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CONTENTS

Vasylyshyn T.V. *Symmetric functions on spaces* $\ell_p(\mathbb{R}^n)$ *and* $\ell_p(\mathbb{C}^n)$ 5

Kravtsiv V.V. *Analogues of the Newton formulas for the block-symmetric polynomials on* $\ell_p(\mathbb{C}^s)$ 17

Azizbayov E.I., Mehraliyev Y.T. *Nonlocal inverse boundary-value problem for a 2D parabolic equation with integral overdetermination condition* 23

Frontczak R., Goy T. *Mersenne-Horadam identities using generating functions* 34

Khats' R.V. *Sufficient conditions for the improved regular growth of entire functions in terms of their averaging* 46

Erdoğan E. *Zero product preserving bilinear operators acting in sequence spaces* 55

Mahi F., Belkhelfa M. *Legendrian normally flat submanifols of* $\mathcal{S}$*-space forms* 69

Malytska G.P., Burtnyak I.V. *Construction of the fundamental solution of a class of degenerate parabolic equations of high order* 79

Fotiy O., Ostrovskii M., Popov M. *Isomorphic spectrum and isomorphic length of a Banach space* 88

Moeini B., Işık H., Aydi H. *Related fixed point results via* C_**-class functions on* C^**-algebra-valued* G_b*-metric spaces* 94

Mytrofanov M.A., Ravsky A.V. *A note on approximation of continuous functions on normed spaces* 107

Lal S., Sharma V.K. *On the estimation of functions belonging to Lipschitz class by block pulse functions and hybrid Legendre polynomials* 111

Bedratyuk L., Luno N. *Some properties of generalized hypergeometric Appell polynomials* . 129

Kal'chuk I.V., Kharkevych Yu.I., Pozharska K.V. *Asymptotics of approximation of functions by conjugate Poisson integrals* 138

Yanchenko S.Ya. *Approximation of the Nikol'skii–Besov functional classes by entire functions of a special form* 148

Antonova T.M. *On convergence criteria for branched continued fraction* 157

Chaikovs'kyi A., Lagoda O. *Bounded solutions of a difference equation with finite number of jumps of operator coefficient* 165

Baranetskij Ya.O., Kalenyuk P.I., Kopach M.I., Solomko A.V. *The nonlocal boundary value problem with perturbations of mixed boundary conditions for an elliptic equation with constant coefficients. II* 173

Chapovskyi Y.Y., Mashchenko L.Z., Petravchuk A.P. *Nilpotent Lie algebras of derivations with the center of small corank* 189

Raievska I.Yu., Raievska M.Yu. *Local nearrings on finite non-abelian 2-generated p-groups* 199

Soltanov K., Sert U. *Certain results for a class of nonlinear functional spaces* 208

Popov M.M. *On two long standing open problems on L_p-spaces* 229

Hentosh O.Ye., Balinsky A.A., Prykarpatski A.K. *The generalized centrally extended Lie algebraic structures and related integrable heavenly type equations* 242

Карпатські математичні публікації

Науковий журнал
Т.12, №1
2020

Зміст

Василишин Т.В. *Симетричні функції на просторах* $\ell_p(\mathbb{R}^n)$ *і* $\ell_p(\mathbb{C}^n)$ 5

Кравців В.В. *Аналог формули Ньютона для блочно-симетричних поліномів на* $\ell_p(\mathbb{C}^n)$ 17

Азізбайов Е.І., Мехралієв Ю.Т. *Нелокальна обернена крайова задача для двовимірного параболічного рівняння з інтегральною переозначеною умовою* 23

Фрончак Р., Гой Т. *Тотожності Мерсенна-Горадама з використанням генератрис* . 34

Хаць Р.В. *Достатні умови покращеного регулярного зростання цілих функцій в термінах їх усереднення* 46

Ердоган Е. *Білінійні оператори, що зберігають нульовий добуток, на просторах послідовностей* 55

Махі Ф., Белхельфа М. *Лежандрові нормально плоскі підмноговиди* $\mathcal{S}$*-просторових форм* 69

Малицька Г.П., Буртняк І.В. *Побудова фундаментального розв'язку одного класу вироджених параболічних рівнянь високого порядку* 79

Фотій О., Островський М., Попов М. *Ізоморфний спектр та ізоморфна довжина банахового простору* 88

Моейні Б., Ішік Г., Айді Г. *Стосовно результатів про нерухому точку для функцій класу* C_* *на* C^**-алгеброзначних* G_b*-метричних просторах* 94

Митрофанов М.А., Равський О.В. *Про апроксисмацію неперервних функції в нормованих просторах* 107

Лал Ш., Шарма В.К. *Про оцінку функцій із класу Ліпшіца блочно-імпульсними функціями та гібридними поліномами Лежандра* 111

Бедратюк Л., Луньо Н. *Деякі властивості узагальнених гіпергеометричних многочленів Аппеля* 129

Кальчук І.В., Харкевич Ю.І., Пожарська К.В. *Асимптотика наближення функцій спряженими інтегралами Пуассона* 138

Янченко С.Я. *Наближення класів функцій Нікольського–Бєсова цілими функціями спеціального вигляду* 148

Антонова Т.М. *Про критерій збіжності для гіллястого ланцюгового дробу* 157

Чайковський А.В., Лагода О.А. *Обмежені розв'язки різницевого рівняння зі скінченною кількістю стрибків операторного коефіцієнта* 165

Баранецький Я.О., Каленюк П.І., Копач М.І., Соломко А.В. *Нелокальна крайова задача зі збуреннями мішаних крайових умов для еліптичного рівняння із сталими коефіцієнтами. II* 173

Чаповський Є.Ю., Мащенко Л.З., Петравчук А.П. *Нільпотентні алгебри Лі диференціювань з центром малого корангу* 189

Раєвська І.Ю., Раєвська М.Ю. *Локальні майже-кільця на скінченних неабелевих неметациклічних 2-породжених p-групах* 199

Солтанов К., Серт У. *Деякі результати для одного класу нелінійних функціональних просторів* 208

Попов М.М. *Про дві давні нерозв'язані проблеми про простори* L_p 229

Гентош О.Є., Балінський О.А., Прикарпатський А.К. *Узагальнені центрально розширені Лі-алгебраїчні структури та асоційовані інтегровні рівняння небесного типу* 242

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VASYLYSHYN T.V.

SYMMETRIC FUNCTIONS ON SPACES $\ell_p(\mathbb{R}^n)$ AND $\ell_p(\mathbb{C}^n)$

This work is devoted to the study of algebras of continuous symmetric polynomials, that is, invariant with respect to permutations of coordinates of its argument, and of $*$-polynomials on Banach spaces $\ell_p(\mathbb{R}^n)$ and $\ell_p(\mathbb{C}^n)$ of p-power summable sequences of n-dimensional vectors of real and complex numbers respectively, where $1 \le p < +\infty$.

We construct the subset of the algebra of all continuous symmetric polynomials on the space $\ell_p(\mathbb{R}^n)$ such that every continuous symmetric polynomial on the space $\ell_p(\mathbb{R}^n)$ can be uniquely represented as a linear combination of products of elements of this set. In other words, we construct an algebraic basis of the algebra of all continuous symmetric polynomials on the space $\ell_p(\mathbb{R}^n)$. Using this result, we construct an algebraic basis of the algebra of all continuous symmetric $*$-polynomials on the space $\ell_p(\mathbb{C}^n)$.

Results of the paper can be used for investigations of algebras, generated by continuous symmetric polynomials on the space $\ell_p(\mathbb{R}^n)$, and algebras, generated by continuous symmetric $*$-polynomials on the space $\ell_p(\mathbb{C}^n)$.

Key words and phrases: polynomial, $*$-polynomial, symmetric polynomial, symmetric $*$-polynomial, algebraic basis.

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INTRODUCTION

Symmetric (invariant with respect to some group of operators) functions on Banach spaces were studied by a number of authors [1–8, 10, 11, 15–21, 23]. In particular, in [15] it was constructed an algebraic basis (see definition below) of the algebra of all continuous symmetric, i.e., invariant with respect to permutations of coordinates of its argument, polynomials on the real Banach space ℓ_p of p-power summable sequences of real numbers, where $1 \le p < +\infty$. In [8] it was generalized this result to continuous symmetric polynomials on real separable rearrangement-invariant sequence Banach spaces. In [11] it was constructed an algebraic basis of the algebra of all continuous symmetric polynomials on the complex Banach space $\ell_p(\mathbb{C}^n)$ of p-power summable sequences of n-dimensional vectors of complex numbers, where $1 \le p < +\infty$. Note that the knowledge of an algebraic basis of an algebra of polynomials is important for the description of spectra (sets of maximal ideals) of completions of this algebra (see, e.g., [1, 4, 5, 7, 9, 17]).

$*$-Polynomials (see definition below) are generalizations of polynomials, acting between complex vector spaces, which were firstly studied in [14]. In [13] it was shown that, in some sense, $*$-polynomials have better approximation properties than polynomials. Symmetric $*$-polynomials on the finite-dimensional complex vector space were studied in [16]. In particular,

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in [16] it was constructed the set of generating elements of the algebra of all symmetric $*$-polynomials on the complex space of finite sequences of n-dimensional complex vectors.

