$, $F^n(x, y) = F(F^{n-1}(x, y), G^{n-1}(y, x))$ *and* $G^n(y, x) = G(G^{n-1}(y, x), F^{n-1}(x, y))$ *where* $F^0(x, y) = x$ *and* $G^0(y, x) = y$ *for all* $x \in X$ *and* $y \in Y$.

In the next section we define FG-coupled fixed point on cone metric spaces and prove existence and uniqueness theorems of FG-coupled fixed point for different contraction type mappings. We consider $d_X : X \times X \to E$ and $d_Y : Y \times Y \to E$, where E is a real Banach space equipped with the partial ordering $\leq$ with respect to the cone $P \subseteq E$ with $intP \neq \emptyset$.

2 Main Results

Three main theorems on FG-coupled fixed point are investigated in this section. We define FG-coupled fixed point in cone metric spaces as follows:

Definition 6. *Let* (X, d_X) *and* (Y, d_Y) *are cone metric spaces and* $F : X \times Y \to X$ *and* $G : Y \times X \to Y$ *are two mappings. An element* $(x, y) \in X \times Y$ *is said to be an* FG*-coupled fixed point if* $F(x, y) = x$ *and* $G(y, x) = y$.

Theorem 1. *Let* (X, d_X) *and* (Y, d_Y) *be two complete cone metric spaces. Suppose that the mappings* $F : X \times Y \to X$ *and* $G : Y \times X \to Y$ *satisfy the following conditions for all* $x, u \in X$, $y, v \in Y$ *:*

$$d_X(F(x, y), F(u, v)) \leq k\, d_X(x, u) + l\, d_Y(y, v), \tag{1}$$

$$d_Y(G(y, x), G(v, u)) \leq k\, d_Y(y, v) + l\, d_X(x, u), \tag{2}$$

where k, l *are non negative constants with* $k + l < 1$. *Then there exist a unique* FG*-coupled fixed point.*

Proof. Take $x_0 \in X$ and $y_0 \in Y$. Construct sequences $\{x_n\}$ and $\{y_n\}$ by defining $x_{n+1} = F(x_n, y_n) = F^{n+1}(x_0, y_0)$ and $y_{n+1} = G(y_n, x_n) = G^{n+1}(y_0, x_0)$ for $n \geq 0$.

We have,

$$d_X(x_{n+1}, x_n) = d_X(F(x_n, y_n), F(x_{n-1}, y_{n-1})) \leq k\, d_X(x_n, x_{n-1}) + l\, d_Y(y_n, y_{n-1}),$$

and

$$\begin{aligned} d_Y(y_{n+1}, y_n) &= d_Y(G(y_n, x_n), G(y_{n-1}, x_{n-1})) \\ &\leq k\, d_Y(y_n, y_{n-1}) + l\, d_X(x_n, x_{n-1}). \end{aligned}$$

By adding the above inequalities we get $d_n \leq (k + l)\, d_{n-1}$, where

$$d_n = d_X(x_{n+1}, x_n) + d_Y(y_{n+1}, y_n).$$

Continuing this process we get $d_n \leq \theta\, d_{n-1} \leq \theta^2\, d_{n-2} \cdots \leq \theta^n\, d_0$, where $\theta = k + l < 1$. If $d_0 = 0$ then (x_0, y_0) is an FG-coupled fixed point. If $d_0 \neq 0$, then we have $d_0 > 0$.

We have for $m > n$, $d_X(x_n, x_m) \leq d_X(x_n, x_{n+1}) + d_X(x_{n+1}, x_{n+2}) + \cdots + d_X(x_{m-1}, x_m)$ and $d_Y(y_n, y_m) \leq d_Y(y_n, y_{n+1}) + d_Y(y_{n+1}, y_{n+2}) + \cdots + d_Y(y_{m-1}, y_m)$

$$\begin{aligned} \text{ie, } d_X(x_n, x_m) + d_Y(y_n, y_m) &\leq d_n + d_{n+1} + \cdots + d_{m-1} \\ &\leq \theta^n\, d_0 + \theta^{n+1}\, d_0 + \cdots + \theta^{m-1}\, d_0 \leq \frac{\theta^n}{1-\theta}\, d_0 \end{aligned}$$

Now, for $0 \ll c$ there exist $r > 0$ such that $y \ll c$ for $\| y \| < r$. Choose a positive integer N_c such that for all $n \geq N_c$, $\| \frac{\theta^n}{1-\theta}\, d_0 \| < r$, which implies $\frac{\theta^n}{1-\theta}\, d_0 \ll c$, for $n \geq N_c$.

Thus $d_X(x_n, x_m) + d_Y(y_n, y_m) \ll c$, for $m > n \geq N_c$. Since $d_X(x_n, x_m) \leq d_X(x_n, x_m) + d_Y(y_n, y_m)$ and $d_Y(y_n, y_m) \leq d_X(x_n, x_m) + d_Y(y_n, y_m)$, $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X and Y respectively. By the completeness of X and Y there exist $(x, y) \in X \times Y$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. ie, for all $0 \ll c$ there exist N' such that $d_X(x_n, x) \ll \frac{c}{2}$ for all $n \geq N'$ and there exist N'' such that $d_Y(y_n, y) \ll \frac{c}{2}$ for all $n \geq N''$. Take $N = \max\{N', N''\}$. We have

$$\begin{aligned} d_X(F(x,y), x) &\leq d_X(F(x,y), x_{N+1}) + d_X(x_{N+1}, x) = d_X(F(x,y), F(x_N, y_N)) + d_X(x_{N+1}, x) \\ &\leq k\, d_X(x, x_N) + l\, d_Y(y, y_N) + d_X(x_{N+1}, x) \ll k\, \frac{c}{2} + l\, \frac{c}{2} + \frac{c}{2} < c \end{aligned}$$

Thus $F(x,y) = x$. Similarly we get $G(y,x) = y$.

Now we prove the uniqueness of FG-coupled fixed point. Let $(x, y) \neq (x', y') \in X \times Y$ such that $F(x', y') = x'$ and $G(y', x') = y'$. Then we have,

$$\begin{aligned} &d_X(x, x') = d_X(F(x,y), F(x', y')) \leq k\, d_X(x, x') + l\, d_Y(y, y') \text{ and} \\ &d_Y(y, y') = d_Y(G(y,x), G(y', x')) \leq k\, d_Y(y, y') + l\, d_X(x, x') \\ &\text{ie, } d_X(x, x') + d_Y(y, y') \leq (k + l)\, [d_X(x, x') + d_Y(y, y')] < d_X(x, x') + d_Y(y, y'). \end{aligned}$$

This is not possible. So $x = x'$ and $y = y'$. Hence the proof. □

Example 1. *Let $X = [0, \infty)$ and $Y = (-\infty, 0]$. Let $E = C^1_{\mathbb{R}}$ with $\| x \| = \| x \|_\infty + \| x' \|_\infty$ and $P = \{x \in E : x(t) \geq 0, t \in [0,1]\}$. Define cone metric $d : X \times X \to E$ by $d(x,y) = |x - y|\varphi$ where $\varphi : [0,1] \to \mathbb{R}$ such that $\varphi(t) = e^t$: see [15] Consider the mappings $F : X \times Y \to X$ and $G : Y \times X \to Y$ defined as $F(x,y) = \frac{x - 4y}{6}$ and $G(y,x) = \frac{y - 4x}{6}$. Clearly F and G satisfy all the conditions given in Theorem 1, and it is easy to see that $(0,0)$ is a unique FG-coupled fixed point.*

Corollary 1 ([4, Theorem 2.2])**.** *Let (X, d) be a complete cone metric space. Suppose that the mapping $F : X \times X \to X$ satisfies the following contractive condition for all $x, y, u, v \in X$:*

$$d(F(x,y), F(u,v)) \leq k\, d(x,u) + l\, d(y,v), \tag{3}$$

where k and l are non negative constants with $k + l < 1$. Then F has a unique coupled fixed point.

Corollary 2 ([4, Corollary 2.3])**.** *Let (X, d) be a complete cone metric space. Suppose that the mapping $F : X \times X \to X$ satisfies the following contractive condition for all $x, y, u, v \in X$:*

$$d(F(x,y), F(u,v)) \leq \frac{k}{2}\,[d(x,u) + d(y,v)], \tag{4}$$

where $k \in [0, 1)$. Then F has a unique coupled fixed point.

Theorem 2. *Let (X, d_X) and (Y, d_Y) be two complete cone metric spaces. Suppose that the mappings $F : X \times Y \to X$ and $G : Y \times X \to Y$ satisfy the following condition for all $x, u \in X$, $y, v \in Y$:*

$$d_X(F(x,y), F(u,v)) \leq k\, d_X(F(x,y), x) + l\, d_X(F(u,v), u), \tag{5}$$

$$d_Y(G(y,x), G(v,u)) \leq k\, d_Y(G(y,x), y) + l\, d_Y(G(v,u), v), \tag{6}$$

where k, l are non negative constants with $k + l < 1$. Then there exist a unique FG-coupled fixed point.

Proof. As in the proof of previous theorem construct sequences $\{x_n\}$ and $\{y_n\}$ defined by $x_{n+1} = F(x_n, y_n) = F^{n+1}(x_0, y_0)$, $y_{n+1} = G(y_n, x_n) = G^{n+1}(y_0, x_0)$ for $n \geq 0$. Then we have

$$\begin{aligned} d_X(x_{n+1}, x_n) &= d_X(F(x_n, y_n), F(x_{n-1}, y_{n-1})) \leq k\, d_X(F(x_n, y_n), x_n) + l\, d_X(F(x_{n-1}, y_{n-1}), x_{n-1}) \\ &= k\, d_X(x_{n+1}, x_n) + l\, d_X(x_n, x_{n-1}). \end{aligned}$$

Therefore $d_X(x_{n+1}, x_n) \leq \dfrac{l}{1-k}\, d_X(x_n, x_{n-1})$. Similarly $d_Y(y_{n+1}, y_n) \leq \dfrac{l}{1-k}\, d_Y(y_n, y_{n-1})$. Repeating this process we get, $d_X(x_{n+1}, x_n) \leq \delta^n\, d_X(x_1, x_0)$ and $d_Y(y_{n+1}, y_n) \leq \delta^n\, d_Y(y_1, y_0)$, where $\delta = \dfrac{l}{1-k}$.

If $x_1 = x_0$ and $y_1 = y_0$, then the result follows. Otherwise for $m > n$ consider,

$$\begin{aligned} d_X(x_n, x_m) &\leq d_X(x_n, x_{n+1}) + d_X(x_{n+1}, x_{n+2}) + \cdots + d_X(x_{m-1}, x_m) \\ &\leq \delta^n\, d_X(x_1, x_0) + \delta^{n+1}\, d_X(x_1, x_0) + \cdots + \delta^{m-1}\, d_X(x_1, x_0) \\ &\leq \frac{\delta^n}{1-\delta}\, d_X(x_1, x_0). \end{aligned}$$

This implies that $\{x_n\}$ is a Cauchy sequence in X. Similarly we can prove that $\{y_n\}$ is a Cauchy sequence in Y. Now by the completeness of the spaces X and Y, there exist $(x, y) \in X \times Y$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. Hence for all $0 \ll c$ there exist N' such that $d_X(x_n, x) \ll \dfrac{(1-k)\,c}{3}$ for all $n \geq N'$ and there exist N'' such that $d_Y(y_n, y) \ll \dfrac{(1-k)\,c}{3}$ for all $n \geq N''$.

Therefor we have

$$\begin{aligned} d_X(F(x,y), x) &\leq d_X(F(x,y), x_{N'+1}) + d_X(x_{N'+1}, x) = d_X(F(x,y), F(x_{N'}, y_{N'})) + d_X(x_{N'+1}, x) \\ &\leq k\, d_X(F(x,y), x) + l\, d_X(F(x_{N'}, y_{N'}), x_{N'}) + d_X(x_{N'+1}, x) \\ &\leq k\, d_X(F(x,y), x) + l\, [d_X(x_{N'+1}, x) + d_X(x, x_{N'})] + d_X(x_{N'+1}, x). \end{aligned}$$

Thus

$$d_X(F(x,y), x) \leq \frac{l+1}{1-k}\, d_X(x_{N'+1}, x) + \frac{l}{1-k}\, d_X(x_{N'}, x) \ll \frac{(l+1)\,c}{3} + \frac{l\,c}{3} < c.$$

Hence we have $F(x,y) = x$. Similarly, $G(y,x) = y$. If (x',y') is another FG-coupled fixed point, then we have

$$\begin{aligned} d_X(x,x') &= d_X(F(x,y),F(x',y')) \leq k\, d_X(F(x,y),x) + l\, d_X(F(x',y'),x') \\ &= k\, d_X(x,x) + l\, d_X(x',x') = 0. \end{aligned}$$

Thus $x = x'$. Similarly $y = y'$. Hence the proof. $\square$

Corollary 3 ([4, Theorem 2.5])**.** *Let (X,d) be a complete cone metric space. Suppose that the mapping $F : X \times X \to X$ satisfies the following contractive condition for all $x,y,u,v \in X$:*

$$d(F(x,y),F(u,v)) \leq k\, d(F(x,y),x) + l\, d(F(u,v),u), \tag{7}$$

where k and l are non negative constants with $k + l < 1$. Then F has a unique coupled fixed point.

Corollary 4 ([4, Corollary 2.7])**.** *Let (X,d) be a complete cone metric space. Suppose that the mapping $F : X \times X \to X$ satisfies the following contractive condition for all $x,y,u,v \in X$:*

$$d(F(x,y),F(u,v)) \leq \frac{k}{2}\, [d(F(x,y),x) + d(F(u,v),u)], \tag{8}$$

where $k \in [0,1)$. Then F has a unique coupled fixed point.

Theorem 3. *Let (X,d_X) and (Y,d_Y) be two complete cone metric spaces. Suppose that the mappings $F : X \times Y \to X$ and $G : Y \times X \to Y$ satisfy the following conditions for all $x,u \in X$, $y,v \in Y$:*

$$d_X(F(x,y),F(u,v)) \leq k\, d_X(F(x,y),u) + l\, d_X(F(u,v),x) \tag{9}$$

$$d_Y(G(y,x),G(v,u)) \leq k\, d_Y(G(y,x),v) + l\, d_Y(G(v,u),y), \tag{10}$$

where $k,l \in [0,\frac{1}{2})$. Then there exist a unique FG-coupled fixed point.

Proof. By defining $x_{n+1} = F(x_n,y_n)$ and $y_{n+1} = G(y_n,x_n)$ as in the above theorems, we construct sequences $\{x_n\}$ and $\{y_n\}$. Now we have,

$$\begin{aligned} d_X(x_{n+1},x_n) &= d_X(F(x_n,y_n),F(x_{n-1},y_{n-1})) \leq k\, d_X(F(x_n,y_n),x_{n-1}) + l\, d_X(F(x_{n-1},y_{n-1}),x_n) \\ &= k\, d_X(x_{n+1},x_{n-1}) + l\, d_X(x_n,x_n) \leq k\, [d_X(x_{n+1},x_n) + d_X(x_n,x_{n-1})]. \end{aligned}$$

Thus $d_X(x_{n+1},x_n) \leq \dfrac{k}{1-k}\, d_X(x_n,x_{n-1})$. Similarly $d_Y(y_{n+1},y_n) \leq \dfrac{k}{1-k}\, d_Y(y_n,y_{n-1})$. Repeating this way we get

$$d_X(x_{n+1},x_n) \leq \delta^n\, d_X(x_1,x_0)$$

and

$$d_Y(y_{n+1},y_n) \leq \delta^n\, d_Y(y_1,y_0), \text{ where } \delta = \frac{k}{1-k}.$$

In the similar lines of Theorem 2 see that the sequences $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X and Y respectively. Since (X,d_X) and (Y,d_Y) are complete, there exist $(x,y) \in X \times Y$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. Hence for all $0 \ll c$ there exist N' such that $d_X(x_n,x) \ll \dfrac{(1-k)\,c}{2}$ for all $n \geq N'$ and there exist N'' such that $d_Y(y_n,y) \ll \dfrac{(1-k)\,c}{2}$ for all $n \geq N''$.

Thus we have

$$\begin{aligned} d_X(F(x,y),x) &\leq d_X(F(x,y),x_{N'+1}) + d_X(x_{N'+1},x) \\ &= d_X(F(x,y),F(x_{N'},y_{N'})) + d_X(x_{N'+1},x) \\ &\leq k\, d_X(F(x,y),x_{N'}) + l\, d_X(F(x_{N'},y_{N'}),x) + d_X(x_{N'+1},x) \\ &\leq k\, [d_X(F(x,y),x) + d_X(x,x_{N'})] + l\, d_X(x_{N'+1},x) + d_X(x_{N'+1},x) \\ \text{ie, } d_X(F(x,y),x) &\leq \frac{k}{1-k}\, d_X(x_{N'},x) + \frac{l+1}{1-k}\, d_X(x_{N'+1},x) \ll \frac{k\,c}{2} + \frac{(l+1)\,c}{2} < c. \end{aligned}$$

Hence we get $F(x,y) = x$. Similarly $G(y,x) = y$. If $(x,y) \neq (x',y')$ is another FG-coupled fixed point, then we have

$$\begin{aligned} d_X(x,x') = d_X(F(x,y),F(x',y')) &\leq k\, d_X(F(x,y),x') + l\, d_X(F(x',y'),x) \\ &= k\, d_X(x,x') + l\, d_X(x',x) = (k+l)\, d_X(x',x) < d_X(x',x). \end{aligned}$$

This is not possible. Thus $x = x'$. Similarly $y = y'$. Hence the proof. □

Corollary 5 ([4, Theorem 2.6]). *Let (X,d) be a complete cone metric space. Suppose that the mapping $F : X \times X \to X$ satisfies the following contractive condition for all $x,y,u,v \in X$:*

$$d(F(x,y),F(u,v)) \leq k\, d(F(x,y),u) + l\, d(F(u,v),x), \tag{11}$$

where k and l are non negative constants with $k + l < 1$. Then F has a unique coupled fixed point.

Corollary 6 ([4, Corollary 2.8]). *Let (X,d) be a complete cone metric space. Suppose that the mapping $F : X \times X \to X$ satisfies the following contractive condition for all $x,y,u,v \in X$:*

$$d(F(x,y),F(u,v)) \leq \frac{k}{2}\, [d(F(x,y),u) + d(F(u,v),x)], \tag{12}$$

where $k \in [0,1)$. Then F has a unique coupled fixed point.

Remark 1. *If $F = G$ and $X = Y$ in Theorems 1, 2 and 3, then we get Corollaries 1, 3 and 5 respectively. In addition to this, if k and l are equal in Theorems 1, 2 and 3 then we get the corollaries 2, 4 and 6 respectively.*

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Праджіша Е., Шаїні П. *Теореми про FG-спарену фіксовану точку в конічних метричних просторах* // Карпатські матем. публ. — 2017. — Т.9, №2. — С. 163–170.

Концепція FG-спареної фіксованої точки, яка введена недавно, є узагальненням спареної фіксованої точки, що введена Ґуо і Лакшмікантамом. Точка $(x, y) \in X \times X$ називається спареною фіксованою точкою відображення $F : X \times X \to X$ якщо $F(x, y) = x$ і $F(y, x) = y$, де X непорожня множина. У цій статті ми вводимо FG-спарену фіксовану точку у конічних метричних просторах для відображень $F : X \times Y \to X$ і $G : Y \times X \to Y$ та встановлюємо деякі теореми про FG-спарену фіксовану точку для різних відображень, як от відображення стискуючого типу, відображення типу Канана та Чатержі. Усі ці теореми стосуються єдиності FG-спареної фіксованої точки. Наші результати узагальнють кілька результатів, в основному результати Сабетхадама та ін., про теореми про спарену фіксовану точку для різних типів стискуючих відображень. Також наведено приклад для того, щоб проілюструвати основну теорему.

Ключові слова і фрази: FG-спарена фіксована точка, конічний метричний простір, відображення стисткуючого типу.

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SOME FIXED POINT RESULTS IN COMPLETE GENERALIZED METRIC SPACES

The Banach contraction principle is the important result, that has many applications. Some authors were interested in this principle in various metric spaces. Branciari A. initiated the notion of the generalized metric space as a generalization of a metric space by replacing the triangle inequality by more general inequality, $d(x,y) \leq d(x,u) + d(u,v) + d(v,y)$ for all pairwise distinct points x, y, u, v of X. As such, any metric space is a generalized metric space but the converse is not true. He proved the Banach fixed point theorem in such a space. Some authors proved different types of fixed point theorems by extending the Banach's result. Wardowski D. introduced a new contraction which generalizes the Banach contraction. Using a mapping $F : \mathbb{R}^+ \to \mathbb{R}$ he introduced a new type of contraction called F-contraction and proved a new fixed point theorem concerning F-contraction.

In this paper, we have dealt with F-contraction and F-weak contraction in complete generalized metric spaces. We prove some results for F-contraction and F-weak contraction and we establish the existence and uniqueness of fixed point for F-contraction and F-weak contraction in complete generalized metric spaces. Some examples are supplied in order to support the usability of our results. The obtained result is an extension and a generalization of many existing results in the literature.

Key words and phrases: F-contraction, F-weak contraction, generalized metric space.

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INTRODUCTION AND PRELIMINARIES

The Banach contraction principle is the simplest result in fixed point theory [4]. This principle has many applications and was extended by several authors (see [5–10, 12, 14–17, 19, 20]). Some authors gave the fundamental linear contractive conditions and the fundamental nonlinear contractive conditions by using the notion of F-contraction, and proved fixed point theorems which generalize Banach contraction principle.

Due to the nature of mathematics science, there have been many attempts to generalize the metric setting by modifying some of the axioms of metric spaces. Thus, several other types of spaces have been introduced and a lot of metric results have been extended to new settings. One of the interesting generalizations of the notion of metric space was introduced by Branciari A. Later, most of the authors dealing with such spaces made some additional requirements in order to deduce their results (see [1–3]).

In this paper, we prove fixed point theorems for F-contraction and F-weak contraction in complete generalized metric spaces. We also present uniqueness of the fixed point.

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Definition 1 ([13]). *Let X be a nonempty set and $d : X \times X \to [0, \infty)$ a mapping such that for all $x, y \in X$ and all distinct points $u, v \in X$, each distinct from x and y:*

(i) $d(x, y) = 0 \Leftrightarrow x = y$,

(ii) $d(x, y) = d(y, x)$,

(iii) $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$ (quadrilateral inequality).

Then X is called a generalized metric space.

The concepts of convergence, Cauchy sequence, completeness, and continuity on a generalized metric space are defined below.

Definition 2 ([1]). *Let (X, d) be a generalized metric space.*

(i) A sequence $\{x_n\}$ is called convergent to $x \in X$ if and only if $d(x_n, x) \to 0$ as $n \to \infty$. In this case, we use the notation $x_n \to x$.

(ii) A sequence $\{x_n\}$ is called Cauchy if and only if for each $\varepsilon > 0$, there exists a natural number $N(\varepsilon)$ such that $d(x_n, x_m) < \varepsilon$ for all $n > m > N(\varepsilon)$.

(iii) A generalized metric space (X, d) is called complete if every Cauchy sequence is convergent in X.

(iv) A mapping $T : (X, d) \to (X, d)$ is continuous if for any sequence $\{x_n\}$ in X such that $d(x_n, x) \to 0$ as $n \to \infty$, we have $d(Tx_n, Tx) \to 0$ as $n \to \infty$.

Lemma 1 ([11]). *Let (X, d) be a generalized metric space and let $\{x_n\}$ be a Cauchy sequence in X such that $x_m \neq x_n$ whenever $m \neq n$. Then the sequence $\{x_n\}$ can converge to at most one point.*

Lemma 2 ([11]). *Let (X, d) be a generalized metric space and let $\{x_n\}$ be a sequence in X which is both Cauchy and convergent. Then the limit x of $\{x_n\}$ is unique. Moreover, if $z \in X$ is arbitrary, then $\lim\limits_{n\to\infty} d(x_n, z) = d(x, z)$.*

Theorem 1 ([13]). *Let (X, d) be a complete generalized metric space and suppose the mapping $f : X \to X$ satisfies $d(f(x), f(y)) \leq kd(x, y)$ for all $x, y \in X$ and fixed $k \in (0, 1)$. Then f has a unique fixed point x^* and $\lim\limits_{n\to\infty} f^n(x) = x^*$ for each $x \in X$.*

Definition 3 ([18]). *Let $\mathcal{F}$ be the family of all functions $F : (0, +\infty) \longrightarrow \mathbb{R}$ such that:*

(F1) F is strictly increasing, that is, for all $\alpha, \beta \in (0, +\infty)$ if $\alpha < \beta$ then $F(\alpha) < F(\beta)$;

(F2) for each sequence $\{\alpha_n\}$ of positive numbers, the following holds: $\lim\limits_{n\to\infty} \alpha_n = 0$ if and only if $\lim\limits_{n\to\infty} F(\alpha_n) = -\infty$;

(F3) there exists $k \in (0, 1)$ such that $\lim\limits_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

Definition 4 ([18]). *Let (X, d) be a metric space. A map $T : X \to X$ is said to be an F-contraction on (X, d) if there exist $F \in \mathcal{F}$ and $\tau > 0$ such that for all $x, y \in X$*

$$\text{from } d(Tx, Ty) > 0 \text{ follows that } \tau + F(d(Tx, Ty)) \leq F(d(x, y)). \tag{1}$$

Theorem 2 ([18]). *Let (X, d) be a complete metric space and let $T : X \to X$ be an F-contraction. Then*

(1) T has a unique fixed point x^*;

(2) for all $x \in X$ the sequence $\{T^n x\}$ is convergent to x^*.

Remark 1 ([18]). *Let T be an F-contraction. Then $d(Tx, Ty) < d(x, y)$ for all $x, y \in X$ such that $Tx \neq Ty$. Also, T is a continuous map.*

1 THE MAIN RESULTS

In this paper, we prove fixed point theorems for F-contraction and F-weak contraction in complete generalized metric spaces. We also present uniqueness of the fixed point.

Theorem 3. *Let (X, d) be a complete generalized metric space and $T : X \to X$ be an F-contraction. If F is continuous, then*
(1) T has a unique fixed point $x^* \in X$;
(2) for all $x \in X$, the sequence $\{T^n x\}$ is convergent to x^*.

Proof. Let $x_0 \in X$ be an arbitrary point. By induction, we easily construct a sequence $\{x_n\}$ such that

$$x_{n+1} = Tx_n = T^{n+1}x_0 \text{ for all } n \in \mathbb{N}. \tag{2}$$

If there exists $n \in \mathbb{N}$, $x_n = x_{n+1}$, the proof is complete. So, we assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$.

Step 1. We shall prove that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

Substituting $x = x_{n-1}$ and $y = x_n$ in (1), we obtain

$$\tau + F(d(Tx_{n-1}, Tx_n)) \leq F(d(x_{n-1}, x_n)),$$

i.e., $F(d(Tx_{n-1}, Tx_n)) \leq F(d(x_{n-1}, x_n)) - \tau$. Repeating this process, we get

$$\begin{aligned} F(d(Tx_{n-1}, Tx_n)) &\leq F(d(x_{n-1}, x_n)) - \tau = F(d(Tx_{n-2}, Tx_{n-1})) - \tau \\ &\leq F(d(x_{n-2}, x_{n-1})) - 2\tau = F(d(Tx_{n-3}, Tx_{n-2})) - 2\tau \\ &\leq F(d(x_{n-3}, x_{n-2})) - 3\tau \leq F(d(x_0, x_1)) - n\tau. \end{aligned} \tag{3}$$

From (3), we obtain $\lim_{n\to\infty} F(d(Tx_{n-1}, Tx_n)) = -\infty$, which together with (F2) and Definition 3 gives $\lim_{n\to\infty} d(Tx_{n-1}, Tx_n) = 0$, which implies that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{4}$$

Step 2. We will prove that $\lim_{n\to\infty} d(x_n, x_{n+2}) = 0$. By (1), we have

$$\begin{aligned} F(d(Tx_{n-1}, Tx_{n+1})) &\leq F(d(x_{n-1}, x_{n+1})) - \tau = F(d(Tx_{n-2}, Tx_n)) - \tau \\ &\leq F(d(x_{n-2}, x_n)) - 2\tau = F(d(Tx_{n-3}, Tx_{n-1})) - 2\tau \\ &\leq F(d(x_{n-3}, x_{n-1})) - 3\tau \leq F(d(x_0, x_2)) - n\tau. \end{aligned} \tag{5}$$

From (5) we obtain $\lim_{n\to\infty} F(d(Tx_{n-1}, Tx_{n+1})) = -\infty$, which together with (F2) and Definition 3 gives $\lim_{n\to\infty} d(Tx_{n-1}, Tx_{n+1}) = 0$, which implies that,

$$\lim_{n\to\infty} d(x_n, x_{n+2}) = 0. \tag{6}$$

Step 3. We will prove that $x_n \neq x_m$ for all $m \neq n$. We argue by contradiction. Suppose that $x_n = x_m$ for some $m, n \in \mathbb{N}$ with $m \neq n$. Since $d(x_p, x_{p+1}) > 0$, for each $p \in \mathbb{N}$, without loss of generality, we may assume that $m > n + 1$. Consider now

$$F(d(x_n,x_{n+1})) = F(d(x_n,Tx_n)) = F(d(x_m,Tx_m)) = F(d(Tx_{m-1},Tx_m))$$
$$\leq F(d(x_{m-1},x_m)) - \tau \leq F(d(x_{n+1},x_n)) - (m-n)\tau.$$

It is a contradiction.

Step 4. We will show that in this case $\{x_n\}$ is a Cauchy sequence. Suppose to the contrary. Then, there is an $\varepsilon > 0$ such that for an integer k, there exist natural numbers $m(k) > n(k) > k$ such that

$$d(x_{n(k)},x_{m(k)}) > \varepsilon. \tag{7}$$

For every integer k let $m(k)$ be the least positive integer exceeding $n(k)$ satisfying (7), we get

$$d(x_{n(k)},x_{m(k)-1}) \leq \varepsilon. \tag{8}$$

Now, using (7), (8) and the quadrilateral inequality, we find that

$$\varepsilon < d(x_{m(k)},x_{n(k)}) \leq d(x_{m(k)},x_{m(k)-2}) + d(x_{m(k)-2},x_{m(k)-1}) + d(x_{m(k)-1},x_{n(k)})$$
$$\leq d(x_{m(k)},x_{m(k)-2}) + d(x_{m(k)-2},x_{m(k)-1}) + \varepsilon.$$

Then, by (4) and (6), it follows that

$$\lim_{k\to\infty} d(x_{n(k)},x_{m(k)}) = \varepsilon. \tag{9}$$

Applying (1) with $x = x_{m(k)-1}$ and $y = x_{n(k)-1}$, we have

$$F(d(x_{m(k)},x_{n(k)})) = F(d(Tx_{m(k)-1},Tx_{n(k)-1})) \leq F(d(x_{m(k)-1},x_{n(k)-1})) - \tau.$$

If $k \to \infty$ in the above inequality and using (9) we obtain$F(\varepsilon) \leq F(\varepsilon) - \tau$.

This contradiction shows that $\{x_n\}$ is a Cauchy sequence. (X,d) is complete, there exists $x^* \in X$ such that

$$\lim_{n\to\infty} d(x_n,x^*) = 0. \tag{10}$$

Since T is continuous, we obtain from (10) that

$$\lim_{n\to\infty} d(x_{n+1},Tx^*) = \lim_{n\to\infty} d(Tx_n,Tx^*) = 0.$$

That is $\lim\limits_{n\to\infty} x_{n+1} = Tx^*$. Taking into account Lemma 2 we conclude that $Tx^* = x^*$. That is x^* is a fixed point of T. Now, let us to show that T has at most one fixed point. Indeed if $x,y \in X$ be two distinct fixed points of T, that is, $Tx = x \neq y = Ty$. Therefore $d(Tx,Ty) = d(x,y) > 0$, then we get

$$F(d(x,y)) = F(d(Tx,Ty)) < \tau + F(d(Tx,Ty)) \leq F(d(x,y)),$$

which is a contradiction. Therefore, the fixed point is unique. □

Definition 5. *Let (X,d) be a generalized metric space. A map $T : X \to X$ is said to be an F-weak contraction on (X,d) if there exist $F \in \mathcal{F}$ and $\tau > 0$ such that for all $x,y \in X$*

$$d(Tx,Ty) > 0 \Longrightarrow \tau + F(d(Tx,Ty)) \leq F(\max\{d(x,y),d(x,Tx),d(y,Ty)\}). \tag{11}$$

Remark 2. *Every F-contraction is an F-weak contraction on (X,d). But the converse is not true.*

Example 1. *Let $X = A \cup B$, where $A = \{1,2,3,4\}, B = [5,6]$. Define the generalized metric d on X as follows:*

$$\begin{aligned}
&d(x,y) = 0, \quad x = y \text{ and } x,y \in A,\\
&d(1,2) = d(3,4) = 2,\ d(1,3) = d(2,3) = 1,\ d(1,4) = d(2,4) = 5,\\
&d(x,y) = |x-y|\,,\ \text{for } x \in A, y \in B \text{ or } x \in B, y \in A \text{ or } x,y \in B.
\end{aligned}$$

It is easy to show that (X,d) is a complete generalized metric space, but (X,d) is not a metric space because d does not satisfy the triangle inequality for all $x,y,z \in X$. Indeed,

$$5 = d(1,4) > d(1,3) + d(3,4) = 1 + 2 = 3.$$

Let $T : X \to X$ be given by

$$Tx = \begin{cases} 3 & if \quad x \in A,\\ 1 & if \quad x \in B.\end{cases}$$

Since T is not continuous, T is not F-contraction by Remark 1. For $x \in A$ and $y \in B$, we have

$$d(Tx,Ty) = d(3,1) = 1 > 0$$

and $\max\{d(x,y), d(x,Tx), d(y,Ty)\} \geq 4$. *Therefore, by choosing $F\alpha = \ln\alpha$, $\alpha \in (0,+\infty)$ and $\tau = \ln 3$, we see that T is F -weak contraction.*

Theorem 4. *Let (X,d) be a complete generalized metric space and $T : X \to X$ be an F-weak contraction. If T or F is continuous, then*

(1) T has a unique fixed point $x^* \in X$;

(2) for all $x \in X$, the sequence $\{T^n x\}$ is convergent to x^*.