In this work we construct an algebraic basis of the algebra of all continuous symmetric polynomials on the real Banach space $\ell_p(\mathbb{R}^n)$ of p-power summable sequences of n-dimensional vectors of real numbers, where $1 \le p < +\infty$. Also we construct an algebraic basis of the algebra of all continuous symmetric $*$-polynomials on the complex Banach space $\ell_p(\mathbb{C}^n)$.

1 Preliminaries

Let $\mathbb{N}$ be the set of all positive integers and $\mathbb{Z}_+$ be the set of all nonnegative integers. Let $\mathcal{S}$ be the set of all bijections $\sigma : \mathbb{N} \to \mathbb{N}$. For $n \in \mathbb{N}$, let $\mathcal{S}_n$ be the set of all bijections $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$.

1.1 Polynomials

Let X and Y be vector spaces over the fields $\mathbb{K}_1$ and $\mathbb{K}_2$ resp., such that $\mathbb{K}_1 \subset \mathbb{K}_2$ and $\mathbb{K}_1, \mathbb{K}_2 \in \{\mathbb{R}, \mathbb{C}\}$. A mapping $A : X^m \to Y$, where $m \in \mathbb{N}$, is called an *m-linear mapping*, if A is linear with respect to every of its m arguments. An m-linear mapping, which is invariant with respect to permutations of its arguments is called *symmetric*. For an m-linear mapping $A : X^m \to Y$, let $A^{(s)} : X^m \to Y$ be defined by

$$A^{(s)}(x_1, \ldots, x_m) = \frac{1}{m!} \sum_{\tau \in \mathcal{S}_m} A\big(x_{\tau(1)}, \ldots, x_{\tau(m)}\big).$$

The mapping $A^{(s)}$ is symmetric and m-linear. It is called the *symmetrization* of the mapping A. A mapping $P : X \to Y$ is called an *m-homogeneous polynomial* if there exists an m-linear mapping $A_P : X^m \to Y$ such that P is the restriction to the diagonal of A_P, i.e., $P(x) = A_P(\underbrace{x, \ldots, x}_{m})$ for every $x \in X$. Note that P is also the restriction to the diagonal of the mapping $A_P^{(s)}$, which is the symmetrization of the mapping A_P. The mapping $A_P^{(s)}$ is called the symmetric m-linear mapping, *associated* with P. By [14, Theorem 1.10, p. 6], the mapping $A_P^{(s)}$ can be recovered by the values of P by means of the formula

$$A_P^{(s)}(x_1, \ldots, x_m) = \frac{1}{2^m m!} \sum_{\varepsilon_1, \ldots, \varepsilon_m = \pm 1} \varepsilon_1 \ldots \varepsilon_m P(\varepsilon_1 x_1 + \ldots + \varepsilon_m x_m). \tag{1}$$

For convenience, we define 0-homogeneous polynomials from X to Y as constant mappings.

A mapping $P : X \to Y$ is called a *polynomial* if it can be represented in the form

$$P = \sum_{j=0}^{K} P_j, \tag{2}$$

where $K \in \mathbb{Z}_+$ and P_j is a j-homogeneous polynomial for every $j \in \{0, \ldots, K\}$. Let $\deg P$ be the maximal number $j \in \mathbb{Z}_+$, such that $P_j \not\equiv 0$.

For, in general, complex numbers $t_1, \ldots, t_m$, let $V_{t_1,\ldots,t_m}$ be the Vandermonde matrix

$$V_{t_1,\ldots,t_m} := \begin{pmatrix} 1 & t_1 & t_1^2 & \ldots & t_1^{m-1} \\ 1 & t_2 & t_2^2 & \ldots & t_2^{m-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & t_m & t_m^2 & \ldots & t_m^{m-1} \end{pmatrix}.$$

The following proposition gives us the method of recovering of homogeneous components of any polynomial P by its values.

Proposition 1 ([12]). *Let P be a polynomial of the form* (2). *Let $\lambda_0, \ldots, \lambda_K$ be distinct nonzero real numbers. Then*

$$P_j(x) = \sum_{s=0}^{K} w_{js} P(\lambda_s x),$$

for every $j \in \{0, \ldots, K\}$, where w_{js} are elements of the matrix $W = (w_{js})_{j,s=\overline{0,K}}$, which is the inverse matrix of the Vandermonde matrix $V_{\lambda_0,\ldots,\lambda_K}$.

Suppose X and Y are normed spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$ respectively. Note that an m-linear mapping $A : X^m \to Y$ is continuous if and only if the value

$$\|A\| = \sup_{\|x_1\|_X \le 1,\ldots,\|x_m\|_X \le 1} \|A(x_1, \ldots, x_m)\|_Y$$

is finite. Similarly, an m-homogeneous polynomial $P : X \to Y$ is continuous if and only if the value

$$\|P\| = \sup_{\|x\|_X \le 1} \|P(x)\|_Y$$

is finite. By definitions of $\|P\|$ and $\|A_P\|$, and by formula (1),

$$\|P\| \le \|A_P^{(s)}\| \le \frac{m^m}{m!}\|P\|. \tag{3}$$

1.2 $*$-Polynomials

Let X and Y be complex vector spaces. A mapping $A : X^{m+n} \to Y$, where $(m, n) \in \mathbb{Z}_+^2 \setminus \{(0, 0)\}$, is called an (m, n)-*linear mapping*, if A is linear with respect to every of first m arguments and it is antilinear with respect to every of last n arguments. An (m, n)-linear mapping, which is invariant with respect to permutations of its first m arguments and last n arguments separately, is called (m, n)-*symmetric*. For (m, n)-linear mapping $A : X^{m+n} \to Y$, let $A^{(s)} : X^{m+n} \to Y$ be defined by

$$A^{(s)}(x_1, \ldots, x_m, x_{m+1}, \ldots, x_{m+n}) = \frac{1}{m!n!} \sum_{\tau \in \mathcal{S}_m} \sum_{\theta \in \mathcal{S}_n} A\big(x_{\tau(1)}, \ldots, x_{\tau(m)}, x_{m+\theta(1)}, \ldots, x_{m+\theta(n)}\big).$$

The mapping $A^{(s)}$ is (m, n)-symmetric and (m, n)-linear. It is called the (m, n)-*symmetrization* of the mapping A. A mapping $P : X \to Y$ is called an (m, n)-*polynomial* if there exists an (m, n)-linear mapping $A_P : X^{m+n} \to Y$ such that P is the restriction to the diagonal of A_P, i.e.,

$$P(x) = A_P(\underbrace{x, \ldots, x}_{m+n})$$

for every $x \in X$. Note that P is also the restriction to the diagonal of the mapping $A_P^{(s)}$, which is the (m, n)-symmetrization of the mapping A_P. The mapping $A_P^{(s)}$ is called the (m, n)-symmetric (m, n)-linear mapping, *associated* with P. By [22, Theorem 3.1], the mapping $A_P^{(s)}$ can be recovered by the values of P by means of the formula

$$\begin{aligned} A_P^{(s)}(x_1, \ldots, x_{m+n}) = \frac{1}{2^{m+n} m! n!} &\sum_{\varepsilon_1,\ldots,\varepsilon_{m+n} = \pm 1} \varepsilon_1 \ldots \varepsilon_{m+n} \sum_{j=1}^{2n+1} \frac{1}{2n+1} \alpha_j^{2n+1-m} \\ &\times P\big(\alpha_j(\varepsilon_1 x_1 + \ldots + \varepsilon_m x_m) + \varepsilon_{m+1} x_{m+1} + \ldots + \varepsilon_{m+n} x_{m+n}\big), \end{aligned} \tag{4}$$

where $\alpha_j = e^{2\pi i j/(2n+1)}$ for $j \in \{1, \ldots, 2n+1\}$.

For convenience, we define $(0,0)$-polynomials from X to Y as constant mappings.

A mapping $P : X \to Y$ is called a *$*$-polynomial* if it can be represented in the form

$$P = \sum_{t=0}^{K} \sum_{j=0}^{t} P_{j,t-j}, \tag{5}$$

where $K \in \mathbb{Z}_+$ and $P_{j,t-j}$ is a $(j, t-j)$-polynomial for every $t \in \{0, \ldots, K\}$ and $j \in \{0, \ldots, t\}$. Let $\deg P$ be the maximal number $t \in \mathbb{Z}_+$, for which there exists $j \in \{0, \ldots, t\}$ such that $P_{j,t-j} \not\equiv 0$.

Results from [16, Proposition 1] and [16, Proposition 2] imply the following proposition, which gives us the method of recovering of components of any $*$-polynomial P by its values.