Proof. Let $x_0 \in X$ be an arbitrary point. By induction, we easily construct a sequence $\{x_n\}$ such that

$$x_{n+1} = Tx_n = T^{n+1}x_0 \text{ for all } n \in \mathbb{N}.$$

If there exists $n \in \mathbb{N}$, $x_n = x_{n+1}$, the proof is complete. So, we assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$.

Step 1. We will prove that

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0.$$

Substituting $x = x_{n-1}$ and $y = x_n$ in (11), we obtain

$$\begin{aligned}
F(d(x_{n+1},x_n)) &= F(d(Tx_n, Tx_{n-1}))\\
&\leq F(\max\{d(x_n,x_{n-1}), d(x_n,Tx_n), d(x_{n-1},Tx_{n-1})\}) - \tau\\
&= F(\max\{d(x_n,x_{n-1}), d(x_n,x_{n+1}), d(x_{n-1},x_n)\}) - \tau\\
&= F(\max\{d(x_n,x_{n-1}), d(x_n,x_{n+1})\}) - \tau.
\end{aligned} \tag{12}$$

If there exists $n \in \mathbb{N}$ such that $\max\{d(x_n,x_{n-1}), d(x_n,x_{n+1})\} = d(x_n,x_{n+1})$, from (12) becomes

$$F(d(x_{n+1},x_n)) \leq F(d(x_{n+1},x_n)) - \tau < F(d(x_{n+1},x_n)).$$

It is a contradiction. Therefore,

$$\max\{d(x_n,x_{n-1}), d(x_n,x_{n+1})\} = d(x_n,x_{n-1}) \tag{13}$$

for all $n \in \mathbb{N}$. That is from (F1), (12) and (13), we get

$$d(x_n, x_{n+1}) < d(x_n, x_{n-1}). \tag{14}$$

Thus, from (12), we have $F(d(x_{n+1}, x_n)) \leq F(d(x_n, x_{n-1})) - \tau$ for all $n \in \mathbb{N}$. It implies that

$$F(d(x_{n+1}, x_n)) \leq F(d(x_1, x_0)) - n\tau \tag{15}$$

for all $n \in \mathbb{N}$. Taking the limit as $n \to \infty$ in (15), we get $\lim\limits_{n\to\infty} F(d(x_{n+1}, x_n)) = -\infty$ that together with (F2) gives

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{16}$$

Step 2. We will prove that

$$\lim_{n\to\infty} d(x_n, x_{n+2}) = 0. \tag{17}$$

By (11), we have

$$\begin{aligned} F(d(x_n, x_{n+2})) &= F(d(Tx_{n-1}, Tx_{n+1})) \\ &\leq F(\max\{d(x_{n-1}, x_{n+1}), d(x_{n-1}, Tx_{n-1}), d(x_{n+1}, Tx_{n+1})\}) - \tau \\ &= F(\max\{d(x_{n-1}, x_{n+1}), d(x_{n-1}, x_n), d(x_{n+1}, x_{n+2})\}) - \tau. \end{aligned} \tag{18}$$

By (14) and from (F2), we have

$$\max\{d(x_{n-1}, x_{n+1}), d(x_{n-1}, x_n), d(x_{n+1}, x_{n+2})\} = \max\{d(x_{n-1}, x_{n+1}), d(x_{n-1}, x_n)\}.$$

Take $a_n = d(x_n, x_{n+2})$ and $b_n = d(x_n, x_{n+1})$. Thus, from (18)

$$\begin{aligned} F(a_n) &= F(d(x_n, x_{n+2})) = F(d(Tx_{n-1}, Tx_{n+1})) \\ &\leq F(\max\{d(x_{n-1}, x_{n+1}), d(x_{n-1}, Tx_{n-1}), d(x_{n+1}, Tx_{n+1})\}) - \tau \\ &= F(\max\{a_{n-1}, b_{n-1})\}) - \tau. \end{aligned} \tag{19}$$

Again, by (14) $b_n \leq b_{n-1} \leq \max\{a_{n-1}, b_{n-1}\}$. Therefore $\max\{a_n, b_n\} \leq \max\{a_{n-1}, b_{n-1}\}$, for all $n \in \mathbb{N}$. Then the sequence $\{\max\{a_n, b_n\}\}$ is monotone nonincreasing, so it converges to some $t \geq 0$. Assume that $t > 0$. Now, by (16)

$$\lim_{n\to\infty} \sup a_n = \lim_{n\to\infty} \sup \max\{a_n, b_n\} = \lim_{n\to\infty} \max\{a_n, b_n\} = t.$$

Taking $n \to \infty$ in (19), since F is continuous,

$$\begin{aligned} F(t) = \lim_{n\to\infty} \sup F(a_n) &\leq \lim_{n\to\infty} \sup(F(\max\{a_{n-1}, b_{n-1}\}) - \tau) \\ &\leq \lim_{n\to\infty} F(\max\{a_{n-1}, b_{n-1}\}) - \tau = F(t) - \tau, \end{aligned}$$

which is a contradiction, that is (17) is proved.

Step 3. We will prove that $x_n \neq x_m$ for all $m \neq n$.

We argue by contradiction. Suppose that $x_n = x_m$ for some $m, n \in \mathbb{N}$ with $m \neq n$. Since $d(x_p, x_{p+1}) > 0$, for each $p \in \mathbb{N}$, without loss of generality, we may assume that $m > n + 1$. Consider now

$$\begin{aligned} F(d(x_n, x_{n+1})) &= F(d(x_n, Tx_n)) = F(d(x_m, Tx_m)) = F(d(Tx_{m-1}, Tx_m)) \\ &\leq F(\max\{d(x_{m-1}, x_m), d(x_{m-1}, Tx_{m-1}), d(x_m, Tx_m)) - \tau \\ &= F(\max\{d(x_{m-1}, x_m), d(x_{m-1}, x_m), d(x_m, x_{m+1})\}) - \tau \\ &= F(\max\{d(x_{m-1}, x_m), d(x_m, x_{m+1})\}) - \tau. \end{aligned} \tag{20}$$

If $\max\{d(x_{m-1},x_m),d(x_m,x_{m+1})\}=d(x_{m-1},x_m)$, then from (20), we get

$$F(d(x_n,x_{n+1}))\leq F(d(x_{m-1},x_m))-\tau\leq F(d(x_n,x_{n+1}))-(m-n)\tau.$$

It is a contradiction. If $\max\{d(x_{m-1},x_m),d(x_m,x_{m+1})\}=d(x_m,x_{m+1})$, then from (20), we get $F(d(x_n,x_{n+1}))\leq F(d(x_m,x_{m+1}))-\tau\leq F(d(x_n,x_{n+1}))-(m-n+1)\tau$. It is a contradiction.

Step 4. We will prove that $\{x_n\}$ is a Cauchy sequence, that is

$$\lim_{n\to\infty} d(x_n,x_{n+p})=0 \text{ for all } p\in\mathbb{N}.$$

From (F3), there exists $k\in(0,1)$ such that

$$\lim_{n\to\infty}((d(x_{n+1},x_n))^k F(d(x_{n+1},x_n)))=0. \tag{21}$$

By using (15) and from (21), we have

$$(d(x_{n+1},x_n))^k(F(d(x_{n+1},x_n))-F(d(x_1,x_0)))\leq -(d(x_{n+1},x_n))^k n\tau\leq 0 \tag{22}$$

for all $n\in\mathbb{N}$. By using (16), (21) and taking the limit as $n\to\infty$ in (22), we get

$$\lim_{n\to\infty}(n(d(x_{n+1},x_n))^k)=0. \tag{23}$$

Then there exists $n_1\in\mathbb{N}$ such that $n(d(x_{n+1},x_n))^k\leq 1$ for all $n\geq n_1$, that is

$$d(x_{n+1},x_n)\leq\frac{1}{n^{\frac{1}{k}}}. \tag{24}$$

From (16) and (17) the cases $p=1$ and $p=2$ are proved. Now, take $p\geq 3$ arbitrary. It is sufficient to examine two cases.

Case 1. Suppose that $p=2m+1$ where $m\geq 1$. Then, by using step 3 and the quadrilateral inequality together with (24), we get

$$\begin{aligned}d(x_n,x_{n+p})=d(x_n,x_{n+2m+1})&\leq d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})+\cdots+d(x_{n+2m},x_{n+2m+1})\\&\leq\sum_{i=n}^{n+2m}d(x_{i+1},x_i)\leq\sum_{i=n}^{\infty}\frac{1}{i^{\frac{1}{k}}}.\end{aligned} \tag{25}$$

Since the series $\sum\limits_{n=1}^{\infty}\frac{1}{n^{\frac{1}{k}}}$ is convergent, taking the limit as $n\to\infty$ in the above inequality, we obtain $\lim\limits_{n\to\infty}d(x_n,x_{n+p})=0$.

Case 2. Suppose that $p=2m$ where $m\geq 2$. Then, by using step 3 and the quadrilateral inequality together with (24), we get

$$\begin{aligned}d(x_n,x_{n+p})=d(x_n,x_{n+2m})&\leq d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})+\cdots+d(x_{n+2m-1},x_{n+2m})\\&\leq\sum_{i=n}^{n+2m-1}d(x_{i+1},x_i)\leq\sum_{i=n}^{\infty}\frac{1}{i^{\frac{1}{k}}}.\end{aligned} \tag{26}$$

Since the series $\sum\limits_{n=1}^{\infty}\frac{1}{n^{\frac{1}{k}}}$ is convergent, taking the limit as $n\to\infty$ in the above inequality, we obtain $\lim\limits_{n\to\infty}d(x_n,x_{n+p})=0$.

This proves that $\{x_n\}$ is Cauchy sequence in X. Since X is complete, there exists x^*, that is a fixed point of T by two following cases.

Case 3. T is continuous. We have $d(x^*, Tx^*) = \lim_{n\to\infty} d(x_n, Tx_n) = \lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. This proves that x^* is a fixed point of T.

Case 4. F is continuous. In this case, we consider two following subcases.

Subcase 1. For each $n \in \mathbb{N}$, there exists $i_n \in \mathbb{N}$ such that $x_{i_{n+1}} = Tx^*$ and $i_n > i_{n-1}$ where $i_0 = 1$. Then we have

$$x^* = \lim_{n\to\infty} x_{i_{n+1}} = \lim_{n\to\infty} Tx^* = Tx^*.$$

This proves that x^* is a fixed point of T.

Subcase 2. There exists $n_0 \in \mathbb{N}$ such that $x_{n+1} \neq Tx^*$ for all $n \geq n_0$. That is $d(Tx_n, Tx^*) > 0$ for all $n \geq n_0$. It follows from (11) that

$$\begin{aligned} \tau + F(d(x_{n+1}, Tx^*)) = \tau + F(d(Tx_n, Tx^*)) &\leq F(\max\{d(x_n, x^*), d(x_n, Tx_n), d(x^*, Tx^*)\}) \\ &= F(\max\{d(x_n, x^*), d(x_n, x_{n+1}), d(x^*, Tx^*)\}). \end{aligned} \tag{27}$$

If $d(x^*, Tx^*) > 0$ then by the fact

$$\lim_{n\to\infty} d(x_n, x^*) = \lim_{n\to\infty} d(x^*, x_{n+1}) = 0,$$

there exists $n_1 \in \mathbb{N}$ such that for all $n \geq n_1$, we have $\max\{d(x_n, x^*), d(x_n, x_{n+1}), d(x^*, Tx^*)\} = d(x^*, Tx^*)$. From (27), we get

$$\tau + F(d(x_{n+1}, Tx^*)) = F(d(x^*, Tx^*)), \tag{28}$$

for all $n \geq \max\{n_0, n_1\}$. Since F is continuous, taking the limit as $n \to \infty$ in (28), we obtain

$$\tau + F(d(x^*, Tx^*)) = F(d(x^*, Tx^*)).$$

It is contradiction. Therefore, $d(x^*, Tx^*) = 0$, that is, x^* is a fixed point of T. By two above cases, T has a fixed point x^*. Now, we prove that the fixed point of T is unique. Let x_1^*, x_2^* be two fixed points of T. Suppose to the contrary that $x_1^* \neq x_2^*$. Then $Tx_1^* \neq Tx_2^*$. It follows from (11) that

$$\begin{aligned} \tau + F(d(x_1^*, x_2^*)) = \tau + F(d(Tx_1^*, Tx_2^*)) &\leq F(\max\{d(x_1^*, x_2^*), d(x_1^*, Tx_1^*), d(x_2^*, Tx_2^*)\}) \\ &= F(\max\{d(x_1^*, x_2^*), d(x_1^*, x_1^*), d(x_2^*, x_2^*)\}) = F(d(x_1^*, x_2^*)). \end{aligned}$$

It is a contradiction. Then $d(x_1^*, x_2^*) = 0$, that is $x_1^* = x_2^*$. This proves that the fixed point of T is unique.

It follows from the proof of Theorem 4 that $\lim_{n\to\infty} T^n x = \lim_{n\to\infty} x_{n+1} = x^*$. □

Example 2. *Let F be given as in Example 1. Then T is an F-weak contraction. Therefore, Theorem 4 can be applicable to T and the unique fixed point of T is 3.*

Example 3. *Let $X = \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}\}$. Define the generalized metric d on X as follows:*

$$d(x, y) = 0, \quad x = y \text{ and } x, y \in X,$$
$$d\Big(\frac{1}{2}, \frac{2}{3}\Big) = d\Big(\frac{3}{4}, \frac{4}{5}\Big) = 0,2,\ d\Big(\frac{1}{2}, \frac{4}{5}\Big) = d\Big(\frac{2}{3}, \frac{3}{4}\Big) = 0,3,\ d\Big(\frac{1}{2}, \frac{3}{4}\Big) = d\Big(\frac{2}{3}, \frac{4}{5}\Big) = 0,6.$$

It is easy to show that (X, d) is a complete generalized metric space, but (X, d) is not a metric space because d does not satisfy the triangle inequality for all $x, y, z \in X$. Indeed,

$$0,6 = d\Big(\frac{1}{2}, \frac{3}{4}\Big) \geq d\Big(\frac{1}{2}, \frac{2}{3}\Big) + d\Big(\frac{2}{3}, \frac{3}{4}\Big) = 0,2 + 0,3 = 0,5.$$

Let $T : X \to X$ be defined as follows:

$$Tx = \begin{cases} \frac{3}{4}, & x \in \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}\}, \\ \frac{2}{3}, & x = \frac{4}{5}. \end{cases}$$

Let $F\alpha = \ln \alpha$, $\alpha \in (0, +\infty)$ and $\tau = \ln \frac{3}{2}$. Then, for $x \in \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}\}$ and $y = \frac{4}{5}$, we get

$$\begin{aligned} F(0,45) &= F\Big(d\Big(T\Big(\frac{1}{2}\Big), T\Big(\frac{4}{5}\Big)\Big)\Big) + \ln\frac{3}{2} \\ &\leq F\Big(\max\Big\{d\Big(\frac{1}{2}, \frac{4}{5}\Big), d\Big(\frac{1}{2}, T\Big(\frac{1}{2}\Big)\Big), d\Big(\frac{4}{5}, T\Big(\frac{4}{5}\Big)\Big)\Big\}\Big) = F(0,6), \end{aligned}$$

$$\begin{aligned} F(0,45) &= F\Big(d\Big(T\Big(\frac{2}{3}\Big), T\Big(\frac{4}{5}\Big)\Big)\Big) + \ln\frac{3}{2} \\ &\leq F\Big(\max\Big\{d\Big(\frac{2}{3}, \frac{4}{5}\Big), d\Big(\frac{2}{3}, T\Big(\frac{2}{3}\Big)\Big), d\Big(\frac{4}{5}, T\Big(\frac{4}{5}\Big)\Big)\Big\}\Big) = F(0,6), \end{aligned}$$

$$\begin{aligned} F(0,45) &= F\Big(d\Big(T\Big(\frac{3}{4}\Big), T\Big(\frac{4}{5}\Big)\Big)\Big) + \ln\frac{3}{2} \\ &\leq F\Big(\max\Big\{d\Big(\frac{3}{4}, \frac{4}{5}\Big), d\Big(\frac{3}{4}, T\Big(\frac{3}{4}\Big)\Big), d\Big(\frac{4}{5}, T\Big(\frac{4}{5}\Big)\Big)\Big\}\Big) = F(0,6). \end{aligned}$$

Therefore, T is a F-weak contraction in generalized metric space. That is, Theorem 4 can be applicable to T and the unique fixed point of T is $\dfrac{3}{4}$.

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Санґурлу С.М., Тюркоґлу Д. *Деякі тереми про фіксовану точку в повних узагальнених метричних просторах* // Карпатські матем. публ. — 2017. — Т.9, №2. — С. 171–180.

Принцип стискуючих відображень є важливим результатом, що має багато застосувань. Деякі автори цікавились цим принципом в різних метричних просторах. Бранчіарі А. ввів поняття узагальненого метричного простору, замінивши нерівність трикутника більш загальною нерівністю $d(x,y) \leq d(x,u) + d(u,v) + d(v,y)$ для всіх попарно різних точок x, y, u, v з X. Таким чином, будь-який метричний простір є узагальненим метричним простором, але не навпаки. Він довів теорему Банаха про фіксовану точку в таких просторах. Деякі автори довели різні типи теорем про фіксовану точку, розширюючи результат Банаха. Так Вардовський Д. представив новий вид стискуючих відображень, який узагальнює поняття стискуючого відображення Банаха. Використовуючи відображення $F : \mathbb{R}^+ \to \mathbb{R}$, він ввів новий тип стискуючих відображень, які називаються F-стиском. Також він довів теорему про фіксовану точку для F-стиску.