Proposition 2. *Let $P : X \to Y$ be a $*$-polynomial of the form* (5)*, where X and Y are complex vector spaces. Let $\lambda_0, \ldots, \lambda_K$ be distinct nonzero real numbers. Let $\varepsilon_0, \ldots, \varepsilon_K$ be complex numbers such that $\varepsilon_0^2, \ldots, \varepsilon_K^2$ are distinct and $|\varepsilon_0| = \ldots = |\varepsilon_K| = 1$. Then*

$$P_{j,t-j}(x) = \sum_{l=0}^{t} u_{jl} \varepsilon_l^t \sum_{s=0}^{K} w_{ts} P(\lambda_s \varepsilon_l x)$$

for every $t \in \{0, \ldots, K\}$, $j \in \{0, \ldots, t\}$ and $x \in X$, where w_{ts} are elements of the matrix $W = (w_{ts})_{t,s=\overline{0,K}}$, which is the inverse matrix of the Vandermonde matrix $V_{\lambda_0,\ldots,\lambda_K}$, and u_{jl} are elements of the matrix $U = (u_{jl})_{j,l=\overline{0,K}}$, which is the inverse matrix of the Vandermonde matrix $V_{\varepsilon_0^2,\ldots,\varepsilon_K^2}$.

Suppose X and Y are complex normed spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$ resp. Note that an (m,n)-linear mapping $A : X^{m+n} \to Y$ is continuous if and only if the value

$$\|A\| = \sup_{\|x_1\|_X \le 1, \ldots, \|x_{m+n}\|_X \le 1} \|A(x_1, \ldots, x_{m+n})\|_Y$$

is finite. Similarly, an (m,n)-polynomial $P : X \to Y$ is continuous if and only if the value

$$\|P\| = \sup_{\|x\|_X \le 1} \|P(x)\|_Y$$

is finite. Formula (4) implies the following inequality

$$\|A_P^{(s)}\| \le \frac{(m+n)^{m+n}}{m!n!} \|P\|. \tag{6}$$

1.3 Algebraic combinations

A mapping $f : T \to \mathbb{K}$, where T is an arbitrary set and $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, is called an *algebraic combination* of mappings $f_1, \ldots, f_m : T \to \mathbb{K}$ over $\mathbb{K}$ if there exists a polynomial $Q : \mathbb{K}^m \to \mathbb{K}$ such that

$$f(x) = Q(f_1(x), \ldots, f_m(x))$$

for every $x \in T$.

A set $\{f_1, \ldots, f_m\}$ of mappings $f_1, \ldots, f_m : T \to \mathbb{K}$ is called *algebraically independent* if

$$Q(f_1(x), \ldots, f_m(x)) = 0$$

for every $x \in T$ if and only if the polynomial Q is identically equal to zero. If a set of mappings $\{f_1, \ldots, f_m\}$ is algebraically independent and polynomials $Q_1, Q_2 : \mathbb{K}^m \to \mathbb{K}$ are such that

$$Q_1(f_1(x), \ldots, f_m(x)) = Q_2(f_1(x), \ldots, f_m(x))$$

for every $x \in T$, then the polynomial Q_1 is identically equal to the polynomial Q_2. Thus, every algebraic combination of elements of an algebraically independent set of mappings is unique. An infinite set of mappings is called algebraically independent if every its finite subset is algebraically independent. A subset $\mathcal{B}$ of some algebra of mappings $\mathcal{A}$ is called an *algebraic basis* of $\mathcal{A}$ if every element of $\mathcal{A}$ can be uniquely represented as an algebraic combination of some elements of $\mathcal{B}$. Evidently, every algebraic basis is algebraically independent.

1.4 The space $\ell_p(\mathbb{K}^n)$

Let $n \in \mathbb{N}$, $p \in [1, +\infty)$ and $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let us denote $\ell_p(\mathbb{K}^n)$ the vector space of all sequences $x = (x_1, x_2, \ldots)$, where $x_j = \big(x_j^{(1)}, \ldots, x_j^{(n)}\big) \in \mathbb{K}^n$ for $j \in \mathbb{N}$, such that the series $\sum_{j=1}^{\infty} \sum_{s=1}^{n} |x_j^{(s)}|^p$ is convergent. The space $\ell_p(\mathbb{K}^n)$ with norm

$$\|x\|_{\ell_p(\mathbb{K}^n)} = \left(\sum_{j=1}^{\infty} \sum_{s=1}^{n} |x_j^{(s)}|^p \right)^{1/p}$$

is a Banach space.

Definition 1. *A function f, defined on $\ell_p(\mathbb{K}^n)$, is called $\mathcal{S}$-symmetric (or just symmetric when the context is clear) if $f(x \circ \sigma) = f(x)$ for every $x \in \ell_p(\mathbb{K}^n)$ and for every bijection $\sigma \in \mathcal{S}$, where $x \circ \sigma = \big(x_{\sigma(1)}, x_{\sigma(2)}, \ldots\big)$.*

For a multi-index $k = (k_1, \ldots, k_n) \in \mathbb{Z}_+^n$, let $|k| = k_1 + \ldots + k_n$. For every $k \in \mathbb{Z}_+^n$ such that $|k| \geq \lceil p \rceil$, where $\lceil p \rceil$ is a ceiling of p, let us define a mapping $H_k^{(\mathbb{K}^n)} : \ell_p(\mathbb{K}^n) \to \mathbb{K}$ by

$$H_k^{(\mathbb{K}^n)}(x) = \sum_{j=1}^{\infty} \prod_{\substack{s=1 \\ k_s > 0}}^{n} \big(x_j^{(s)}\big)^{k_s}.$$

Note that $H_k^{(\mathbb{K}^n)}$ is an $\mathcal{S}$-symmetric $|k|$-homogeneous polynomial. We will use following result, proven in [11].

Proposition 3 ([11], Proposition 2)**.** *For $p \in [1, +\infty)$ and for every $k \in \mathbb{Z}_+^n$ such that $|k| \geq \lceil p \rceil$, the polynomial $H_k^{(\mathbb{C}^n)}$ on $\ell_p(\mathbb{C}^n)$ is continuous and $\|H_k^{(\mathbb{C}^n)}\| \leq 1$.*

Theorem 1. *Polynomials $H_k^{(\mathbb{C}^n)}$, where $k \in \mathbb{Z}_+^n$ are such that $|k| \geq \lceil p \rceil$, form an algebraic basis of the algebra of all $\mathcal{S}$-symmetric continuous complex-valued polynomials on $\ell_p(\mathbb{C}^n)$.*

Note that Proposition 3 implies that for $p \in [1, +\infty)$ and for every $k \in \mathbb{Z}_+^n$ such that $|k| \geq \lceil p \rceil$, the polynomial $H_k^{(\mathbb{R}^n)}$ on $\ell_p(\mathbb{R}^n)$ is continuous and $\|H_k^{(\mathbb{R}^n)}\| \leq 1$.

2 THE ALGEBRAIC BASIS OF THE ALGEBRA OF ALL SYMMETRIC CONTINUOUS POLYNOMIALS ON $\ell_p(\mathbb{R}^n)$

Let $n \in \mathbb{N}$ and $p \in [1, +\infty)$. For every continuous m-homogeneous polynomial P on $\ell_p(\mathbb{R}^n)$, which is, in general, complex-valued (we need this assumption for the sake of the applicability of results of the current section in section 3), let us define an m-homogeneous polynomial $\widehat{P} : \ell_p(\mathbb{C}^n) \to \mathbb{C}$ in the following way. Let $A_P^{(s)}$ be the m-linear symmetric mapping associated with P. Let $A_{\widehat{P}}^{(s)} : \underbrace{\ell_p(\mathbb{C}^n) \times \ldots \times \ell_p(\mathbb{C}^n)}_{m} \to \mathbb{C}$ be defined by

$$A_{\widehat{P}}^{(s)}(z_1, \ldots, z_m) = \sum_{j_1=0}^{1} \cdots \sum_{j_m=0}^{1} i^{j_1+\ldots+j_m} A_P^{(s)}\Big(w_{j_1}(z_1), \ldots, w_{j_m}(z_m)\Big), \tag{7}$$

where operators $w_0, w_1 : \ell_p(\mathbb{C}^n) \to \ell_p(\mathbb{R}^n)$ are defined by

$$w_0(z) = \big((\operatorname{Re} x_1^{(1)}, \ldots, \operatorname{Re} x_1^{(n)}), (\operatorname{Re} x_2^{(1)}, \ldots, \operatorname{Re} x_2^{(n)}), \ldots\big),$$
$$w_1(z) = \big((\operatorname{Im} x_1^{(1)}, \ldots, \operatorname{Im} x_1^{(n)}), (\operatorname{Im} x_2^{(1)}, \ldots, \operatorname{Im} x_2^{(n)}), \ldots\big)$$

for every $z = \big((x_1^{(1)}, \ldots, x_1^{(n)}), (x_2^{(1)}, \ldots, x_2^{(n)}), \ldots\big) \in \ell_p(\mathbb{C}^n)$. Note that operators w_1 and w_2 are linear, continuous and $\|w_0\| = \|w_1\| = 1$. It can be checked that $A_{\widehat{P}}^{(s)}$ is an m-linear symmetric mapping. By the continuity of mappings $A_P^{(s)}, w_0$ and w_1, the mapping $A_{\widehat{P}}^{(s)}$ is continuous. By (7), taking into account $\|w_0\| = \|w_1\| = 1$,

$$\|A_{\widehat{P}}^{(s)}\| \le 2^m \|A_P^{(s)}\|. \tag{8}$$

Let $\widehat{P}$ be the restriction of $A_{\widehat{P}}^{(s)}$ to the diagonal. Since the mapping $A_P^{(s)}$ is continuous and m-linear, it follows that the mapping $\widehat{P}$ is a continuous m-homogeneous polynomial. By (3), (7) and (8),

$$\|\widehat{P}\| \le \|A_{\widehat{P}}^{(s)}\| \le 2^m \|A_P^{(s)}\| \le \frac{(2m)^m}{m!} \|P\|. \tag{9}$$

It can be checked that for every m_1-homogeneous polynomial P_1 and for every m_2-homogeneous polynomial P_2, which acts from $\ell_p(\mathbb{R}^n)$ to $\mathbb{C}$, where $m_1, m_2 \in \mathbb{N}$, we have $\widehat{P_1 P_2} = \widehat{P}_1 \widehat{P}_2$.

For every continuous polynomial $P : \ell_p(\mathbb{R}^n) \to \mathbb{C}$ of the form (2), let

$$\widehat{P} = P_0 + \widehat{P}_1 + \ldots + \widehat{P}_K.$$

Proposition 4. *Let Γ be an arbitrary index set. For every $\gamma \in \Gamma$, let $P_\gamma : \ell_p(\mathbb{R}^n) \to \mathbb{C}$ be a continuous m_γ-homogeneous polynomial, where $m_\gamma \in \mathbb{N}$. Suppose the set of polynomials $\{\widehat{P}_\gamma : \gamma \in \Gamma\}$ is algebraically independent. Then the set of polynomials $\{P_\gamma : \gamma \in \Gamma\}$ is algebraically independent.*

Proof. Let Γ_0 be an arbitrary finite nonempty subset of Γ. Let us show that the set of polynomials $\{P_\gamma : \gamma \in \Gamma_0\}$ is algebraically independent. Suppose

$$\alpha_0 + \sum_{\mu=1}^{\mu'} \sum_{\substack{l:\Gamma_0 \to \mathbb{Z}_+ \\ \varkappa(l)=\mu}} \alpha_l \prod_{\substack{\gamma \in \Gamma_0 \\ l(\gamma)>0}} \Big(P_\gamma(x)\Big)^{l(\gamma)} = 0 \tag{10}$$

for every $x \in \ell_p(\mathbb{R}^n)$, where $\alpha_0, \alpha_l \in \mathbb{C}$, $\mu' \in \mathbb{N}$, $\varkappa(l) = \sum_{\gamma\in\Gamma_0} l(\gamma)m_\gamma$. For $\mu \in \{1, \ldots, \mu'\}$, let

$$Q_\mu(x) = \sum_{\substack{l:\Gamma_0\to\mathbb{Z}_+\\ \varkappa(l)=\mu}} \alpha_l \prod_{\substack{\gamma\in\Gamma_0\\ l(\gamma)>0}} \Big(P_\gamma(x)\Big)^{l(\gamma)}$$

for every $x \in \ell_p(\mathbb{R}^n)$. By Proposition 1, taking into account (10), $\alpha_0 = 0$ and, for every $\mu \in \{1, \ldots, \mu'\}$, the polynomial Q_μ is identically equal to zero, i.e., $\|Q_\mu\| = 0$. By (9), $\|\widehat{Q}_\mu\| \le \frac{(2\mu)^\mu}{\mu!}\|Q_\mu\|$. Therefore $\|\widehat{Q}_\mu\| = 0$. Consequently, $\widehat{Q}_\mu$ is identically equal to zero, i.e.,

$$\sum_{\substack{l:\Gamma_0\to\mathbb{Z}_+\\ \varkappa(l)=\mu}} \alpha_l \prod_{\substack{\gamma\in\Gamma_0\\ l(\gamma)>0}} \Big(\widehat{P}_\gamma(z)\Big)^{l(\gamma)} = 0$$

for every $z \in \ell_p(\mathbb{C}^n)$. Since the set of polynomials $\{\widehat{P}_\gamma : \gamma \in \Gamma_0\}$ is algebraically independent, it follows that every coefficient α_l is equal to zero. Thus, the set of polynomials $\{P_\gamma : \gamma \in \Gamma_0\}$ is algebraically independent.

Since every finite nonempty subset of the set of polynomials $\{P_\gamma : \gamma \in \Gamma\}$ is algebraically independent, it follows that the set $\{P_\gamma : \gamma \in \Gamma\}$ is algebraically independent. □

Theorem 2. *Let $P : \ell_p(\mathbb{R}^n) \to \mathbb{C}$ be a continuous m-homogeneous $\mathcal{S}$-symmetric polynomial. Then, in the case $1 \le m < \lceil p \rceil$, the polynomial P is identically equal to zero. In the case $m \ge \lceil p \rceil$, the polynomial P can be uniquely represented in the form*

$$P(x) = \sum_{\substack{l:\Gamma_m\to\mathbb{Z}_+\\ \varkappa(l)=m}} \alpha_l \prod_{\substack{k\in\Gamma_m\\ l(k)>0}} \Big(H_k^{(\mathbb{R}^n)}(x)\Big)^{l(k)},$$

where $x \in \ell_p(\mathbb{R}^n)$, $\alpha_l \in \mathbb{C}$, $\Gamma_m = \{k \in \mathbb{Z}_+^n : \lceil p \rceil \le |k| \le m\}$ and $\varkappa(l) = \sum_{k\in\Gamma_m} |k| l(k)$.

Proof. Let P be a continuous m-homogeneous $\mathcal{S}$-symmetric complex-valued polynomial on $\ell_p(\mathbb{R}^n)$, where $m \in \mathbb{N}$. Then $\widehat{P}$ is a continuous m-homogeneous complex-valued polynomial on $\ell_p(\mathbb{C}^n)$. Let us show that the polynomial $\widehat{P}$ is $\mathcal{S}$-symmetric. Let $z \in \ell_p(\mathbb{C}^n)$ and $\sigma \in \mathcal{S}$. Let us show that $\widehat{P}(z\circ\sigma) = \widehat{P}(z)$. By (1), taking into account that P is $\mathcal{S}$-symmetric,

$$\begin{aligned} A_P^{(s)}(x_1\circ\sigma, \ldots, x_m\circ\sigma) &= \frac{1}{2^m m!}\sum_{\varepsilon_1,\ldots,\varepsilon_m=\pm1} \varepsilon_1\ldots\varepsilon_m P(\varepsilon_1 x_1\circ\sigma + \ldots + \varepsilon_m x_m\circ\sigma)\\ &= \frac{1}{2^m m!}\sum_{\varepsilon_1,\ldots,\varepsilon_m=\pm1} \varepsilon_1\ldots\varepsilon_m P((\varepsilon_1 x_1 + \ldots + \varepsilon_m x_m)\circ\sigma)\\ &= \frac{1}{2^m m!}\sum_{\varepsilon_1,\ldots,\varepsilon_m=\pm1} \varepsilon_1\ldots\varepsilon_m P(\varepsilon_1 x_1 + \ldots + \varepsilon_m x_m) = A_P^{(s)}(x_1, \ldots, x_m) \end{aligned} \tag{11}$$

for every $x_1, \ldots, x_m \in \ell_p(\mathbb{R}^n)$. By (7) and (11), taking into account the equalities $w_0(z\circ\sigma) = w_0(z)$ and $w_1(z\circ\sigma) = w_1(z)$,

$$\begin{aligned} \widehat{P}(z\circ\sigma) = A_{\widehat{P}}(\underbrace{z\circ\sigma, \ldots, z\circ\sigma}_{m}) &= \sum_{j_1=0}^{1}\ldots\sum_{j_m=0}^{1} i^{j_1+\ldots+j_m} A_P\Big(w_{j_1}(z\circ\sigma), \ldots, w_{j_m}(z\circ\sigma)\Big)\\ &= \sum_{j_1=0}^{1}\ldots\sum_{j_m=0}^{1} i^{j_1+\ldots+j_m} A_P\Big(w_{j_1}(z)\circ\sigma, \ldots, w_{j_m}(z)\circ\sigma\Big)\\ &= \sum_{j_1=0}^{1}\ldots\sum_{j_m=0}^{1} i^{j_1+\ldots+j_m} A_P\Big(w_{j_1}(z), \ldots, w_{j_m}(z)\Big) = \widehat{P}(z). \end{aligned}$$

Thus, $\widehat{P}$ is $\mathcal{S}$-symmetric. So, $\widehat{P}$ is an $\mathcal{S}$-symmetric continuous m-homogeneous complex-valued polynomial on $\ell_p(\mathbb{C}^n)$.