У даній роботі ми розглянули F-стиск та слабкий F-стиск у повних узагальнених метричних просторах. Доведено деякі результати для F-стисків і слабких F-стисків і встановлено існування та єдиність фіксованої точки для F-стискуючих і слабких F-стискуючих відображень у повних узагальних метричних просторах. Наведено деякі приклади для ілюстрації використання отриманих результатів. Дані результати є розширенням і узагальненням багатьох отриманих у літературі результатів.

Ключові слова і фрази: F-стиск, слабкий F-стиск, узагальнений метричний простір.

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SHEREMETA M.M.

ON THE GROWTH OF A COMPOSITION OF ENTIRE FUNCTIONS

Let γ be a positive continuous on $[0, +\infty)$ function increasing to $+\infty$ and f and g be arbitrary entire functions of positive lower order and finite order.

In order to

$$\lim_{r\to+\infty} \frac{\ln\ln M_{f(g)}(r)}{\ln\ln M_f(\exp\{\gamma(r)\})} = +\infty, \quad M_f(r) = \max\{|f(z)| : |z| = r\},$$

it is necessary and sufficient $(\ln \gamma(r))/(\ln r) \to 0$ as $r \to +\infty$. This statement is an answer to the question posed by A.P. Singh and M.S. Baloria in 1991.

Also in order to

$$\lim_{r\to+\infty} \frac{\ln\ln M_F(r)}{\ln\ln M_f(\exp\{\gamma(r)\})} = 0, \quad F(z) = f(g(z)),$$

it is necessary and sufficient $(\ln \gamma(r))/(\ln r) \to \infty$ as $r \to +\infty$.

Key words and phrases: entire function, composition of functions, generalized order.

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INTRODUCTION

For an entire function $f \not\equiv \text{const}$ we put $M_f(r) = \max\{|f(z)| : |z| = r\}$. The quantities

$$\varrho[f] = \overline{\lim_{r\to+\infty}} \frac{\ln\ln M_f(r)}{\ln r}, \quad \lambda[f] = \underline{\lim_{r\to+\infty}} \frac{\ln\ln M_f(r)}{\ln r} \tag{1}$$

are called [7, p. 61] the order and the lower order of f accordingly.

G.D. Song and C.C. Yang [6] have proved that if f and g are transcendental entire functions, $0 < \lambda[f] \le \varrho[f] < +\infty$ and $F(z) = f(g(z))$ then

$$\lim_{r\to+\infty} \frac{\ln\ln M_F(r)}{\ln\ln M_f(r)} = +\infty.$$

A.P. Singh and M.S. Baloria [3] posed a question: how to find $R = R(r)$ such that

$$\underline{\lim}_{r\to+\infty} \frac{\ln\ln M_F(r)}{\ln\ln M_f(R)} < +\infty\,?$$

They have proved the following theorems.

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Theorem A. *Let f and g be entire functions of positive lower order and of finite order, and $F(z) = f(g(z))$. Then* $\displaystyle\lim_{r\to+\infty} \frac{\ln\ln M_F(r)}{\ln\ln M_f(r^A)} = +\infty$ *for every positive constant A.*

Theorem B. *Let f and g be entire functions of finite order with $\varrho[g] < \varrho[f]$ and $F(z) = f(g(z))$. Then* $\displaystyle\lim_{r\to+\infty} \frac{\ln\ln M_F(r)}{\ln\ln M_f(\exp\{r^{\varrho[f]}\})} = 0.$

The aim of proposed article is research of the above mentioned problem from [4].

1 MAIN RESULTS

Next theorem gives an answer to the question of A.P. Singh and M.S. Baloria.

Theorem 1. *Let γ be a positive continuous on $[0, +\infty)$ function increasing to $+\infty$. Let f and g be arbitrary entire functions with $0 < \lambda[f] \le \varrho[f] < +\infty$ and $\lambda[g] > 0$. In order to*

$$\lim_{r\to+\infty} \frac{\ln\ln M_F(r)}{\ln\ln M_f(\exp\{\gamma(r)\})} = +\infty, \quad F(z) = f(g(z)), \tag{2}$$

it is necessary and sufficient

$$\lim_{r\to+\infty} \frac{\ln \gamma(r)}{\ln r} = 0. \tag{3}$$

Proof. G. Polya [2] has proved that if f and g are entire functions, $|g(0)| = 0$ and $F(z) = f(g(z))$ then there exists a constant $c \in (0, 1)$ independent of f and g such that for all $r > 0$

$$M_F(r) \ge M_f\left(cM_g\left(\frac{r}{2}\right)\right) \text{ and} \tag{4}$$

$$M_F(r) \le M_f(M_g(r)). \tag{5}$$

J. Clunie [1] defines more precisely inequality (4). He proved that

$$M_F(r) \ge M_f\left(\frac{1}{8}M_g\left(\frac{r}{2}\right) - |g(0)|\right). \tag{6}$$

We assume that the function γ satisfies (3), that is $\ln \gamma(r) = o(\ln r)$ as $r \to +\infty$. If the lower orders $\lambda[f]$ and $\lambda[g]$ are positive then for $\lambda \in (0, \min\{\lambda[f], \lambda[g]\})$ and all $r \ge r_0(\lambda)$ the inequalities $\ln\ln M_f(r) \ge \lambda \ln r$ and $\ln\ln M_g(r) \ge \lambda \ln r$ are true. Therefore, in view of (6)

$$\begin{aligned} \ln\ln M_F(r) &\ge \ln\ln M_f\left(\frac{1}{8}M_g\left(\frac{r}{2}\right) - |g(0)|\right) \ge \lambda \ln\left(\frac{1}{8}M_g\left(\frac{r}{2}\right) - |g(0)|\right) \\ &= \lambda(1+o(1)) \ln M_g\left(\frac{r}{2}\right) \ge (1+o(1))\lambda 2^{-l} r^{\lambda}, \quad r \to +\infty. \end{aligned} \tag{7}$$

On the other hand, if $\varrho[f] < +\infty$ then $\ln\ln M_f(\exp\{\gamma(r)\}) \le \varrho\gamma(r)$ for $\varrho > \varrho[f]$ and all $r \ge r_0(\varrho)$. Therefore, in view of (7)

$$\frac{\ln\ln M_F(r)}{\ln\ln M_f(\exp\{\gamma(r)\})} \ge (1+o(1)) \frac{\lambda}{2^{\lambda}(\varrho[f]+\varepsilon)} \frac{r^{\lambda}}{\gamma(r)} \to +\infty, \quad r \to +\infty, \tag{8}$$

because $\lambda \ln r - \ln \gamma(r) = (1+o(1))\lambda \ln r \to +\infty$ as $r \to +\infty$. The sufficiency of (3) is proved.

To prove the necessity of (3) we assume that (3) does not hold. Then $\ln \gamma(r_n) \geq \delta \ln r_n$ for some $\delta > 0$ and an increasing to $+\infty$ sequence (r_n). We choose $f(z) = e^z$ and $g(z) = E_\varrho(z)$ with $\varrho < \delta$, where E_ϱ is the Mittag-Leffler function. Then $M_f(r) = e^r$ and [7, p. 115]

$$M_{E_\varrho}(r) = E_\varrho(r) = (1 + o(1))\varrho e^{r^\varrho}, \quad r \to +\infty. \tag{9}$$

Therefore,

$$\ln \ln M_F(r) = \ln M_g(r) = r^\varrho + \ln \varrho + o(1), \quad r \to +\infty. \tag{10}$$

Thus,

$$\begin{aligned} \varliminf_{r\to+\infty} \frac{\ln\ln M_F(r)}{\ln\ln M_f(\exp\{\gamma(r)\})} &\leq \varliminf_{n\to+\infty} \frac{\ln\ln M_F(r_n)}{\ln\ln M_f(\exp\{\gamma(r_n)\})} \\ &= \varliminf_{n\to+\infty} \frac{r_n^\varrho}{\gamma(r_n)} \leq \varliminf_{n\to+\infty} \frac{r_n^\varrho}{r_n^\delta} = 0, \end{aligned} \tag{11}$$

that is, if (3) does not hold then there exist entire functions f and g with $\lambda[f] = \varrho[f] = 1$ and $\lambda[g] = \varrho[g] = \varrho \in (0, +\infty)$, for which (2) is false. Theorem 1 is proved. □

The following theorem complements Theorem 1.

Theorem 2. *Let γ be a positive continuous on $[0, +\infty)$ function increasing to $+\infty$. Let f and g be arbitrary entire functions with $0 < \lambda[g] \leq \varrho[g] < +\infty$ and $\lambda[f] > 0$. In order to*

$$\lim_{r\to+\infty} \frac{\ln\ln M_F(r)}{\ln\ln M_g(\exp\{\gamma(r)\})} = +\infty, \quad F(z) = f(g(z)),$$

it is necessary and sufficient that (3) holds.

Proof. As in the proof of Theorem 1 we obtain (7) and for the function g we have $\ln\ln M_g(\exp\{\gamma(r)\}) \leq \varrho \ln \gamma(r)$ for every $\varrho > \varrho[g]$ and all $r \geq r_0(\varrho)$. Therefore, estimate (8) is true with $\varrho[g]$ instead $\varrho[f]$ and the sufficiency of (3) is proved.

If there exists a sequence (r_n) such that $\ln \gamma(r_n) \geq \delta \ln r_n$, $\delta > 0$, then again we choose f and g as in the proof of Theorem 1. Then (9) holds and

$$\ln\ln M_g(\exp\{\gamma(r)\}) = \ln\ln\left((1 + o(1))\varrho e^{\varrho\gamma(r)}\right) = \varrho\gamma(r) + o(1), \quad r \to +\infty.$$

In view of (9) as above we have

$$\varliminf_{r\to+\infty} \frac{\ln\ln M_F(r)}{\ln\ln M_g(\exp\{\gamma(r)\})} \leq \varliminf_{n\to+\infty} \frac{r_n^\varrho}{\varrho\gamma(r_n)} \leq \varliminf_{n\to+\infty} \frac{r_n^\varrho}{\varrho r_n^\delta} = 0.$$

Theorem 2 is proved. □

For the functions $f(z) = e^z$, $g(z) = E_\varrho(z)$ and $F(z) = f(g(z))$ chose the proof of Theorems 1 and 2 the following equalities are true

$$\varliminf_{r\to+\infty} \frac{\ln\ln M_F(r)}{\ln\ln M_f(\exp\{\gamma(r)\})} = \varliminf_{r\to+\infty} \frac{\ln\ln M_F(r)}{\ln\ln M_g(\exp\{\gamma(r)\})} = 0.$$

The following question arises: what is condition on γ providing existence of the limit

$$\lim_{r\to+\infty} \frac{\ln\ln M_F(r)}{\ln\ln M_f(\exp\{\gamma(r)\})} \left(\lim_{r\to+\infty} \frac{\ln\ln M_F(r)}{\ln\ln M_g(\exp\{\gamma(r)\})} \right) = 0.$$

The following theorem gives an answer to this question.

Theorem 3. *Let γ be a positive continuous on $[0, +\infty)$ function increasing to $+\infty$. Let f and g be arbitrary entire functions with $0 < \lambda[f] \le \varrho[f] < +\infty$ and $\varrho[g] < +\infty$. In order to*

$$\lim_{r\to+\infty} \frac{\ln\ln M_F(r)}{\ln\ln M_f(\exp\{\gamma(r)\})} = 0, \quad F(z) = f(g(z)), \tag{12}$$

it is necessary and sufficient that

$$\lim_{r\to+\infty} \frac{\ln \gamma(r)}{\ln r} = +\infty. \tag{13}$$

Proof. We assume that the function γ satisfies (13), that is $\ln r = o(\ln \gamma(r))$ as $r \to +\infty$. If the orders $\varrho[f]$ and $\varrho[g]$ are finite then $\ln \ln M_f(r) \le \varrho \ln r$ and $\ln \ln M_g(r) \le \varrho \ln r$ for $\varrho > \max\{\varrho[f], \varrho[g]\}$ and all $r \ge r_0(\varrho)$. Therefore, in view of (5)

$$\ln \ln M_F(r) \le \ln \ln M_f(M_g(r)) \le \varrho \ln M_g(r) \le \varrho r^{\varrho}, \quad r \ge r_0(\varrho).$$

On the other hand, for $\lambda < \lambda[f]$ and all $r \ge r_0(\lambda)$ $\ln \ln M_f(e^{\gamma(r)}) \ge l\gamma(r)$. Therefore,

$$\frac{\ln\ln M_F(r)}{\ln\ln M_f(\exp\{\gamma(r)\})} \le \frac{\varrho r^{\varrho}}{\lambda\gamma(r)} \to 0, \quad r \to +\infty,$$

because $\varrho \ln r - \ln \gamma(r) = (1 + o(1)) \ln \gamma(r) \to -\infty$ as $r \to +\infty$. The sufficiency of (13) is proved.

Now we assume that (13) does not hold, that is for some $\delta < +\infty$ and an increasing to $+\infty$ sequence (r_n) the inequality $\ln \gamma(r_n) \le \delta \ln r_n$ is true. We choose $f(z) = e^z$ and $g(z) = E_{\varrho}(z)$ with $\varrho > \delta$. Then in view of (10)

$$\begin{aligned} \overline{\lim_{r\to+\infty}} \frac{\ln\ln M_F(r)}{\ln\ln M_f(\exp\{\gamma(r)\})} &\ge \overline{\lim_{n\to+\infty}} \frac{\ln\ln M_F(r_n)}{\ln\ln M_f(\exp\{\gamma(r_n)\})} \\ &= \overline{\lim_{n\to+\infty}} \frac{r_n^{\varrho}}{\gamma(r_n)} \ge \lim_{n\to+\infty} \frac{r_n^{\varrho}}{r_n^{\delta}} = +\infty, \end{aligned} \tag{14}$$

that is equality (12) does not hold. Theorem 3 is proved. □

The following theorem is proved similarly.

Theorem 4. *Let γ be a positive continuous on $[0, +\infty)$ function increasing to $+\infty$. Let f and g be arbitrary entire functions with $0 < \lambda[g] \le \varrho[g] < +\infty$ and $\varrho[f] < +\infty$. In order to*

$$\lim_{r\to+\infty} \frac{\ln\ln M_F(r)}{\ln\ln M_g(\exp\{\gamma(r)\})} = 0, \quad F(z) = f(g(z)),$$

it is necessary and sufficient that (13) *holds.*

Remark 1.1. *From the proofs of Theorems 1 and 3 one can see that equality* (3) *is true provided, γ is an arbitrary slowly increasing function, and* (12) *holds if γ increase rapidly than power functions.*

Remark 1.2. *If we choose f and g as in the proofs of Theorem 1 and 2 and $\gamma(r) = ar^{\varrho}$, then there exists the limit*

$$\lim_{r\to+\infty} \frac{\ln\ln M_F(r)}{\ln\ln M_f(\exp\{\alpha(r)\})} = \lim_{r\to+\infty} \frac{r^{\varrho}}{\alpha(r)} = \frac{1}{a},$$

that is for each $K \in (0, +\infty)$ there exist entire functions of a finite order and a positive lower order and a positive continuous on $[0, +\infty)$ function γ such that

$$\lim_{r\to+\infty} \frac{\ln\ln M_F(r)}{\ln\ln M_f(\exp\{\gamma(r)\})} = K.$$

2 OTHER RESULTS

In [5] the following analogue of Theorem A is proved.

Theorem C. *Let f, g, h be entire functions of positive lower order and of finite order and $F(z) = f(g(z))$, $\Phi(z) = f(h(z))$. If $\varrho[h] < \lambda[g]$ then for every $A \in (0, \lambda[g]/\varrho[h])$*

$$\lim_{r\to+\infty} \frac{\ln\ln M_F(r)}{\ln\ln M_H(r^A)} = +\infty.$$

We will complement this theorem by two next statements.

Proposition 2.1. *Let γ be a positive continuous on $[0, +\infty)$ function increasing to $+\infty$. Let f, g and h be arbitrary entire functions with $0 < \lambda[f] \le \varrho[f] < +\infty$, $\lambda[g] > 0$ and $\varrho[h] < +\infty$. In order to*

$$\lim_{r\to+\infty} \frac{\ln\ln M_F(r)}{\ln\ln M_\Phi(e^{\gamma(r)})} = +\infty, \quad F(z) = f(g(z)),\ \Phi(z) = f(h(z)), \tag{15}$$

it is necessary and sufficient that

$$\lim_{r\to+\infty} \frac{\gamma(r)}{\ln r} = 0. \tag{16}$$

Proof. In view of (5) for arbitrary $\varrho > \max\{\varrho[f], \varrho[h]\}$ and all $r \ge r_0(\varrho)$ we have

$$\ln\ln M_\Phi(e^{\gamma(r)}) \le \varrho \ln M_h(e^{\gamma(r)}) \le \varrho e^{\varrho\gamma(r)}.$$

Therefore, in view of (7) $\dfrac{\ln\ln M_F(r)}{\ln\ln M_\Phi(e^{\gamma(r)})} \ge (1+o(1))\dfrac{l2^{-\lambda}}{\varrho}\dfrac{r^\lambda}{e^{\varrho\gamma(r)}} \to +\infty, \quad r \to +\infty$, because by the condition (16) $\dfrac{r^l}{e^{\varrho\gamma(r)}} = \exp\{\lambda \ln r - \varrho\gamma(r)\} \to +\infty$ as $r \to +\infty$. The sufficiency of (16) is proved.