Therefore, by Theorem 1, the polynomial $\widehat{P}$ can be uniquely represented as an algebraic combination of polynomials $H_k^{(\mathbb{C}^n)}$, where $k \in \mathbb{Z}_+^n$ are such that $|k| \geq \lceil p \rceil$. Since every $H_k^{(\mathbb{C}^n)}$ is a $|k|$-homogeneous polynomial and $|k| \geq \lceil p \rceil$, it follows that, in the case $m < \lceil p \rceil$, the polynomial $\widehat{P}$ is identically equal to zero. In the case $m \geq \lceil p \rceil$, the polynomial $\widehat{P}$ is an algebraic combination of polynomials $H_k^{(\mathbb{C}^n)}$, where $k \in \mathbb{Z}_+^n$ are such that $m \geq |k| \geq \lceil p \rceil$, i.e.,

$$\widehat{P}(z) = \sum_{\substack{l:\Gamma_m \to \mathbb{Z}_+ \\ \varkappa(l)=m}} \alpha_l \prod_{\substack{k \in \Gamma_m \\ l(k)>0}} \left(H_k^{(\mathbb{C}^n)}(z) \right)^{l(k)}, \tag{12}$$

for every $z \in \ell_p(\mathbb{C}^n)$, where $\alpha_l \in \mathbb{C}$, $\Gamma_m = \left\{ k \in \mathbb{Z}_+^n : \ \lceil p \rceil \leq |k| \leq m \right\}$ and $\varkappa(l) = \sum_{k \in \Gamma_m} |k| l(k)$. Since polynomials P and $H_k^{(\mathbb{R}^n)}$ are restrictions to the space $\ell_p(\mathbb{R}^n)$ of polynomials $\widehat{P}$ and $H_k^{(\mathbb{C}^n)}$ respectively, by (12),

$$P(x) = \sum_{\substack{l:\Gamma_m \to \mathbb{Z}_+ \\ \varkappa(l)=m}} \alpha_l \prod_{\substack{k \in \Gamma_m \\ l(k)>0}} \left(H_k^{(\mathbb{R}^n)}(x) \right)^{l(k)} \tag{13}$$

for every $x \in \ell_p(\mathbb{R}^n)$. By Theorem 1, the set of polynomials $\left\{ H_k^{(\mathbb{C}^n)} : \ k \in \Gamma_m \right\}$ is algebraically independent. Consequently, by Proposition 4, taking into account the equality $\widehat{H}_k^{(\mathbb{R}^n)} = H_k^{(\mathbb{C}^n)}$, the set of polynomials $\left\{ H_k^{(\mathbb{R}^n)} : \ k \in \Gamma_m \right\}$ is algebraically independent over $\mathbb{C}$. Therefore, the representation (13) is unique. $\square$

Theorem 3. *Polynomials $H_k^{(\mathbb{R}^n)}$, where $k \in \mathbb{Z}_+^n$ are such that $|k| \geq \lceil p \rceil$, form an algebraic basis of the algebra of all $\mathcal{S}$-symmetric continuous real-valued polynomials on $\ell_p(\mathbb{R}^n)$.*

Proof. Let P be a continuous $\mathcal{S}$-symmetric real-valued polynomial on $\ell_p(\mathbb{R}^n)$ of the form (2). Let us show that P can be uniquely represented as an algebraic combination of some elements of the set $\left\{ H_k^{(\mathbb{R}^n)} : k \in \mathbb{Z}_+^n, |k| \geq \lceil p \rceil \right\}$. By Proposition 1, for every $j \in \{1, \ldots, \deg P\}$, the j-homogeneous polynomial P_j is continuous, $\mathcal{S}$-symmetric and real-valued. Therefore, by Theorem 2, if $1 \leq j < \lceil p \rceil$, then the polynomial P_j is identically equal to zero, otherwise

$$P_j(x) = \sum_{\substack{l:\Gamma_j \to \mathbb{Z}_+ \\ \varkappa_j(l)=j}} \alpha_l \prod_{\substack{k \in \Gamma_j \\ l(k)>0}} \left(H_k^{(\mathbb{R}^n)}(x) \right)^{l(k)}$$

for every $x \in \ell_p(\mathbb{R}^n)$, where $\alpha_l \in \mathbb{C}$, $\Gamma_j = \left\{ k \in \mathbb{Z}_+^n : \ \lceil p \rceil \leq |k| \leq j \right\}$ and $\varkappa_j(l) = \sum_{k \in \Gamma_j} |k| l(k)$. Let us show that all the coefficients α_l are real. Since the polynomial P_j is real-valued, it follows that $P_j(x) - \overline{P_j(x)} = 0$ for every $x \in \ell_p(\mathbb{R}^n)$, i.e.,

$$2i \sum_{\substack{l:\Gamma_j \to \mathbb{Z}_+ \\ \varkappa_j(l)=j}} \operatorname{Im} \alpha_l \prod_{\substack{k \in \Gamma_j \\ l(k)>0}} \left(H_k^{(\mathbb{R}^n)}(x) \right)^{l(k)} = 0 \tag{14}$$

for every $x \in \ell_p(\mathbb{R}^n)$. By Proposition 4, the set of polynomials

$$\left\{ H_k^{(\mathbb{R}^n)} : \ k \in \mathbb{Z}_+^n, |k| \geq \lceil p \rceil \right\}$$

is algebraically independent over $\mathbb{C}$, therefore it is algebraically independent over $\mathbb{R}$. Consequently, by (14), $\operatorname{Im}\alpha_l = 0$ for every coefficient α_l, i.e., every α_l is real. Thus, we have that for every $x \in \ell_p(\mathbb{R}^n)$, $P(x) = P_0$ in the case $\deg P < \lceil p \rceil$, and

$$P(x) = P_0 + \sum_{j=\lceil p\rceil}^{\deg P} \sum_{\substack{l:\Gamma_j \to \mathbb{Z}_+ \\ \varkappa_j(l)=j}} \alpha_l \prod_{\substack{k\in\Gamma_j \\ l(k)>0}} \left(H_k^{(\mathbb{R}^n)}(x)\right)^{l(k)} \tag{15}$$

otherwise. Since the set of polynomials $\{H_k^{(\mathbb{R}^n)} : k \in \mathbb{Z}_+^n, |k| \geq \lceil p \rceil\}$ is algebraically independent over $\mathbb{R}$, it follows that the representation (15) is unique. □

3 SYMMETRIC $*$-POLYNOMIALS ON $\ell_p(\mathbb{C}^n)$

Let $n \in \mathbb{N}$ and $p \in [1, +\infty)$. Let the mapping $J : \ell_p(\mathbb{C}^n) \to \ell_p(\mathbb{R}^{2n})$ be defined by

$$J(z) = \big((\operatorname{Re} z_1^{(1)}, \operatorname{Im} z_1^{(1)}, \ldots, \operatorname{Re} z_1^{(n)}, \operatorname{Im} z_1^{(n)}), (\operatorname{Re} z_2^{(1)}, \operatorname{Im} z_2^{(1)}, \ldots, \operatorname{Re} z_2^{(n)}, \operatorname{Im} z_2^{(n)}), \ldots\big),$$

where $z = \big((z_1^{(1)}, \ldots, z_1^{(n)}), (z_2^{(1)}, \ldots, z_2^{(n)}), \ldots\big) \in \ell_p(\mathbb{C}^n)$. Let us show that the mapping J is well-defined and bijective. Since all norms on $\mathbb{R}^2$ are equivalent, it follows that there exist constants $C > 0$ and $c > 0$ such that

$$c\sqrt{|t_1|^2 + |t_2|^2} \leq \big(|t_1|^p + |t_2|^p\big)^{1/p} \leq C\sqrt{|t_1|^2 + |t_2|^2} \tag{16}$$

for every $(t_1, t_2) \in \mathbb{R}^2$. Therefore

$$\begin{aligned}\sum_{j=1}^{\infty}\sum_{s=1}^{n}\Big(|\operatorname{Re} z_j^{(s)}|^p + |\operatorname{Im} z_j^{(s)}|^p\Big) &\leq C^p \sum_{j=1}^{\infty}\sum_{s=1}^{n}\left(\sqrt{|\operatorname{Re} z_j^{(s)}|^2 + |\operatorname{Im} z_j^{(s)}|^2}\right)^p \\ &= C^p \sum_{j=1}^{\infty}\sum_{s=1}^{n} |z_j^{(s)}|^p = C^p \|z\|^p_{\ell_p(\mathbb{C}^n)}.\end{aligned}$$

Thus, for every $z \in \ell_p(\mathbb{C}^n)$ the sequence $J(z)$ belongs to the space $\ell_p(\mathbb{R}^{2n})$ and $\|J(z)\|^p_{\ell_p(\mathbb{R}^{2n})} \leq C^p\|z\|^p_{\ell_p(\mathbb{C}^n)}$, i.e.,

$$\|J(z)\|_{\ell_p(\mathbb{R}^{2n})} \leq C\|z\|_{\ell_p(\mathbb{C}^n)}. \tag{17}$$

Thus, the mapping J is well-defined. Evidently, J is injective. Let us show that J is surjective. Let $x = \big((x_1^{(1)}, x_1^{(2)}, \ldots, x_1^{(2n-1)}, x_1^{(2n)}), (x_2^{(1)}, x_2^{(2)}, \ldots, x_2^{(2n-1)}, x_2^{(2n)}), \ldots\big) \in \ell_p(\mathbb{R}^{2n})$. Let us construct $z_x \in \ell_p(\mathbb{C}^n)$ such that $J(z_x) = x$. Let $z_x = \big((x_1^{(1)} + ix_1^{(2)}, \ldots, x_1^{(2n-1)} + ix_1^{(2n)}), (x_2^{(1)} + ix_2^{(2)}, \ldots, x_2^{(2n-1)} + ix_2^{(2n)}), \ldots\big)$. Let us show that z_x belongs to $\ell_p(\mathbb{C}^n)$. By (16),

$$\begin{aligned}\sum_{j=1}^{\infty}\sum_{s=1}^{n}|x_j^{(2s-1)} + ix_j^{(2s)}|^p &= \sum_{j=1}^{\infty}\sum_{s=1}^{n}\left(\sqrt{|x_j^{(2s-1)}|^2 + |x_j^{(2s)}|^2}\right)^p \\ &\leq \sum_{j=1}^{\infty}\sum_{s=1}^{n}\left(\frac{1}{c}\Big(|x_j^{(2s-1)}|^p + |x_j^{(2s)}|^p\Big)^{1/p}\right)^p \\ &= \frac{1}{c^p}\sum_{j=1}^{\infty}\sum_{s=1}^{n}\Big(|x_j^{(2s-1)}|^p + |x_j^{(2s)}|^p\Big) = \frac{1}{c^p}\|x\|^p_{\ell_p(\mathbb{R}^{2n})}.\end{aligned}$$

Thus, z_x belongs to the space $\ell_p(\mathbb{C}^n)$ and $\|z_x\|^p_{\ell_p(\mathbb{C}^n)} \leq \frac{1}{c^p}\|x\|^p_{\ell_p(\mathbb{R}^{2n})}$, i.e., taking into account the equality $J(z_x) = x$,

$$\|J^{-1}(x)\|_{\ell_p(\mathbb{C}^n)} \leq \frac{1}{c}\|x\|_{\ell_p(\mathbb{R}^{2n})} \tag{18}$$

for every $x \in \ell_p(\mathbb{R}^{2n})$. Hence, the mapping J is bijective. Note that the mapping J is real-linear, i.e., it is additive and $J(\lambda z) = \lambda J(z)$ for every $\lambda \in \mathbb{R}$ and $z \in \ell_p(\mathbb{C}^n)$. By (17) and (18), both mappings J and J^{-1} are continuous.

Proposition 5. *For every continuous $\mathcal{S}$-symmetric (m_1, m_2)-polynomial $P : \ell_p(\mathbb{C}^n) \to \mathbb{C}$ the mapping $P \circ J^{-1}$ is a continuous $\mathcal{S}$-symmetric $(m_1 + m_2)$-homogeneous polynomial, acting from $\ell_p(\mathbb{R}^{2n})$ to $\mathbb{C}$.*

Proof. Let $P : \ell_p(\mathbb{C}^n) \to \mathbb{C}$ be a continuous $\mathcal{S}$-symmetric (m_1, m_2)-polynomial. Let $A_P^{(s)}$ be the (m_1, m_2)-symmetric (m_1, m_2)-linear mapping, associated with P. Let the mapping $B_{\widetilde{P}} : \big(\ell_p(\mathbb{R}^{2n})\big)^{m_1+m_2} \to \mathbb{C}$ be defined by

$$B_{\widetilde{P}}(x_1, \ldots, x_{m_1+m_2}) = A_P(J^{-1}(x_1), \ldots, J^{-1}(x_{m_1+m_2})),$$

where $x_1, \ldots, x_{m_1+m_2} \in \ell_p(\mathbb{R}^{2n})$. Since J^{-1} is real-linear and A_P is (m_1, m_2)-linear, it follows that $B_{\widetilde{P}}$ is an $(m_1 + m_2)$-linear mapping. By (6) and (18),

$$\begin{aligned}
\|B_{\widetilde{P}}\| &= \sup_{\|x_1\|_{\ell_p(\mathbb{R}^{2n})} \leq 1, \ldots, \|x_{m_1+m_2}\|_{\ell_p(\mathbb{R}^{2n})} \leq 1} |B_{\widetilde{P}}(x_1, \ldots, x_{m_1+m_2})| \\
&= \sup_{\|x_1\|_{\ell_p(\mathbb{R}^{2n})} \leq 1, \ldots, \|x_{m_1+m_2}\|_{\ell_p(\mathbb{R}^{2n})} \leq 1} |A_P(J^{-1}(x_1), \ldots, J^{-1}(x_{m_1+m_2}))| \\
&\leq \sup_{\|x_1\|_{\ell_p(\mathbb{R}^{2n})} \leq 1, \ldots, \|x_{m_1+m_2}\|_{\ell_p(\mathbb{R}^{2n})} \leq 1} \|A_P\| \|J^{-1}(x_1)\|_{\ell_p(\mathbb{C}^n)} \cdots \|J^{-1}(x_{m_1+m_2})\|_{\ell_p(\mathbb{C}^n)} \\
&\leq \frac{\|A_P\|}{c^{m_1+m_2}} \sup_{\|x_1\|_{\ell_p(\mathbb{R}^{2n})} \leq 1, \ldots, \|x_{m_1+m_2}\|_{\ell_p(\mathbb{R}^{2n})} \leq 1} \|x_1\|_{\ell_p(\mathbb{R}^{2n})} \cdots \|x_{m_1+m_2}\|_{\ell_p(\mathbb{R}^{2n})} \\
&= \frac{\|A_P\|}{c^{m_1+m_2}} \leq \frac{(m_1+m_2)^{m_1+m_2}\|P\|}{m_1! m_2! c^{m_1+m_2}}.
\end{aligned}$$

Thus, $\|B_{\widetilde{P}}\|$ is finite and, consequently, $B_{\widetilde{P}}$ is continuous. Let $\widetilde{P}$ be the restriction to the diagonal of $B_{\widetilde{P}}$. Then $\widetilde{P}$ is the $(m_1 + m_2)$-homogeneous polynomial. Since $\|\widetilde{P}\| \leq \|B_{\widetilde{P}}\|$, it follows that $\widetilde{P}$ is continuous. Note that $\widetilde{P} = P \circ J^{-1}$. Let us show that $\widetilde{P}$ is $\mathcal{S}$-symmetric. Let $x \in \ell_p(\mathbb{R}^{2n})$ and $\sigma \in \mathcal{S}$. Note that $J^{-1}(x \circ \sigma) = J^{-1}(x) \circ \sigma$. Therefore, since P is $\mathcal{S}$-symmetric,

$$\widetilde{P}(x \circ \sigma) = P(J^{-1}(x \circ \sigma)) = P(J^{-1}(x) \circ \sigma) = P(J^{-1}(x)) = \widetilde{P}(x).$$

Thus, $\widetilde{P}$ is $\mathcal{S}$-symmetric. □

Theorem 4. *The set of mappings $\big\{H_k^{(\mathbb{R}^{2n})} \circ J : \ k \in \mathbb{Z}_+^{2n}, |k| \geq \lceil p \rceil\big\}$ is an algebraic basis of the algebra of all continuous $\mathcal{S}$-symmetric $*$-polynomials, acting from $\ell_p(\mathbb{C}^n)$ to $\mathbb{C}$.*

Proof. Let $P : \ell_p(\mathbb{C}^n) \to \mathbb{C}$ be a continuous $\mathcal{S}$-symmetric $*$-polynomial of the form (5). By Proposition 2, taking into account the continuity and the $\mathcal{S}$-symmetry of P, for every $t \in \{0, \ldots, K\}$ and $j \in \{0, \ldots, t\}$, the $(j, t-j)$-polynomial $P_{j,t-j}$ is continuous and $\mathcal{S}$-symmetric.

Therefore, by Proposition 5, the mapping $P_{j,t-j} \circ J^{-1}$ is a continuous $\mathcal{S}$-symmetric t-homogeneous polynomial, acting from $\ell_p(\mathbb{R}^{2n})$ to $\mathbb{C}$. Consequently, by Theorem 2, the polynomial $P_{j,t-j} \circ J^{-1}$ is identically equal to zero in the case $1 \le t < \lceil p \rceil$, and, otherwise, the polynomial $P_{j,t-j} \circ J^{-1}$ can be uniquely represented in the form

$$(P_{j,t-j} \circ J^{-1})(x) = \sum_{\substack{l:\Gamma_t \to \mathbb{Z}_+ \\ \varkappa_t(l)=t}} \alpha_l^{(j,t-j)} \prod_{\substack{k \in \Gamma_t \\ l(k)>0}} \left(H_k^{(\mathbb{R}^{2n})}(x) \right)^{l(k)},$$

where $x \in \ell_p(\mathbb{R}^{2n})$, $\alpha_l^{(j,t-j)} \in \mathbb{C}$, $\Gamma_t = \{k \in \mathbb{Z}_+^{2n} : \lceil p \rceil \le |k| \le t\}$ and $\varkappa_t(l) = \sum_{k \in \Gamma_t} |k| l(k)$. Therefore, taking into account that J is a bijection, the mapping $P_{j,t-j}$ is identically equal to zero in the case $1 \le t < \lceil p \rceil$, and