Now we assume that (16) does not hold, that is for some $\delta < +\infty$ and an increasing to $+\infty$ sequence (r_n) the inequality $\gamma(r_n) \ge \delta \ln r_n$ is true. We choose $f(z) = h(z) = e^z$ and $g(z) = E_\varrho(z)$ with $\varrho < \delta$. Then $\ln\ln M_\Phi(r) = r$ and in view of (10)

$$\begin{aligned} \varliminf_{r\to+\infty} \frac{\ln\ln M_F(r)}{\ln\ln M_\Phi(\exp\{\gamma(r)\})} &\le \varliminf_{n\to+\infty} \frac{\ln\ln M_F(r_n)}{\ln\ln M_\Phi(\exp\{\gamma(r_n)\})} \\ &= \varliminf_{n\to+\infty} \frac{r_n^\varrho}{\exp\{\gamma(r)\}} \le \varliminf_{n\to+\infty} \frac{r_n^\varrho}{r_n^\delta} = 0, \end{aligned} \tag{17}$$

that is there exist entire functions f, g and h for which (13) is false. Proposition 1 is proved. □

Proposition 2.2. *Let γ be a positive continuous on $[0, +\infty)$ function increasing to $+\infty$. Let f, g and h be arbitrary entire functions with $0 < l[f] \le \varrho[f] < +\infty$, $\varrho[g] < +\infty$ and $\lambda[h] > 0$. In order to*

$$\lim_{r\to+\infty} \frac{\ln\ln M_F(r)}{\ln\ln M_\Phi(\exp\{\gamma(r)\})} = 0, \quad F(z) = f(g(z)),\ \Phi(z) = f(h(z)), \tag{18}$$

it is necessary and sufficient that

$$\lim_{r\to+\infty} \frac{\gamma(r)}{\ln r} = +\infty. \tag{19}$$

Proof. We assume that the function γ satisfies (19), that is $\ln r = o(\gamma(r))$ as $r \to +\infty$. If the orders $\varrho[f]$ and $\varrho[g]$ are finite then for $\varrho > \max\{\varrho[f], \varrho[g]\}$ and all $r \geq r_0(\varrho)$ in view of (5) we have $\ln\ln M_F(r) \leq \varrho r^{\varrho}$ for $r \geq r_0(\varrho)$. On the other hand, using (6) for $0 < \lambda < \min\{\lambda[f], \lambda[f]\}$ and $r \geq r_0(\lambda)$ we obtain

$$\ln\ln M_{\Phi}(e^{\gamma(r)}) \geq \ln\ln M_f\left(\frac{1}{8}M_g\left(\frac{r}{2}\right) - |g(0)|\right) \geq (1+o(1))\lambda 2^{-\lambda} e^{\lambda\gamma(r)}, \quad r \to +\infty.$$

Therefore, $\dfrac{\ln\ln M_F(r)}{\ln\ln M_{\Phi}(\exp\{\gamma(r)\})} \leq \dfrac{(1+o(1))\lambda}{\varrho 2^{\lambda}} e^{\varrho\ln r - \lambda\gamma(r)} \to 0, \quad r \to +\infty$. The sufficiency of (19) is proved.

Now we assume that (19) does not hold, that is for some $\delta < +\infty$ and an increasing to $+\infty$ sequence (r_n) the inequality $\gamma(r_n) \leq \delta \ln r_n$ is true. We choose $f(z) = h(z) = e^z$ and $g(z) = E_{\varrho}(z)$ with $\varrho > \delta$. Then in view of (10)

$$\begin{aligned}\varlimsup_{r\to+\infty} \frac{\ln\ln M_F(r)}{\ln\ln M_{\Phi}(\exp\{\gamma(r)\})} &\geq \varlimsup_{n\to+\infty} \frac{\ln\ln M_F(r_n)}{\ln\ln M_{\Phi}(\exp\{\gamma(r_n)\})} \\ &= \varlimsup_{n\to+\infty} \frac{r_n^{\varrho}}{\exp\{\gamma(r_n)\}} \geq \varliminf_{n\to+\infty} \frac{r_n^{\varrho}}{r_n^{\delta}} = +\infty,\end{aligned} \tag{20}$$

that is (18) does not hold. Proposition 2 is proved. □

Finally, we will prove a result on the growth of a composition of entire functions in the terms of generalized orders. By L we denote a class of all positive continuous on $(-\infty, +\infty)$ functions α such that $\alpha(x) = \alpha(x_0)$ for $-\infty < x \leq x_0$ and $\alpha(x) \uparrow +\infty$ as $x_0 \leq x \to +\infty$.

For $\alpha \in L$ and $\beta \in L$ the generalized order $\varrho_{\alpha\beta}[f]$ and a lower generalized order $\lambda_{\alpha\beta}[f]$ of an entire function f are defined [3] by the formulas

$$\varrho_{\alpha,\beta}[f] = \varlimsup_{r\to+\infty} \frac{\alpha(\ln M_f(r))}{\beta(\ln r)}, \quad l_{\alpha,\beta}[f] = \varliminf_{r\to+\infty} \frac{\alpha(\ln M_f(r))}{\beta(\ln r)}.$$

Proposition 2.3. *Let $\alpha \in L$, $\beta \in L$, $\beta(x + O(1)) = (1 + o(1)\beta(x)$ as $x \to +\infty$ and f, g be entire functions with $0 < \lambda_{\alpha,\beta}[f] \leq \varrho_{\alpha,\beta}[f] < +\infty$ and $0 < l_{\alpha,\beta}[g] \leq \varrho_{\alpha,\beta}[g] < +\infty$. In order to*

$$\lim_{r\to+\infty} \frac{\alpha(\ln M_F(r))}{\alpha(\ln M_f(r))} = +\infty, \quad F(z) = f(g(z)), \tag{21}$$

it is necessary and sufficient that

$$\lim_{x\to+\infty} \frac{\beta(x)}{\alpha(x)} = +\infty. \tag{22}$$

Proof. If (22) holds then from (6) and the definition of the lower generalized order it follows that for each $0 < \lambda < \lambda_1 < \min\{\lambda_{\alpha,\beta}[f], \lambda_{\alpha,\beta}[g]\}$ and $r \geq r_0(\lambda)$

$$\begin{aligned}\alpha(\ln M_F(r)) &\geq \alpha\left(\ln M_f\left(\frac{1}{8}M_g\left(\frac{r}{2}\right) - |g(0)|\right)\right) \geq \lambda_1\beta\left(\ln M_g\left(\frac{r}{2}\right) + O(1)\right) \\ &= \lambda_1(1+o(1))\beta\left(\ln M_g\left(\frac{r}{2}\right)\right) = \lambda_1(1+o(1))\beta\left(\alpha^{-1}\left(\alpha\left(\ln M_g\left(\frac{r}{2}\right)\right)\right)\right) \\ &\geq \lambda_1(1+o(1))\beta(\alpha^{-1}(\lambda_1(1+o(1))\beta(\ln r))) \geq \lambda\beta(\alpha^{-1}(\lambda\beta(\ln r))).\end{aligned}$$

On the other hand, for $\varrho > \varrho_{\alpha,\beta}[f]$ and all $r \geq r_0(\varrho)$ we have $\alpha(\ln M_f(r)) \leq \varrho\beta(\ln r)$. Therefore,

$$\varliminf_{r\to+\infty} \frac{\alpha(\ln M_F(r))}{\alpha(\ln M_f(r))} \geq \varliminf_{r\to+\infty} \frac{\lambda\beta(\alpha^{-1}(\lambda\beta(\ln r)))}{\varrho\beta(\ln r)} = \frac{l^2}{\varrho} \varliminf_{x\to+\infty} \frac{\beta(x)}{\alpha(x)} = +\infty,$$

that is (21) is true. If (22) does not hold, that is $\varliminf\limits_{x\to+\infty} \beta(x)/\alpha(x) < +\infty$ then in view of (5) for $\lambda < \lambda_{\alpha,\beta}[f]$, $\varrho > \max\{\varrho_{\alpha,\beta}[f],\, \varrho_{\alpha,\beta}[f]\}$ and all r enough large

$$\varliminf_{r\to+\infty} \frac{\alpha(\ln M_F(r))}{\alpha(\ln M_f(r))} \leq \varliminf_{r\to+\infty} \frac{\varrho\beta(\ln M_g(r))}{\lambda\beta(\ln r)} = \varliminf_{r\to+\infty} \frac{\varrho\beta(\alpha^{-1}(\alpha(\ln M_g(r))))}{\lambda\beta(\ln r)}$$
$$\leq \varliminf_{r\to+\infty} \frac{\varrho\beta(\alpha^{-1}(\varrho\beta(\ln r)))}{l\beta(\ln r)} = \frac{\varrho^2}{l} \varliminf_{x\to+\infty} \frac{\beta(x)}{\alpha(x)} < +\infty,$$

that is (21) is false. Proposition 3 is proved. □

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Нехай γ — додатна, неперервна на $[0, +\infty)$ і зростаюча до $+\infty$ функція, а f і g — довільні цілі функції додатного нижнього порядку і скінченного порядку.

Для того, щоб

$$\varlimsup_{r\to+\infty} \frac{\ln\ln M_{f(g)}(r)}{\ln\ln M_f(\exp\{\gamma(r)\})} = +\infty, \quad M_f(r) = \max\{|f(z)| : |z| = r\},$$

необхідно і досить, щоб $(\ln \gamma(r))/(\ln r) \to 0$ при $r \to +\infty$. Це твердження є відповіддю на питання, поставлене А. Сінхом і М. Балорія у 1991 р.

Також для того, щоб

$$\lim_{r\to+\infty} \frac{\ln\ln M_F(r)}{\ln\ln M_f(\exp\{\gamma(r)\})} = 0, \quad F(z) = f(g(z)),$$

необхідно і достатньо, щоб $(\ln \gamma(r))/(\ln r) \to \infty$ при $r \to +\infty$.

Ключові слова і фрази: ціла функція, композиція функцій, узагальнений порядок.

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SKEW SEMI-INVARIANT SUBMANIFOLDS OF GENERALIZED QUASI-SASAKIAN MANIFOLDS

In the present paper, we study a new class of submanifolds of a generalized Quasi-Sasakian manifold, called skew semi-invariant submanifold. We obtain integrability conditions of the distributions on a skew semi-invariant submanifold and also find the condition for a skew semi-invariant submanifold of a generalized Quasi-Sasakian manifold to be mixed totally geodesic. Also it is shown that a skew semi-invariant submanifold of a generalized Quasi-Sasakian manifold will be anti-invariant if and only if $A_{\xi} = 0$; and the submanifold will be skew semi-invariant submanifold if $\nabla w = 0$. The equivalence relations for the skew semi-invariant submanifold of a generalized Quasi-Sasakian manifold are given. Furthermore, we have proved that a skew semi-invariant $\xi^{\perp}$-submanifold of a normal almost contact metric manifold and a generalized Quasi-Sasakian manifold with non-trivial invariant distribution is CR-manifold. An example of dimension 5 is given to show that a skew semi-invariant $\xi^{\perp}$ submanifold is a CR-structure on the manifold.

Key words and phrases: skew semi-invariant submanifold, generalized quasi-Sasakian manifold, integrability conditions of the distributions, CR-structure.

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INTRODUCTION

The theory of submanifolds in spaces endowed with additional structure is very interesting topic in the field of differential geometry [5]. The theory of CR-submanifolds has been introduced by A. Bejancu for almost contact geometry [1] and also for almost complex geometry [2], after that several papers have been appeared in this field. M. Barros et al. [5], B. Y. Chen [6, 7], A. Bejancu and N. Papaghuic [3], V. Mangione [10] and N. Papaghiuc [11] have studied semi-invariant submanifolds in Sasakian manifolds and the study was also extended to other ambient spaces. Moreover, some related topics were studied by V. V. Goldberg and R. Rosca [16–20]. In 2012, C. Calin et al. [8] have studied the semi-invariant $\xi^{\perp}$-submanifold of a generalized quasi-Sasakian manifold. Later on, A. Bejancu defined and studied a semi-invariant submanifold of a locally product manifold [4]. Recently, L. Ximin and F. M. Shao [12] have discussed a new class of submanifolds of a locally product manifold, that is, known as skew semi-invariant submanifolds. The purpose of the present work is to investigate some interseting results on the skew semi-invariant submanifolds of a generalized Quasi-Sasakian manifold.

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1 Preliminaries

Let $\bar{M}$ be a real $(2n+1)$-dimensional smooth manifold equipped with an almost contact metric structure (φ, ξ, η, g), where φ is $(1,1)$-tensor field, ξ is a vector field, η is a 1-form and g is a Riemannian metric such that [1]

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \; \varphi(\xi) = 0, \; \eta \circ \varphi = 0, \tag{1}$$

$$g(\varphi X, Y) = -g(X, \varphi Y), \quad g(X, \xi) = \eta(X), \quad g(\xi, \xi) = 1 \tag{2}$$

for all X, Y on space M. The almost contact manifold $\bar{M}(\varphi, \xi, \eta)$ is said to be normal, if

$$N_\varphi(X, Y) + 2d\eta(X, Y)\xi = 0$$

for all $X, Y \in (TM)$, where

$$N_\varphi(X, Y) = [\varphi X, \varphi Y] + \varphi^2[X, Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y]$$

is the Nijenhuis tensor field corresponding to the tensor field φ. The fundamental 2-form Φ on $\bar{M}$ is defined by

$$\Phi(X, Y) = g(X, \varphi Y). \tag{3}$$

S. S. Eum [9], considered the hypersurface of an almost contact metric manifold $\bar{M}$ whose structure tensor field satisfy the following relation:

$$(\bar{\nabla}_X \varphi)Y = g(\bar{\nabla}_{\varphi X}\xi, Y)\xi - \eta(Y)\bar{\nabla}_{\varphi X}\xi, \tag{4}$$

where $\bar{\nabla}$ is the Levi-Civita connection of metric tensor g. For the sake of simplicity we say that a manifold $\bar{M}$ with an almost contact metric structure satisfying (4) is a generalized Quasi-Sasakian manifold. We define a $(1,1)$-tensor field F by

$$FX = -\bar{\nabla}_X \xi.$$

Now, we assume that $\bar{M}$ is a generalized Quasi-Sasakian manifold and M is an m-dimensional submanifold isometrically immersed in $\bar{M}$. Denote by g the induced metric on M and by ∇ its Levi-Civita connection. For $p \in M$ and the tangent vector $X_p \in T_pM$, we can write

$$FX_p = PX_p + QX_p, \tag{5}$$

where $PX_p \in T_pM$ and $QX_p \in T^\perp{}_pM$. For any two vectors $X_p, Y_p \in T_pM$, we have $g(FX_p, Y_p) = g(PX_p, Y_p)$, which implies that $g(PX_p, Y_p) = g(X_p, PY_p)$. Therefore P and P^2 are all symmetric operators on the tangent space T_pM. If $\alpha(p)$ is the eigen value of P^2 at $p \in M$, since P^2 is a composition of an isometry and a projection, then $\alpha(p) \in [0, 1]$.

For each $p \in M$, we set

$$D_p^\alpha = \mathcal{K}er(P^2 - \alpha(p)I),$$

where I is an identity transformation on T_pM and $\alpha(p)$ an eigenvalue of P^2 at $p \in M$. Obviously, we have $D_p^0 = \mathcal{K}erP$ and $D_p^1 = \mathcal{K}erQ$, where D_p^1 is the maximal F-invariant subspace of T_pM and D_p^0 is the maximal F-anti invariant subspace of T_pM. If $\alpha_1(p), \ldots, \alpha_k(p)$ are all eigenvalues of P^2 at p, then T_pM can be decomposed as the direct sum of the mutually orthogonal eigenspaces, *i.e.*,

$$T_pM = D_p^{\alpha_1} \oplus \cdots \oplus D_p^{\alpha_k}.$$

Definition 1 ([12]). *A submanifold M of a generalized quasi-Sasakian manifold $\bar{M}$ is said to be skew semi-invariant submanifold, if there exists an integer k and the functions $\alpha_1, \cdots, \alpha_k$ defined on M with values in $(0,1)$ such that*

(1) *each $\alpha_1(x), \cdots, \alpha_k(x)$ are distinct eigenvalues of P^2 at each $p \in M$ with*

$$T_pM = D_p^1 \oplus D_p^0 \oplus D_p^{\alpha_1} \oplus \cdots \oplus D_p^{\alpha_k};$$

(2) *the dimensions of $D_p^0, D_p^1, D_p^{\alpha_1}, \cdots, D_p^{\alpha_k}$ are independent of $p \in M$.*

Remark 1. *(i) From the second case of Definition 1, we can also define P-invariant mutually orthogonal distributions*

$$D^\alpha = \bigcup_{p \in M} D_p^\alpha, \quad \alpha \in \{0, \alpha_1, \cdots, \alpha_k, 1\}$$

on M and $TM = D^1 \oplus D^0 \oplus D^{\alpha_1} \oplus \cdots \oplus D^{\alpha_k}$ are differentiable (see [7]).

(ii) If $k = 0$ in Definition 1, then it follows that P is a structure of type $f(3,-1)$ on M [13] and $\dim(D_p^1) = rank(P_p)$, $\dim(D_p^0)$ are independent of $p \in M$ [14].

(iii) If $k = 0$, (1) implies (2), then M is called a semi-invariant $\xi^\perp$-submanifold.

(iv) If $k = 0$ and $D_p^1 = \{0\}$ (resp., $D_p^0 = \{0\}$), then M is called an anti-invariant (resp., invariant) $\xi^\perp$-submanifold.

(v) If $D_p^1 = \{0\} = D_p^0$, $k = 1$ and $\alpha_1^2(x)$ is constant, then M may be said to be a θ-slant submanifold with slant angle $\cos\theta = \alpha_1$.

Example 1. *We consider the Euclidean space R^9 and denote its points by $y = (y^i)$. Let $(e_j), j = 1, \ldots, 9$ be the natural basis defined by $e^j = \partial/\partial y^j$. We define a vector field ξ and a 1-form η by $\xi = \partial/\partial y^9$ and $\eta = dy^9$ respectively and φ is $(1,1)$ tensor field defined by*

$$\begin{aligned}
&\varphi e_1 = e_2, \ \ \varphi e_2 = e_1, \ \ \varphi e_3 = e_8, \ \ \varphi e_8 = e_3,\\
&\varphi e_4 = cost(y)e_5 - sint(y)e_6, \varphi e_5 = cost(y)e_4 + sint(y)e_7,\\
&\varphi e_6 = -sint(y)e_4 + cost(y)e_7, \varphi e_7 = sint(y)e_5 + cost(y)e_6, \varphi e_9 = 0,
\end{aligned}$$

where $t : R^9 \to (0, \pi/2)$ is a smooth function. Then it is easy to verify that R^9 is an almost contact metric manifold with almost contact structure (φ, ξ, η, g) with associated metric g given by $g(e_i, e_j) = \delta_{ij}$. The submanifold

$$M = \left\{ (y^1, \ldots, y^9) \in R^9 | y^6, y^7, y^8, y^9 = 0 \right\}$$

of R^9 is a skew semi-invariant submanifold with

$$D^1 = Span\{e_1, e_2\}, \ \ D^0 = Span\{e_3\}, \ \ D^\alpha = Span\{e_4, e_5\},$$

where for $x \in M$ one has $\alpha(y) = cost(y)$.

Denote the induced connection in M by ∇, then the Gauss and Weingarten eqautions are given respectively by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y), \quad \bar{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \quad X,Y \in TM; N \in T^{\perp}M, \tag{6}$$

where $\bar{\nabla}$, ∇ and $\nabla^{\perp}$ are the Riemannian, induced Riemannian and induced normal connections in M, $\bar{M}$ and the normal bundle $T^{\perp}M$ of $\bar{M}$ *respectively* and h is the second fundamental form related to A by the equation

$$g(h(X,Y),N) = g(A_N X, Y). \tag{7}$$

Let M be a submanifold of a generalized Quasi-Sasakian manifold $\bar{M}$ for $X,Y \in TM$, $N \in T^{\perp}M$. By using

$$\varphi X = tX + wX, \quad tX \in TM,\ wX \in T^{\perp}M, \tag{8}$$

$$\varphi N = BN + CN, \quad BN \in TM,\ CN \in T^{\perp}M, \tag{9}$$

we have

$$(\bar{\nabla}_X \varphi)Y = ((\nabla_X t)Y - A_{wY}X - Bh(X,Y)) + ((\nabla_X w)Y + h(X,tY) - Ch(X,Y)), \tag{10}$$

$$(\bar{\nabla}_X \varphi)N = ((\nabla_X B)N - A_{BN}X + BA_N X)) + ((\nabla_X B)N + h(X,BN) + wA_N X)),$$

where

$$(\nabla_X t)Y = \nabla_X tY - t\nabla_X Y, \quad (\nabla_X w)Y = \nabla^{\perp}_X wY - w\nabla_X Y,$$

$$(\nabla_X B)N = \nabla_X BN - t\nabla^{\perp}_X N, \quad (\nabla_X C)N = \nabla^{\perp}_X CN - C\nabla^{\perp}_X N.$$

Comparing the tangential and normal components in (10), we obtain

$$t\nabla_X Y = \nabla_X tY - Bh(X,Y) - A_{wY}X, \tag{11}$$

$$\nabla_X tY = h(X,tY) + \nabla^{\perp}_X wY - Ch(X,Y). \tag{12}$$

From (11) and (12) we have

$$t[X,Y] = \nabla_X tY - \nabla_Y tY + A_{wX}Y - A_{wY}X, \tag{13}$$

$$w[X,Y] = h(X,tY) - h(tX,Y) + \nabla^{\perp}_Y wX - \nabla^{\perp}_X wY. \tag{14}$$

Thus from (11), (12), (13) and (14), we have the following lemmas.

Lemma 1 ([8]). *Let M be a skew semi-invariant submanifold of a generalized quasi-Sasakian manifold $\bar{M}$. Then, we have*

$$(\nabla_X t)Y = A_{wY}X + Bh(X,Y), \quad (\nabla_X w)Y = Ch(X,Y) - h(X,tY) + g(FX,\varphi Y)\xi \tag{15}$$

for all $X,Y \in TM$.

Proof. The Lemma follows from (10)–(11) by taking into the consideration decomposition of $TM^{\perp}$. □

Lemma 2 ([8]). *Let M be a skew semi-invariant submanifold of a generalized quasi-Sasakian manifold $\bar{M}$. Then we have for any $N \in TM^{\perp}$*

1) $BN \in D^0$,

2) $CN \in D^1$.

Lemma 3 ([8]). *Let M be a skew semi-invariant submanifold of a generalized quasi-Sasakian manifold $\bar{M}$, then the distribution D^0 is integrable if and only if*

$$A_{wZ}W = A_{wW}Z, \quad \text{for all } X, Y \in D^0. \tag{16}$$

The following results give necessary and sufficient conditions for the integrability of the distributions D^0 and D^1.

Theorem 1. *Let M be a skew semi-invariant submanifold of a generalized quasi-Sasakian manifold $\bar{M}$. Then the distribution D^0 is integrable.*

Proof. Let $Z, W \in D^0$, then from (8), (15) and (16), we deduce that

$$t[Z, W] = A_{wZ}W - A_{wW}Z = 0.$$

Hence the conclusion. □

Theorem 2. *Let M be a skew semi-invariant submanifold of a generalized quasi-Sasakian manifold $\bar{M}$, then the distribution D^1 is integrable if and only if*

$$h(tX, Y) - h(X, tY) = (L_\xi g)(X, \varphi Y)\xi \quad \text{for all } X, Y \in D^1. \tag{17}$$

Proof. The statement yields from (15)

$$w([tX, Y]) = h(X, tY) - h(tX, Y) + (L_\xi g)(X, \varphi Y)\xi \quad \text{for all } X, Y \in D^1. \tag{18}$$

□

Proposition 1. *If M is a skew semi-invariant submanifold of a generalized quasi-Sasakian manifold $\bar{M}$, then the following relations take place:*

$$-A_\xi X = t^2 X, \tag{19}$$

$$\nabla^\perp_X \xi = w^2 X, \tag{20}$$

$$\eta(h(X, Y)) = g(X, t^2 Y), \tag{21}$$

$$\eta(H) = -\frac{1}{n}\, trace(t^2)$$

for any $X, Y \in TM$, where H is the mean curvature vector.

Proof. Form equation (18), it follows that $\bar{\nabla}_X \xi = \varphi^2 X = -X + \eta(X)\xi$.

Using (19), (8) and $\eta(X) = 0$ in (6), we get

$$-A_\xi X + \nabla^\perp_X \xi = t^2 X + w^2 X. \tag{22}$$

Equating tangential and normal part of (22), we get (19) and (20), *respectively*. From (2), (7) and (15) it follows that

$$\eta(h(X, Y)) = g(h(X, Y), \xi) = g(A_\xi X, Y) = -g(t^2 X, Y),$$

which gives (21). If $\{e_1, e_2, \ldots, e_n\}$, $n = dimM$ is a local orthonormal frame field, then from (17) we get

$$\eta(H) = \frac{1}{n}\eta\left(\sum_{i=1}^{n} h(e_i, e_i)\right) = -\frac{1}{n}\left(\sum_{i=1}^{n} g(P^2 e_i, e_i)\right).$$

Therefore (16) holds. □

From (16), we have the following.

Corollary 1. *Let M be skew semi-invariant submanifold of a generalized quasi-Sasakian manifold $\bar{M}$. If $trace(t^2) \neq 0$, then M can not be minimal.*

In view of (16), we have the following theorem.

Theorem 3 ([8]). *Let M be a skew semi-invariant submanifold of a generalized quasi-Sasakian manifold $\bar{M}$. Then M is anti-invariant if and only if $A_\xi = 0$.*

Let D^1 and D^2 be two distributions defined on a manifold $\bar{M}$. We say that D^1 is parallel to D^2 for all $X \in D^2$ and $Y \in D^1$, we have

$$\nabla_X Y \in D^1.$$

If D^1 is parallel then for $X \in TM$ and $Y \in D^1$, we have $\nabla_X Y \in D^1$. It is easy to verify that D^1 is parallel if and only if the orthogonal complementary distribution of D^1 is also parallel.

Let M be a skew semi-invariant submanifold of $\bar{M}$. A distribution D is said to be totally geodesic, if $h(X,Y) = 0$ for all $X, Y \in D$. The distributions D^1 and D^2 are said to be D^1-D^2-mixed totally geodesic, if $h(X,Y) = 0$ for all $X \in D^1$ and $Y \in D^2$.

Proposition 2. *Let M be a skew semi-invariant submanifold of generalized quasi-Sasakian manifold $\bar{M}$. For any distribution D^α, if*

$$A_N tX = tA_N X \text{ for all } X \in D^\alpha,\ N \in T^\perp M,$$

then M is D^α-D^β-mixed totally geodesic, where $\alpha \neq \beta$.

Proof. From the assumption, we have

$$t^2 A_N X - \alpha A_N X = 0.$$

This implies that $A_N X \in D^\alpha$. So for all $Y \in D^\beta$, $N \in T^\perp M$, $\alpha \neq \beta$, we have

$$g(A_N X, Y) = g(h(X,Y), N) = 0.$$

Therefore $h(X,Y) = 0$. Hence M is D^α-D^β-mixed totally geodesic. □

Now from (5), (8) and (9), we find

$$CwX_p = -wtX_p, \quad wBN = N - C^2 N \tag{23}$$

for all $X_p \in T_pM,\ N \in T_p^\perp M$. Furthermore for $X_p \in D_p^{\alpha_i}$, $\alpha \in \{\alpha_1, \ldots, \alpha_k\}$, we have

$$C^2 wX_p = \alpha_i wX_p.$$

Also, if $X_p \in D_p^0$, then it is clear that $t^2 wX_p = 0$. Thus if X_p is an eigenvector of t^2 corresponding to the eigenvalue $\alpha(p) \neq 1$, then wX_p is an eigenvector of C^2 with the same eigenvalue $\alpha(p)$. Thus, (23) implies that $\alpha(p)$ is an eigenvalue of B^2 if and only if $\gamma(p) = 1 - \alpha(p)$ is an eigenvalue of wt. Since wB and f^2 are symmetric operators on the normal bundle $T^\perp M$, then their eigenspaces are orthogonal. The dimension of the eigenspace of wB corresponding to the eigenvalue $1 - \alpha(p)$ is equal to the dimension of D_p^α if $\alpha(p) \neq 1$. Consequently, we have the following lemma.

Lemma 4. *A submanifold M is a skew semi-invariant submanifold of generalized quasi-Sasakian manifold $\bar{M}$ if and only if the eigenvalues of wB are constant and the eigenspaces of wB have constant dimension.*

2 SKEW SEMI-INVARIANT SUBMANIFOLD

Theorem 4. *Let M be a submanifold of a generalized quasi-Sasakian manifold $\bar{M}$. If $\nabla t = 0$, then M is a skew semi-invariant submanifold. Furthermore each of the t-invariant distributions D^0, D^1 and D^{α_i}, $1 \leq i \leq k$ are parallel.*

Proof. For a fix $p \in M$ any $Y_p \in D^{\alpha_{ip}}$ and $X \in TM$. Let Y be the parallel translation of Y_p along with the integral curve of X. Since $(\nabla_X t)Y = 0$ and from (11) we have

$$\nabla_X(t^2 - \alpha(p)Y) = t^2\nabla_X Y - \alpha(p)\nabla_X Y = 0.$$

Since $(t^2 Y - \alpha(p)Y) = 0$ at p, it is identical to 0 on $\bar{M}$. Thus the eigenvalues of t^2 are constant. Moreover, parallel translation of T_pM along any curve is an isometry which preserves each D^α. Thus the dimension of D^α is constant and $\bar{M}$ is a skew semi-invariant submanifold.

Now if Y is any vector field in D^α, then we have $t^2 Y = \alpha Y$ (α constant), *i.e*, $t^2\nabla_X Y = \alpha\nabla_X Y$ which implies that D^α is parallel. □

Now, we see the vanishing of ∇w. For $X, Y \in TM$ if $(\nabla_X w)Y = 0$, then (21) yields

$$Ch(X,Y) = h(X,tY) - g(FX,\varphi Y)\xi. \tag{24}$$

In particular if $Y \in D^\alpha$, then (24) implies

$$C^2 h(X,Y) = \alpha h(X,Y) - \alpha g(FX,\varphi Y)\xi.$$

Consequently we have the following proposition.

Proposition 3. *Let M be a skew semi-invariant submanifold of a generalized quasi-Sasakian manifold $\bar{M}$, if $\nabla w = 0$, then M is D^α-D^β -mixed totally geodesic for all $\alpha \neq \beta$. Moreover, if $X \in D^\alpha$ then either $h(X,X) = 0$ or $h(X,X)$ is an eigenvector of C^2 with eigenvalue α.*

Lemma 5. *Let M be a submanifold of a generalized quasi-Sasakian manifold $\bar{M}$, then $\nabla w = 0$ if and only if $\nabla_X BN = B\nabla^\perp N$ for all $X \in TM$ and $N \in T^\perp M$.*

Theorem 5. *Let M be a submanifold of a generalized quasi-Sasakian manifold $\bar{M}$. If $\nabla w = 0$, then M is a skew semi-invariant submanifold.*

Proof. If $TM = D^1$, then we are done. Otherwise, we may find a point $p \in M$ and a vector $X_p \in D^\alpha_p$, $\alpha \neq 1$. Set $N_p = wX_p$, then N_p is an eigenvector of wB with eigenvalue $\mu(p) = 1 - \alpha(p)$. Now, let $Y \in TM$ and N be the translation of N_p in the normal bundle $T^\perp M$ along with integral curve of Y, we have

$$\nabla^\perp_Y(wBN - \mu(p)N) = \nabla^\perp_Y wBN - \mu(p)\nabla^\perp_Y N = w(\nabla_Y BN) - \mu(p)\nabla^\perp_Y N.$$

In view of Lemma 5,

$$\nabla^\perp_Y(wBN - \mu(p)N) = \nabla^\perp_Y wBN - \mu(p)\nabla^\perp_Y N = 0.$$

Since $wBN - \mu(p)N = 0$ at p and $tBN - \mu(p)N = 0$ on M. It follows from Lemma 4 that M is a skew semi-invariant submanifold. □

Theorem 6. *Let M be a skew semi-invariant submanifold of a generalized quasi-Sasakian manifold $\bar{M}$, then the following relations are equivalent.*

1. $(\nabla_X w)Y - (\nabla_Y w)X = 0$ for all $X, Y \in D^\alpha$.
2. $h(tX, Y) = h(X, tY)$ for all $X, Y \in D^\alpha$.
3. $w[X, Y] = \nabla^\perp_X wY - \nabla^\perp_Y wX$ for all $X, Y \in D^\alpha$.
4. $A_N tY - tA_N Y$ *is perpendicular to* D^α *for all* $Y \in D^\alpha$ *and* $N \in T^\perp N$.

Proof. The proof is trivial, hence we omit it. □

3 CR-STRUCTURE

Let $\bar{M}$ be a differentiable manifold and $T^c\bar{M}$ be the complexified tangent bundle to $\bar{M}$. A CR-structure on M is complex subbundle H of $T^c\bar{M}$ such that $H \cap \bar{H} = \{0\}$ and H is involutive [15]. A manifold endowed with a CR-structure is called a CR-manifold. It is known that a differentiable manifold $\bar{M}$ admits a CR-structure [1] if and only if there is a differentiable distribution $\bar{D}$ and a $(1,1)$ tensor field P on M such that for all $X, Y \in \bar{D}$

$$P^2X = -X, [P, P](X, Y) \equiv [PX, PY] - [X, Y] - P[PX, Y] - P[X, PY] = 0, [PX, PY] - [X, Y] \in \bar{D}.$$

Definition 2. *A differentiable manifold $\bar{M}$ is said to admit a CR-structure if there is a differentiable distribution $\bar{D}$ and a $(1,1)$ tensor field P on $\bar{M}$ such that for all $X, Y \in \bar{D}$*

$$P^2X = X, [P, P](X, Y) \equiv [PX, PY] + [X, Y] - P[PX, Y] - P[X, PY] = 0, [PX, PY] = [X, Y] \in D.$$

A manifold equipped with a CR-structure is called a CR-manifold.