$$P_{j,t-j}(z) = \sum_{\substack{l:\Gamma_t \to \mathbb{Z}_+ \\ \varkappa_t(l)=t}} \alpha_l^{(j,t-j)} \prod_{\substack{k \in \Gamma_t \\ l(k)>0}} \left((H_k^{(\mathbb{R}^{2n})} \circ J)(z) \right)^{l(k)},$$

for every $z \in \ell_p(\mathbb{C}^n)$, otherwise. Consequently, $P = P_0$ in the case $\deg P < \lceil p \rceil$, and

$$P(z) = P_0 + \sum_{t=\lceil p \rceil}^{\deg P} \sum_{j=0}^{t} \sum_{\substack{l:\Gamma_t \to \mathbb{Z}_+ \\ \varkappa_t(l)=t}} \alpha_l^{(j,t-j)} \prod_{\substack{k \in \Gamma_t \\ l(k)>0}} \left((H_k^{(\mathbb{R}^{2n})} \circ J)(z) \right)^{l(k)} \tag{19}$$

for every $z \in \ell_p(\mathbb{C}^n)$, otherwise. By Proposition 4, the set of polynomials $\{H_k^{(\mathbb{R}^{2n})} : k \in \mathbb{Z}_+^{2n}, |k| \ge \lceil p \rceil\}$ is algebraically independent. Since J is a bijection, it follows that the set of $*$-polynomials $\{H_k^{(\mathbb{R}^{2n})} \circ J : k \in \mathbb{Z}_+^{2n}, |k| \ge \lceil p \rceil\}$ is algebraically independent. Therefore, the representation (19) is unique. $\square$

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Василишин Т.В. *Симетричні функції на просторах* $\ell_p(\mathbb{R}^n)$ *і* $\ell_p(\mathbb{C}^n)$ // Карпатські матем. публ. — 2020. — Т.12, №1. — C. 5–16.

Дана робота присвячена вивченню алгебр неперервних симетричних, тобто, інваріантних відносно перестановок координат їхніх аргументів, поліномів і $*$-поліномів на банахових просторах $\ell_p(\mathbb{R}^n)$ і $\ell_p(\mathbb{C}^n)$ всіх сумовних у степені p послідовностей n-вимірних векторів дійсних і комплексних чисел відповідно, де $1 \le p < +\infty$.

Сконструйовано підмножину алгебри всіх неперервних симетричних поліномів на просторі $\ell_p(\mathbb{R}^n)$ таку, що кожен неперервний симетричний поліном на просторі $\ell_p(\mathbb{R}^n)$ може бути єдиним чином поданий у вигляді лінійної комбінації добутків елементів цієї множини. Іншими словами, сконструйовано алгебраїчний базис алгебри всіх неперервних симетричних поліномів на просторі $\ell_p(\mathbb{R}^n)$. Використовуючи даний результат, сконструйовано алгебраїчний базис алгебри всіх неперервних симетричних $*$-поліномів на просторі $\ell_p(\mathbb{C}^n)$.

Результати даної роботи можуть бути використані для досліджень алгебр, згенерованих неперервними симетричними поліномами на просторі $\ell_p(\mathbb{R}^n)$, і алгебр, згенерованих неперервними симетричними $*$-поліномами на просторі $\ell_p(\mathbb{C}^n)$.

Ключові слова і фрази: поліном, $*$-поліном, симетричний поліном, симетричний $*$-поліном, алгебраїчний базис.

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KRAVTSIV V. V.

ANALOGUES OF THE NEWTON FORMULAS FOR THE BLOCK-SYMMETRIC POLYNOMIALS ON $\ell_p(\mathbb{C}^s)$

The classical Newton formulas gives recurrent relations between algebraic bases of symmetric polynomials. They are true, of course, for symmetric polynomials on infinite-dimensional Banach sequence spaces.

In this paper, we consider block-symmetric polynomials (or MacMahon symmetric polynomials) on Banach spaces $\ell_p(\mathbb{C}^s)$, $1 \le p \le \infty$. We prove an analogue of the Newton formula for the block-symmetric polynomials for the case $p = 1$. In the case $1 < p$ we have no classical elementary block-symmetric polynomials. However, we extend the obtained Newton type formula for $\ell_1(\mathbb{C}^s)$ to the case of $\ell_p(\mathbb{C}^s)$, $1 < p \le \infty$, and in this way we found a natural definition of elementary block-symmetric polynomials on $\ell_p(\mathbb{C}^s)$.

Key words and phrases: symmetric polynomials, block-symmetric polynomials, algebraic basis, Newton's formula.

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1 INTRODUCTION

Let X be a Banach space, and let $\mathcal{P}(X)$ be the algebra of all continuous polynomials defined on X. Let $\mathcal{P}_0(X)$ be a subalgebra of $\mathcal{P}(X)$. A sequence $(Q_i)_i$ of polynomials is called an algebraic basis of $\mathcal{P}_0(X)$ if for every $P \in \mathcal{P}_0(X)$ there is a unique polynomial $q \in \mathcal{P}(\mathbb{C}^n)$ for some n such that $P(x) = q(Q_1(x), \ldots, Q_n(x))$. In other words, if Q is mapping $x \in X \rightsquigarrow (Q_1(x), \ldots, Q_n(x)) \in \mathbb{C}^n$, then $P = q \circ Q$ and this representation is unique. Subalgebras of polynomials with countable algebraic bases were considered by many authors (see e. g. [4, 8, 9, 11, 12]). Typical examples of such kind of algebras are algebras of polynomials which are invariant with respect to a (semi)group $\mathcal{S}$ of operators on X. If X has an unconditional basis (e_n), we can consider the group $\mathcal{S} = S_\infty$ of all permutations of natural numbers $\mathbb{N}$ acting on X by

$$\sigma \colon x = \sum_{n=1}^{\infty} x_n e_n \rightsquigarrow \sum_{n=1}^{\infty} x_{\sigma(n)} e_n.$$

S_∞-invariant polynomials on X are called *symmetric*. Symmetric polynomials and analytic functions on ℓ_p were investigated in [1–3, 5, 6, 8]. Linear bases of symmetric polynomials on ℓ_1 were considered in [7].

Let $\mathcal{P}_s(\ell_p)$ be the algebra of all symmetric polynomials on ℓ_1. In [10], it is proved that polynomials

$$F_k = \sum_{i=1}^{\infty} x_i^k,$$

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$k \geq \lceil p \rceil$ form an algebraic basis in $\mathcal{P}_s(\ell_p)$, where $\lceil p \rceil$ is the smallest integer, greater than p. Polynomials F_k are called *power* symmetric polynomials. In the case $p = 1$ we can consider another natural algebraic basis in $\mathcal{P}_s(\ell_1)$, which is called the *basis of elementary symmetric polynomials*, $\{G_k\}_{k=1}^{\infty}$,

$$G_k = \sum_{i_1 < i_2 < \ldots < i_k}^{\infty} x_{i_1} x_{i_2} \ldots x_{i_k}, \tag{1}$$

The relation between power symmetric polynomials and elementary symmetric polynomials can be given by the well-known Newton formulas (see, e.g., [17]):

$$nG_n = F_1 G_{n-1} - F_2 G_{n-2} + F_3 G_{n-3} - \ldots + (-1)^{n-2} F_{n-1} G_1 + (-1)^{n-1} F_n, \quad n \in \mathbb{N}.$$

In the case $p > 1$ we have no elementary symmetric polynomials, because the series (1) does not converge for any k. But putting in the Newton formulas $F_k = 0$ for $k < p$, we can define elementary symmetric polynomials on ℓ_p by

$$G_n^{(p)} = \sum_{k=\lceil p \rceil}^{n-\lceil p \rceil} (-1)^{k-1} F_k G_{n-k}.$$

It is easy to check that the sequence $\{G_n^{(p)}\}_{n>p}$ forms an algebraic basis in $\mathcal{P}_s(\ell_p)$.

There are other natural representations of S_∞ in Banach spaces with bases. For example, if $\mathcal{X}$ is a directs sum of infinite many of "blocks" which are copies of a Banach space X, then S_∞ acts permutating the "blocks". For this case we can consider the algebra of block-symmetric analytic functions consisting of invariants of this group. Note that this algebra is much more complicated and in the finitely-dimensional case has no algebraic basis (see, e.g., [15, 19]).

A generalization of the Newton formula for block-symmetric polynomials on $\ell_1(\mathbb{C}^2)$ was proved in [13]. In this paper we propose a generalization of this formula for block-symmetric polynomials on $\ell_p(\mathbb{C}^s)$.