Lemma 6. *An almost contact metric structure (φ, ξ, η, g) is normal if the Nijenhuis tensor $[\varphi, \varphi]$ of φ satisfies [3]*

$$[\varphi, \varphi] + 2d\eta \otimes \xi = 0. \tag{25}$$

Now, we prove the following theorem.

Theorem 7. *If M is a skew semi-invariant $\xi^\perp$-submanifold of a normal almost contact metric manifold $\bar{M}$ with non-trivial invariant distribution, then $\bar{M}$ possesses a CR-structure.*

Proof. Since M is normal for $X, Y \in \bar{D}^\perp$, we get $P^2X = -X$ and in view of (25), we have

$$0 = [P, P](X, Y) - Q([X, PY] + [PX, Y])$$

from which it follows that $Q([PX, Y] + [X, PY]) = 0$, that is, $[PX, Y] + [X, PY] \in \bar{D}^1$. Thus

$$[PX, PY] + [X, Y] = P([PX, Y] + [X, PY]) \in \bar{D}^1$$

and hence $(\bar{D}^1, P)$ is a CR-structure on M. □

Theorem 8. *A skew semi-invariant $\xi^\perp$-submanifold of a generalized quasi-Sasakian manifold with non-trivial invariant distribution is a CR-manifold.*

Proof. Since every generalized quasi-Sasakian manifold is normal (see [8], Theorem 7), the proof is obvious. □

From Theorem 7, it is obvious that normality of $\bar{M}$ is a sufficient condition for a skew semi-invariant submanifold with nontrivial invariant distribution to carry a CR-structure. However, this is not neccessary, and now we give an example of skew semi-invariant submanifold.

Example 2. *We consider the Euclidean space R^5 and denote its points by $x = (x^i)$. Let $(e_j), j = 1, \dots, 5$ be the natural basis defined by $e^j = \partial/\partial x^j$. We define a vector field ξ and a 1-form η by $\xi = \partial/\partial x^5$ and $\eta = dx^5$ respectively. For each $x \in R^5$, and g the canonical metric defined by $g(e_i, e_j) = \delta_{i,j}, i, j = 1, \dots, 5$, the set E_j defined by*

$$E_1 = e_1, \ \ E_2 = cos(x^1)e_2 + sin(x^1)e_3, \ \ E_3 = -sin(x^1)e_2 + cos(x^1)e_3, \ \ E_4 = e_4, \ \ E_5 = e_5$$

forms an orthonormal basis. As the point x varies in R^5, the above set of equations defines 5 vector fields also denoted by (E_j) and φ is $(1, 1)$ tensor field defined by

$$\varphi(E_1) = E_2, \ \ \varphi(E_2) = E_1, \ \ \varphi(E_3) = E_4, \ \ \varphi(E_4) = E_3 \ \ \varphi(E_5) = 0.$$

Then (φ, ξ, η, g) defines an almost contact metric structure on R^5. Since

$$[\varphi, \varphi](E_1, E_4) + 2d\eta(E_1, E_4)\xi = E_1 \neq 0,$$

then, the almost contact structure is not normal. The submanifold

$$M = \left\{x \in R^5 : x^4, x^5 = 0\right\}$$

is a skew semi-invariant submanifold of R^5 with $D^1 = Span\{E_1, E_2\}$ and $D^0 = Span\{E_3\}$ such that (D^1, φ) is a CR-structure on M. Moreover, D^1 is not integrable because $D^0 = E_3$.

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Сіддікі М.Д., Хасіб А., Ахмад М. *Антинапівінваріантні підмноговиди узагальнених квазі-Сасакянових многовидів* // Карпатські матем. публ. — 2017. — Т.9, №2. — С. 188–197.

У цій роботі ми вивчаємо новий клас підмноговидів узагальнених квазі-Сасакянових многвидів, що називаються антинапівінваріантними підмноговидами. Нами отримано умови інтегровності розподілів на антинапівінваріантному підмноговиді, а також знайдемо умову того, що антинапівінваріантний підмноговид узагальненого квазі-Сасакянового многовиду є змішаним цілком геодезичним. Також показано, що антинапівінваріантний підмноговид узагальненого квазі-Сасакянового многовиду буде антиінваріантним тоді і тільки тоді, якщо $A_(\xi) = 0$; і підмноговид буде антинапівінваріантним підмноговидом, якщо $\nabla w = 0$. Отримано співвідношення еквівалентності для антинапівінваріантного підмноговиду узагальненого квазі-Сасакянового многовиду. Більше того, ми довели, що антинапівінваріантний $\xi^\perp$-підмноговид нормального майже контактного метричного многовиду та узагальненого квазі-Сасакянового многовиду з нетривіальним інваріантним розподілом є CR-многовидом. Наведено приклад розмірності 5 для того, щоб показати, що антинапівінваріантний $\xi^\perp$-підмноговид є CR-структурою на многовиді.

Ключові слова і фрази: антинапівінваріантний многовид, узагальнений квазі-Сасакяновий многовид, умови інтегровності розподілів, CR-структура.

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VASYLYSHYN T.V.

METRIC ON THE SPECTRUM OF THE ALGEBRA OF ENTIRE SYMMETRIC FUNCTIONS OF BOUNDED TYPE ON THE COMPLEX L_∞

It is known that every complex-valued homomorphism of the Fréchet algebra $H_{bs}(L_\infty)$ of all entire symmetric functions of bounded type on the complex Banach space L_∞ is a point-evaluation functional δ_x (defined by $\delta_x(f) = f(x)$ for $f \in H_{bs}(L_\infty)$) at some point $x \in L_\infty$. Therefore, the spectrum (the set of all continuous complex-valued homomorphisms) M_{bs} of the algebra $H_{bs}(L_\infty)$ is one-to-one with the quotient set $L_\infty/_\sim$, where an equivalence relation "$\sim$" on L_∞ is defined by $x \sim y \Leftrightarrow \delta_x = \delta_y$. Consequently, M_{bs} can be endowed with the quotient topology. On the other hand, M_{bs} has a natural representation as a set of sequences which endowed with the coordinate-wise addition and the quotient topology forms an Abelian topological group. We show that the topology on M_{bs} is metrizable and it is induced by the metric $d(\xi,\eta) = \sup_{n\in\mathbb{N}} \sqrt[n]{|\xi_n - \eta_n|}$, where $\xi = \{\xi_n\}_{n=1}^\infty, \eta = \{\eta_n\}_{n=1}^\infty \in M_{bs}$.

Key words and phrases: symmetric function, spectrum of the algebra.

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INTRODUCTION

Symmetric functions on Banach spaces were studied by a number of authors [1, 3–8, 10, 12, 13] (see also a survey [2]). In particular, symmetric polynomials and symmetric analytic functions on L_∞ (see definition below) were studied in [6, 12, 13].

Let L_∞ be the complex Banach space of all Lebesgue measurable essentially bounded complex-valued functions x on $[0,1]$ with norm $\|x\|_\infty = \operatorname{ess\,sup}_{t\in[0,1]}|x(t)|$.

Let Ξ be the set of all measurable bijections of $[0,1]$ that preserve the measure. A function $f : L_\infty \to \mathbb{C}$ is called symmetric if $f(x \circ \sigma) = f(x)$ for every $x \in L_\infty$ and for every $\sigma \in \Xi$.

Let $H_{bs}(L_\infty)$ be the Fréchet algebra of all entire symmetric functions $f : L_\infty \to \mathbb{C}$ which are bounded on bounded sets endowed with the topology of uniform convergence on bounded sets. By [6, Theorem 4.3], polynomials $R_n : L_\infty \to \mathbb{C}$, $R_n(x) = \int_{[0,1]}(x(t))^n\,dt$ for $n \in \mathbb{N}$ form an algebraic basis in the algebra of all symmetric continuous polynomials on L_∞. Since every $f \in H_{bs}(L_\infty)$ can be described by its Taylor series of continuous symmetric homogeneous polynomials, it follows that f can be uniquely represented as

$$f(x) = f(0) + \sum_{n=1}^{\infty} \sum_{k_1+2k_2+\ldots+nk_n=n} \alpha_{k_1,\ldots,k_n} R_1^{k_1}(x)\cdots R_n^{k_n}(x).$$

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Consequently, for every non-trivial continuous homomorphism $\varphi : H_{bs}(L_\infty) \to \mathbb{C}$, taking into account $\varphi(1) = 1$, we have

$$\varphi(f) = f(0) + \sum_{n=1}^{\infty} \sum_{k_1+2k_2+\ldots+nk_n=n} \alpha_{k_1,\ldots,k_n} \varphi(R_1)^{k_1} \cdots \varphi(R_n)^{k_n}.$$

Therefore, φ is completely determined by the sequence of its values on $R_n : (\varphi(R_1), \varphi(R_2), \ldots)$. By the continuity of φ, the sequence $\{\sqrt[n]{|\varphi(R_n)|}\}_{n=1}^{\infty}$ is bounded. On the other hand, we have the following

Theorem 1 ([6, Section 3]). *For every sequence $\xi = \{\xi_n\}_{n=1}^{\infty} \subset \mathbb{C}$ such that $\sup_{n\in\mathbb{N}} \sqrt[n]{|\xi_n|} < +\infty$, there exists $x_\xi \in L_\infty$ such that $R_n(x_\xi) = \xi_n$ for every $n \in \mathbb{N}$ and $\|x_\xi\|_\infty \le \frac{2}{M} \sup_{n\in\mathbb{N}} \sqrt[n]{|\xi_n|}$, where*

$$M = \prod_{n=1}^{\infty} \cos\left(\frac{\pi}{2}\frac{1}{n+1}\right). \tag{1}$$

Hence, for every sequence $\xi = \{\xi_n\}_{n=1}^{\infty}$ such that $\sup_{n\in\mathbb{N}} \sqrt[n]{|\xi_n|} < +\infty$, there exists the point-evaluation functional $\varphi = \delta_{x_\xi}$ such that $\varphi(R_n) = \xi_n$ for every $n \in \mathbb{N}$. Since every such a functional is a continuous homomorphism, it follows that the spectrum (the set of all continuous complex-valued homomorphisms) of the algebra $H_{bs}(L_\infty)$, which we denote by M_{bs}, can be identified with the set of all sequences $\xi = \{\xi_n\}_{n=1}^{\infty} \subset \mathbb{C}$ such that $\{\sqrt[n]{|\xi_n|}\}_{n=1}^{\infty}$ is bounded.

Let $\nu : L_\infty \to M_{bs}$ be defined by

$$\nu(x) = (R_1(x), R_2(x), \ldots).$$

Let τ_∞ be the topology on L_∞, generated by $\|\cdot\|_\infty$. Let us define an equivalence relation on L_∞ by $x \sim y \Leftrightarrow \nu(x) = \nu(y)$. Let τ be the quotient topology on M_{bs} :

$$\tau = \{\nu(V) : V \in \tau_\infty\}.$$

Note that ν is a continuous open mapping.

The operation of coordinate-wise addition $+ : M_{bs}^2 \to M_{bs}$ is defined by

$$a + b = (a_1 + b_1, a_2 + b_2, \ldots)$$

for $a = (a_1, a_2, \ldots), b = (b_1, b_2, \ldots) \in M_{bs}$. In [13] it is shown that $(M_{bs}, +, \tau)$ is an Abelian topological group. In this work we show that (M_{bs}, τ) is a metrizable topological space. Also we explicitly construct the metric which induces τ.

1 THE MAIN RESULT

Let us denote $B(x, r)$ the open ball of radius r and center x in L_∞.

Proposition 1. *The identity element $0 = (0, 0, \ldots)$ of the topological group $(M_{bs}, +, \tau)$ has a countable local basis of neighborhoods.*

Proof. For $n \in \mathbb{N}$ let $U_n = \nu(B(0, \frac{1}{n}))$. Since ν is an open mapping, it follows that $U_n \in \tau$. Note that $0 \in U_n$. Thus, U_n is an open neighborhood of 0 for every $n \in \mathbb{N}$. Let us show that a family $\{U_n : n \in \mathbb{N}\}$ form a local basis of neighborhoods of 0. Let $W \subset M_{bs}$ be an arbitrary open neighborhood of 0. Then $\nu^{-1}(W)$ is open in L_∞ and $\nu^{-1}(W)$ contains 0. Therefore, there exists $r > 0$ such that $B(0, r) \subset \nu^{-1}(W)$. Let $n \in \mathbb{N}$ be such that $\frac{1}{n} < r$. Then $B(0, \frac{1}{n}) \subset B(0, r) \subset \nu^{-1}(W)$. Therefore, $\nu(B(0, \frac{1}{n})) \subset W$, i. e. $U_n \subset W$. $\square$

We will use Birkhoff-Kakutani theorem.

Theorem 2 ([9, p.34]). *Let G be a Hausdorff topological group whose open sets at the identity element have a countable basis. Then G is metrizable and, moreover, there exists a metric which is right-invariant.*

Corollary 1. *There exists an invariant metric d on M_{bs} which induces topology τ.*

Proof. By [13, Corollary 1], $(M_{bs}, +, \tau)$ is an Abelian topological group. By [13, Theorem 2], τ is Hausdorff. By Proposition 1, the identity element of M_{bs} has a countable local basis. Therefore by Theorem 2 there exists a right-invariant metric d on M_{bs} which induces topology τ. Since $(M_{bs}, +, \tau)$ is Abelian, the metric d is also left-invariant. □

For $a = (a_1, a_2, \ldots)$ and $b = (b_1, b_2, \ldots) \in M_{bs}$ let

$$d_I(a, b) = \sup_{n\in\mathbb{N}} \sqrt[n]{|a_n - b_n|}.$$

Note that analogical metric is defined on spaces of entire functions of one complex variable (where a role of sequences a and b play sequences of coefficients of the Taylor series of functions) and it is called Iyer metric (see e. g. [11]). Also note that a metric space (M_{bs}, d_I) is isometric to the space of entire functions $f : \mathbb{C} \to \mathbb{C}$ of the exponential type such that $f(0) = 0$ with Iyer metric.

Let $V(a, r)$ be the open ball in M_{bs} of radius r and center $a \in M_{bs}$ with respect to the metric d_I.

Lemma 1. *Let $r > 0$ and $0 < \rho < \frac{Mr}{2}$, where M is defined by* (1). *Then $V(0, \rho) \subset \nu(B(0, r))$.*

Proof. Let $a = (a_1, a_2, \ldots) \in V(0, \rho)$. Let us show that $a \in \nu(B(0, r))$. By Theorem 1, there exists $x_a \in L_\infty$ such that $\nu(x_a) = a$ and $\|x_a\|_\infty < \frac{2}{M} \sup_{n\in\mathbb{N}} \sqrt[n]{|a_n|}$. Since $a \in V(0, \rho)$, it follows that $d_I(0, a) < \rho$, i. e. $\sup_{n\in\mathbb{N}} \sqrt[n]{|a_n|} < \rho$. Thus, $\|x_a\|_\infty < \frac{2}{M}\rho$. Since $\rho < \frac{Mr}{2}$, it follows that $\|x_a\|_\infty < r$, i. e. $x_a \in B(0, r)$. Therefore $\nu(x_a) \in \nu(B(0, r))$, i. e. $a \in \nu(B(0, r))$. □

Theorem 3. *The metric d_I induces the topology τ.*

Proof. Since both metrics d_I and d (given by Corollary 1) are invariant with respect to translations (in the sense that $d(a + c, b + c) = d(a, b)$ for every $a, b, c \in M_{bs}$), it suffices to prove that every open neighborhood of 0 with respect to τ contains some open ball with center 0 with respect to d_I and vice versa.

Let $W \in \tau$ such that $0 \in W$. Then $\nu^{-1}(W)$ is the open neighborhood of 0 in L_∞. Therefore, there exists $r > 0$ such that $B(0, r) \subset \nu^{-1}(W)$. By Lemma 1, for $0 < \rho < \frac{2r}{M}$ we have $V(0, \rho) \subset \nu(B(0, r))$. Since $\nu(B(0, r)) \subset W$, it follows that $V(0, \rho) \subset W$.

Let us show that for every open ball $V(0, r)$ there exists $W \in \tau$ such that $0 \in W$ and $W \subset V(0, r)$. Set $W = \nu(B(0, r))$. Let us show that $W \subset V(0, r)$. It suffices to prove that $\nu(x) \in V(0, r)$ for every $x \in B(0, r)$. For $x \in B(0, r)$ we have $\|x\|_\infty < r$ and, consequently,

$$|R_n(x)| \leq \|x\|_\infty^n < r^n.$$

Therefore

$$d_I(0, \nu(x)) = \sup_{n\in\mathbb{N}} \sqrt[n]{|R_n(x)|} < r.$$

Thus, $\nu(x) \in V(0, r)$. □

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Василишин Т.В. *Метрика на спектрі алгебри цілих симетричних функцій обмеженого типу на комплексному просторі* L_∞ // Карпатські матем. публ. — 2017. — Т.9, №2. — С. 198–201.

Відомо, що кожен комплекснозначний гомоморфізм алгебри Фреше $H_{bs}(L_\infty)$ усіх цілих симетричних функцій обмеженого типу на комплексному банаховому просторі L_∞ є функціоналом обчислення значення в точці δ_x (визначеного як $\delta_x(f) = f(x)$ для $f \in H_{bs}(L_\infty)$) у деякій точці $x \in L_\infty$. Тому спектр (множина усіх неперервних комплекснозначних гомоморфізмів) M_{bs} алгебри $H_{bs}(L_\infty)$ є у взаємно однозначній відповідності із фактор-множиною $L_\infty/_\sim$, де відношення еквівалентності "$\sim$" на просторі L_∞ визначене наступним чином: $x \sim y \Leftrightarrow \delta_x = \delta_y$. Як наслідок, на M_{bs} можна задати фактор-топологію. З іншого боку, для M_{bs} існує природне подання у вигляді множини послідовностей, яка разом із заданими на ній операцією покоординатного додавання і фактор-топологією утворює абелеву топологічну групу. У статті доведено, що топологія на M_{bs} є метризовною і породжується метрикою $d(\xi,\eta) = \sup_{n\in\mathbb{N}} \sqrt[n]{|\xi_n - \eta_n|}$, де $\xi = \{\xi_n\}_{n=1}^{\infty}, \eta = \{\eta_n\}_{n=1}^{\infty} \in M_{bs}$.

Ключові слова і фрази: симетрична функція, спектр алгебри.

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FAITHFUL GROUP ACTIONS AND SCHREIER GRAPHS

Each action of a finitely generated group on a set uniquely defines a labelled directed graph called the Schreier graph of the action. Schreier graphs are used mainly as a tool to establish geometrical and dynamical properties of corresponding group actions. In particilar, they are widely used in order to check amenability of different classed of groups. In the present paper Schreier graphs are utilized to construct new examples of faithful actions of free products of groups. Using Schreier graphs of group actions a sufficient condition for a group action to be faithful is presented. This result is applied to finite automaton actions on spaces of words i.e. actions defined by finite automata over finite alphabets. It is shown how to construct new faithful automaton presentations of groups upon given such a presentation. As an example a new countable series of faithful finite automaton presentations of free products of finite groups is constructed. The obtained results can be regarded as another way to construct new faithful actions of groups as soon as at least one such an action is provided.

Key words and phrases: group action, faithful action, Schreier graph, free product, automaton permutation.

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INTRODUCTION

O. Shreier introduced in [8] graphs to represent cosets by finite index subgroups in free groups. Such kind of graphs were later named after Schreier and they naturally arise in geometric group theory. In particular they were used to produce exotic example of group actions and to establish rare properties of graphs and groups [1,2,4].

In this paper we use Schreier graphs of group actions to give a sufficient condition for a group action to be faithful. This approach gives an alternative way to construct faithful group actions of free products compared to a well-known method based on ping-pong lemma (see e.g. [5,7]). As an application we construct a new countable series of faithful finite automaton presentations of free products of finite groups.

This result generalizes our previous construction from [3] and its proof explores the main theorem from [6].

The paper is organized as follows. In the first section we recall the definition of Schreier graphs and introduce Schreier embedding of group actions. Then we prove the main theorem, which allows to built new faithful group actions upon given one.

In the second section we recall basic definitions about automaton permutations and define a countable series of finite automaton actions of free products of finite groups. In the last section we prove the result about Schreier embeddability of constructed actions and apply the main theorem to obtain faithfulness of them.

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1 SCHREIER GRAPHS

Let G be a group with a finite generation set S, acting on a set M.

Definition 1. *The Schreier graph $\Gamma(G, S, M)$ of the action of the group G on the set M is a directed graph with the set of vertices M and the set of edges $M \times S$, where for every $m \in M$ and $s \in S$ there is an edge (m, s) from m to $s(m)$ and this edge has a label s.*

Definition 2. *The Schreier graphs Γ_1 and Γ_2 of the group G with the generation set S acting on the sets M_1 and M_2 respectively are called isomorphic if they are isomorphic as oriented edge-labeled graphs, i.e., there is a one-to-one function $f : M_1 \to M_2$ such that for arbitrary vertices v_1, v_2 of the graph Γ_1 there is an arrow from v_1 to v_2 with the label $s \in S$ if and only if the graph Γ_2 contains an arrow from $\mathbf{f}(v_1)$ to $\mathbf{f}(v_2)$ with the label s.*

It immediately follows from the definition that for isomorphic Schreier graphs Γ_1 and Γ_2 there exists a path between two vertices v_1 and v_2 in Γ_1 with the labels of the edges $g_1, \ldots, g_n$ if and only if Γ_2 contains a path between $\mathbf{f}(v_1)$ and $\mathbf{f}(v_2)$ with the labels of the edges $g_1, \ldots, g_n$.

It is possible to give a natural sufficient condition for the faithfulness of the group actioni in terms of the Schreier graphs. Namely, let the group G act on the sets M_1 and M_2, that is, the actions ψ_1 and ψ_2 of the group G are given on these sets respectively.

Definition 3. *The action ψ_1 is Schreier-embedded into the action ψ_2 if a group G has a generation set A such that each connected component of the Schreier graph of action ψ_1 of this group with respect to the generation set A is isomorphic to some component of the Schreier graph of the action ψ_2 of this group with respect to the same generation set A.*

We call actions ψ_1, ψ_2 Schreier-equivalent if ψ_1 is Schreier-embedded into ψ_2 and vice versa.

We have the following useful observation.

Theorem 1. *Let ψ_1 and ψ_2 be actions of a group G such that ψ_1 is Shreier-embedded into ψ_2. If the action ψ_1 is faithful then the action ψ_2 is faithful as well.*

Proof. Let ψ_1 and ψ_2 be actions on sets M_1 and M_2 respectively. Denote by A a generating set of G used to construct Schreier-embedding of the action ψ_1 in the action ψ_2.

Assume that the action ψ_2 is not faithful. Then there exists a non-identity element g of the group G that fixes an arbitrary element of the set M_2. Then $g = g_1 \ldots g_n$ for some $g_1, \ldots, g_n \in A$. So paths with the edges labeled $g_1, \ldots, g_n$ in the Schreier graph of the action ψ_2 are cycles.

By the assumption of the theorem, the action ψ_1 of the group G is Schreier-embedded into the action ψ_2 of the same group. Therefore all paths with the labels $g_1, \ldots, g_n$ in the Schreier graph of the action ψ_1 of the group G are cycles as well. This implies that the non-identity element $g = g_1 \ldots g_n$ of the group G fixes arbitrary element from the set M_1. This contradicts with the faithfulness of the action ψ_1. □

2 AUTOMATON ACTIONS OF FREE PRODUCTS

Let an *alphabet* be a finite set X, $|X| > 1$. A sequence $x_1 \ldots x_n$ of elements from the alphabet is the *word* of length n. An *empty word* Λ has a length equal to zero. Denote by X^n the set of words of length n over the alphabet X. Consider X^* and X^ω — words of finite and infinite length respectively. For arbitrary words $u, v \in X^* \bigcup X^\omega$ one can define the product of two words v and u by concatenation $uv \in X^* \bigcup X^\omega$.

Definition 4. *An initial automaton is a tuple $A = \langle X, Q, \varphi, \psi, q_0 \rangle$,*

- *where X is a finite input and output alphabet, $|X| = n$,*
- *Q is a nonempty set, the set of inner states of the automaton A,*
- *φ and ψ are transition and output functions, acting from $Q \times X$ into Q and X, respectively,*
- *$q_0 \in Q$ is an initial state.*

In particular, a *finite* automaton is an automaton with a finite set of states: $|Q| < \infty$. An automaton is called *permutational* if for each state of the automaton the restriction of the output function in this state determines some permutation on the alphabet.

The transformation of the set X^* of all finite words over the alphabet X defined by the finite initial permutational automata form a group $FGA(X)$ with respect to a superposition. Elements of this group are called finite automaton permutations over the alphabet X.

Consider the group G generated by a finite set S of finite automaton permutations over the alphabet X. It acts on the set X^* of all finite words over the alphabet X.

The sets X^n, $n \geq 1$, are invariant under the action of G. Thus the sequence Γ_n of finite Schreier graphs of the action of G over X^n, $n \geq 1$, naturally arise. We call these graphs the Schreier graphs of the action of G on the levels.

Let $G_1, \ldots, G_s$ be s ($s \geq 2$) finite groups of orders $p_1, \ldots, p_s$ respectively. Without loss of generality suppose $1 < p_1 \leq \ldots \leq p_s$ and denote $n = p_s$. Let us remind the construction from [6] of an embedding of the free product $G_1 * \ldots * G_s$ into the group $FGA(X)$ of finite automaton permutations over the alphabet $X = \{x_1, \ldots, x_n\}$.

For every i, $1 \leq i \leq s$ fix a regular action of the group G_i on the first p_i symbols of X and fix remain letters. Denote the letter x_1 by 0 and the word $0 \ldots 0 \in X^s$ of all words of length s by $\overline{0}$. Consider subsets M_i, $1 \leq i \leq s$, in X^s:

$$M_i = \{\underbrace{x \ldots x}_{i} 0 \ldots 0 : x \in X, x \neq 0\}.$$

For each $i, 1 \leq i \leq s$, we define the set $D_i = \bigcup\limits_{j \neq i} M_j^{G_i}$, where

$$M_j^{G_i} = \{\omega^g : \omega \in M_j, g \in G_i\}, 1 \leq i, j \leq s, \quad D_i = \{\overline{x_1}^{hg} : h \in G_j, g \in G_i, h \neq e_j, j \neq i\}.$$

Let φ_{1i} be functions, which assign to each element $g \in G_i$ a map $\varphi_{1i}(g)$ on the set X^∞ of all infinite words over the alphabet X. An infinite word $\omega \in X^\infty$ can be divided into syllables of arbitrary length $k \in \mathbb{N}$:

$$\omega = \omega[k, 1]\omega[k, 2] \ldots$$

For all $g \in G_i$, $u \in X^\infty$ we construct $v_1 = (\varphi_{1i}(g))(u)$ as follows. Let $v_1[s, 1] = u[s, 1]$, and for all $j \geq 2$

$$v_1[s, j] = \begin{cases} (u[s, j])^g, & \text{if } u[s, j-1] \in D_i \\ u[s, j] & \text{otherwise.} \end{cases} \tag{1}$$

Hence we have constructed everywhere defined transformations of the set of infinite words over X.

In [6] it is proved that for each element $g \in G_i$ the transformation $\varphi_{1i}(g)$ is a finite automaton permutation over the alphabet X and the function φ_{1i} is a monomorphism from the group G_i into the group of finite automaton permutations $FGA(X)$. Denoted by $\mathbb{G}_1(G_1, \dots, G_s)$ a subgroup of $FGA(X)$ generated by the images of these monomorphisms.

Theorem 2 ([6]). *The group $\mathbb{G}_1(G_1, \dots, G_s)$ splits into the free product as follows:*

$$\mathbb{G}_1(G_1, \dots, G_s) \simeq G_1 * \dots * G_s.$$

We proceed to the construction of a class of actions in each of which the action (1) is Schreier-embedded. We define a series of sets of functions $\varphi_{ti}, t \geq 1$ on $G_i, 1 \leq i \leq s$. The function $\varphi_{ti}, t \geq 1$ assigns to each element $g \in G_i$ a finite automaton transformations $\varphi_{ti}(g)$ of the set X^∞ of all infinite words over X. For arbitrary $g \in G_i$, $u \in X^\infty$, we define $v_t = (\varphi_{ti}(g))(u)$ as follows. For arbitrary $1 \leq j < t+1$ we put $v_t[s,j] = u[s,j]$, and for all $j \geq t+1$

$$v_t[s,j] = \begin{cases} (u[s,j])^g, & \text{if } u[s,j-t] \in D_i \\ u[s,j] & \text{otherwise.} \end{cases} \tag{2}$$

It is directly verified that for each $t \geq 1$ and $i, 1 \leq i \leq s$ the function φ_{ti} is a homomorphism on the group G_i. Hence, for each $t \geq 1$ we obtain an action of the free product $G_1 * \dots * G_s$ by finite state automaton permutations.

Let $\mathbb{G}_t(G_1, \dots, G_s)$ be a subgroup of the group of finite automaton permutations over X generated by $\varphi_{t1}(G_1), \dots, \varphi_{ts}(G_s), t \geq 1$. Note that for $t = 1$ we obtain the action given by A. Oliynyk in [6], and for $t = 2$ — by the author in [3].

3 Properties of actions

The Schreier-embeddability of the constructed actions of a free product of finite number of finite groups is proved by the next theorem.

Lemma 1. *The action given by equation* (1) *is Schreier-embedded into each action of the series given by equation* (2).

Proof. To prove the statement of the lemma we will express the first action in terms of the second one. We fix $t > 1$.

We will use representation of an infinite word ω as a product of subwords of length s:

$$\omega = \omega[s,1]\omega[s,2]\omega[s,3]\dots.$$

Then we construct t infinite words as follows

$$\begin{aligned} \omega_1 &= \omega[s,1]\omega[s,t+1]\omega[s,2t+1]\dots \\ &\dots \\ \omega_i &= \omega[s,i]\omega[s,t+i]\omega[s,2t+i]\dots \\ &\dots \\ \omega_t &= \omega[s,t]\omega[s,2t]\omega[s,3t]\dots \end{aligned}$$

In other terms, the representation of infinite word ω_i as a product of subwords of length s consists of those subwords of length s of ω which numbers have the form $tk + i$, $k \geq 0$.

Let $\varphi_{ti}(g)$, $g \in G_i$, defined by (2), acts on infinite words ω. Denote by v_t the word obtained as the result of this action. Denote by $v_{1,i}$, $0 \leq i < t$ infinite words that are the results of the action $\varphi_{1i}(g)$ on ω_i respectively. Comparing the words v_t and $v_{1,i}$, we have the following equations for arbitrary $k \geq 1$:

$$\begin{cases} v_t[s, tk-1] = v_{1,1}[s,k], \\ v_t[s, tk-2] = v_{1,2}[s,k], \\ \dots \\ v_t[s, tk-t+1] = v_{1,t-1}[s,k]. \end{cases} \tag{3}$$

Thus, in order to express a second action in terms of the first one, it is sufficient to decompose the word ω, on which the second mapping acts, on the words ω_i, apply the first transformation to them, and create a new word using equalities (3).

Consider arbitrary connected component of the Schreier graph of the first action. Then fix arbitrary vertex of this component. This vertex correspond to some infinite word ω_1. Let us prove that in the Schreier graph of the second action there is an isomorphic connected component to the selected one. For that purpose we consider the infinite word $\omega_{00,1}$, that for all $k \geq 1$ satisfies the equalities

$$\begin{cases} \omega_{00,1}[s, tk-1] = 00, \\ \dots \\ \omega_{00,1}[s, tk-k+1] = 00, \\ \omega_{00,1}[s, tk] = \omega_1[s,k]. \end{cases} \tag{4}$$

Since $00 \notin D_i, 1 \leq i \leq s$, the $\omega_{00,1}$ blocks whose numbers are not divisible by t will not be changed under the action of the second map. And the blocks which numbers are divisible by t will be changed in the same way as ω_1 under the action of the first map. Thus, the connected component of the Schreier graph of the second action which contains the vertex corresponding to the word $\omega_{00,1}$ will be isomorphic to the connected component of the Schreier graph of the first-action which contains the vertex corresponding to the word ω_1.

Consequently, for arbitrary connected component of the Schreier graph of the first action one can find an isomorphic connected component of the Schreier graph of the second action. That is, the first action is Schreier-embedded into the second one. □

Note that we leave as open a question about Schreier equivalence of these actions.

The main result now can be formulated as follows.

Theorem 3. *Each action of series* (2) *is faithful.*

Proof. Theorem 2 implies that the action (1) is a faithful action of the free product

$$G_1 * \ldots * G_s.$$

Theorem 1 implies that the action (1) is Schreier-embedded into each action of the series given by equation (2). Hence by theorem 1 action (2) for all $t \geq 2$ is faithful as well. □

Then we obtain as a corollary the following result.

Corollary 1. *For each $t \geq 1$ the group $\mathbb{G}_t(G_1, \ldots, G_s)$ splits into the free product*

$$G_1 * \ldots * G_s.$$

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Кожна дія скінченно породженої групи на множині однозначно визначає помічений орієнтований граф, який називається графом Шраєра цієї дії. Графи Шраєра переважно використовуються як інструмент для встановлення геометричних і динамічних властивостей відповідних групових дій. Зокрема, їх вони широко вживані для перевірки аменабельності різноманітних класів груп. В даній статті графи Шраєра вжито для побудови нових прикладів точних дій вільних добутків груп. Використовуючи графи Шраєра дії груп наведено достатню умову того, коли дія групи є точною. Цей результат застосовано до скінченно автоматних дій на просторах слів, тобто до дій, визначених скінченними автоматами над скінченними алфавітами. Показно, як будувати нові точні автоматні зображення груп за умови існування такого зображення. Як приклад, побудовано нову зліченну серію точних скінченно автоматних зображень вільних добутків скінченних груп. Отримані результати можна розглядати, як ще один спосіб побудувати нові точні дії груп за умови існування хоча б однієї такої дії.

Ключові слова і фрази: дія групи, точна дія, граф Шраєра, вільний добуток, автоматна підстановка.