2 MAIN RESULT

Let us denote by $\ell_p(\mathbb{C}^s)$, $1 \leq p < \infty$, the vector space of all sequences

$$x = (x_1, x_2, \ldots, x_m, \ldots),$$

where $x_j = (x_j^{(1)}, \ldots, x_j^{(s)}) \in \mathbb{C}^s$ for $j \in \mathbb{N}$, such that the series $\sum\limits_{j=1}^{\infty} \sum\limits_{r=1}^{s} \left| x_j^{(r)} \right|^p$ is convergent. The space $\ell_p(\mathbb{C}^s)$ with norm

$$\|x\| = \left(\sum_{j=1}^{\infty} \sum_{r=1}^{s} \left| x_j^{(r)} \right|^p \right)^{1/p}$$

is a Banach space. A polynomial P on the space $\ell_p(\mathbb{C}^s)$ is called block-symmetric (or vector-symmetric) if

$$P(x_1, x_2, \ldots, x_m, \ldots) = P(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(m)}, \ldots)$$

for every permutation $\sigma \in S_\infty$, where $x_j \in \mathbb{C}^s$ for all $j \in \mathbb{N}$. Let us denote by $\mathcal{P}_{vs}(\ell_p(\mathbb{C}^s))$ the algebra of all block-symmetric polynomials on $\ell_p(\mathbb{C}^s)$.

The algebra $\mathcal{P}_{vs}(\ell_p(\mathbb{C}^s))$ was considered in [14, 16]. Note that in Combinatorics, block-symmetric polynomials on finite-dimension spaces are called *MacMahon symmetric polynomials* (see [18]).

For a multi-index $\mathbf{k} = (k_1, k_2, \ldots, k_s) \in \mathbb{Z}_+^s$ let $m = |\mathbf{k}| = k_1 + k_2 + \ldots + k_s$.

In [14] it was proved that polynomials

$$H_m^{\mathbf{k}}(x) = H_m^{k_1,k_2,\ldots,k_s}(x) = \sum_{j=1}^{\infty} \prod_{\substack{r=1 \\ |\mathbf{k}| \geq \lceil p \rceil}}^{s} (x_j^{(r)})^{k_r} \tag{2}$$

form an algebraic basis in $\mathcal{P}_{vs}(\ell_p(\mathbb{C}^s))$, $1 \leq p < \infty$, where $x = (x_1, \ldots, x_m, \ldots) \in \ell_p(\mathbb{C}^s)$, $x_j = (x_j^{(1)}, \ldots, x_j^{(s)}) \in \mathbb{C}^s$.

In the case of the space $\ell_1(\mathbb{C}^s)$ there are elementary block-symmetric polynomials

$$R_m^{\mathbf{k}}(x) = R_m^{k_1,k_2,\ldots,k_s}(x) = \sum_{\substack{i_1 < \ldots < i_{k_1} \\ j_1 < \ldots < j_{k_2} \\ \cdots \\ l_1 < \ldots < l_{k_s} \\ i_{k_p} \neq j_{k_q} \neq \ldots \neq l_{k_r}}}^{\infty} x_{i_1}^{(1)} \ldots x_{i_{k_1}}^{(1)} x_{j_1}^{(2)} \ldots x_{j_{k_2}}^{(2)} \ldots x_{l_1}^{(s)} \ldots x_{l_{k_s}}^{(s)}, \tag{3}$$

where $(x_i^{(1)}, x_i^{(2)}, \ldots, x_i^{(s)}) \in \mathbb{C}^s$.

Combining (2) and (3), we can get an analog of Newton's formula for block-symmetric polynomials on $\ell_1(\mathbb{C}^s)$.

Theorem 1. *The following formula is true for the algebraic bases of symmetric polynomials on $\ell_1(\mathbb{C}^s)$.*

$$\begin{aligned} nR_n^{k_1,k_2,\ldots,k_s} = & \sum_{\substack{|\mathbf{q}|=1 \\ k_r \geq q_r}} H_1^{q_1,q_2,\ldots,q_s} R_{n-1}^{k_1-q_1,k_2-q_2,\ldots,k_s-q_s} \\ & - \sum_{\substack{|\mathbf{q}|=2 \\ k_r \geq q_r}} \frac{2!}{q_1! q_2! \ldots q_s!} H_2^{q_1,q_2,\ldots,q_s} R_{n-2}^{k_1-q_1,k_2-q_2,\ldots,k_s-q_s} + \ldots \\ & + (-1)^{n-2} \sum_{\substack{|\mathbf{q}|=n-1 \\ k_r \geq q_r}} \frac{(n-1)!}{q_1! q_2! \ldots q_s!} H_{n-1}^{q_1,q_2,\ldots,q_s} R_1^{k_1-q_1,k_2-q_2,\ldots,k_s-q_s} \\ & + (-1)^{n-1} \frac{n!}{k_1! k_2! \ldots k_s!} H_n^{k_1,k_2,\ldots,k_s}, \end{aligned} \tag{4}$$

where $\mathbf{q} = (q_1, q_2, \ldots, q_s)$, $R_0^{k_1,k_2,\ldots,k_s} \equiv 1$ and if $k_r < q_r$ for some $r = 1, \ldots, s$, then $R_m^{k_1-q_1,k_2-q_2,\ldots,k_s-q_s} \equiv 0$.

Proof. Let us consider the polynomial $P(t_1 x^{(1)} + t_2 x^{(2)} + \ldots + t_s x^{(s)})$, which is symmetric on the space ℓ_1 with respect to simultaneously permutations of $t_1 x_i^{(1)} + t_2 x_i^{(2)} + \ldots + t_s x_i^{(s)}$, $i \geq 1$. Let us denote by $\widetilde{tx} = t_1 x^{(1)} + t_2 x^{(2)} + \ldots + t_s x^{(s)}$. For the algebraic bases $F_k(\widetilde{tx})$ and $G_k(\widetilde{tx})$ of this polynomial the Newton formula holds

$$\begin{aligned} nG_n(\widetilde{tx}) = & F_1(\widetilde{tx}) G_{n-1}(\widetilde{tx}) - F_2(\widetilde{tx}) G_{n-2}(\widetilde{tx}) \\ & + F_3(\widetilde{tx}) G_{n-3}(\widetilde{tx}) - \ldots + (-1)^{n-2} F_{n-1}(\widetilde{tx}) G_1(\widetilde{tx}) + (-1)^{n-1} F_n(\widetilde{tx}). \end{aligned} \tag{5}$$

Each of polynomials $F_m(\widetilde{tx})$ and $G_m(\widetilde{tx})$ can be represented as a linear combination of polynomials $H_m^{k_1,k_2,\dots,k_s}(x)$ and $R_m^{k_1,k_2,\dots,k_s}(x)$ respectively. Indeed,

$$\begin{aligned} G_n(\widetilde{tx}) &= G_n(t_1x^{(1)} + t_2x^{(2)} + \ldots + t_sx^{(s)}) \\ &= \sum_{i_1<\ldots<i_n} (t_1x^{(1)} + t_2x^{(2)} + \ldots + t_sx^{(s)})_{i_1} \ldots (t_1x^{(1)} + t_2x^{(2)} + \ldots + t_sx^{(s)})_{i_n} \\ &= \sum_{p_1+p_2+\ldots+p_s=n} t_1^{p_1}t_2^{p_2}\ldots t_s^{p_s} R_n^{p_1,p_2,\dots,p_s}(x) \end{aligned} \tag{6}$$

and

$$\begin{aligned} F_n(\widetilde{tx}) &= F_n(t_1x^{(1)} + t_2x^{(2)} + \ldots + t_sx^{(s)}) = \sum_{i=1}^{\infty}(t_1x^{(1)} + t_2x^{(2)} + \ldots + t_sx^{(s)})_i^n \\ &= \sum_{k_1+k_2+\ldots+k_s=n} \frac{n!}{k_1!k_2!\ldots k_s!} t_1^{k_1}t_2^{k_2}\ldots t_s^{k_s} H_n^{k_1,k_2,\dots,k_s}(x). \end{aligned} \tag{7}$$

So each term of equality (5) can be represented by polynomials $H_m^{k_1,k_2,\dots,k_s}$ and $R_m^{p_1,p_2,\dots,p_s}$. Then we obtain

$$\begin{aligned} F_1(\widetilde{tx})G_{n-1}(\widetilde{tx}) &= \left(\sum_{k_1+k_2+\ldots+k_s=1} \frac{1!}{k_1!k_2!\ldots k_s!} t_1^{k_1}t_2^{k_2}\ldots t_s^{k_s} H_1^{k_1,k_2,\dots,k_s}(x)\right) \\ &\quad\times \left(\sum_{p_1+p_2+\ldots+p_s=n-1} t_1^{p_1}t_2^{p_2}\ldots t_s^{p_s} R_{n-1}^{p_1,p_2,\dots,p_s}(x)\right) \\ &= \sum_{\substack{k_1+k_2+\ldots+k_s=1\\ p_1+p_2+\ldots+p_s=n-1}} \frac{1!}{k_1!k_2!\ldots k_s!} t_1^{k_1+p_1}t_2^{k_2+p_2}\ldots t_s^{k_s+p_s} H_1^{k_1,k_2,\dots,k_s}(x) R_{n-1}^{p_1,p_2,\dots,p_s}(x), \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ F_r(\widetilde{tx})G_{n-r}(\widetilde{tx}) &= \left(\sum_{k_1+k_2+\ldots+k_s=r} \frac{r!}{k_1!k_2!\ldots k_s!} t_1^{k_1}t_2^{k_2}\ldots t_s^{k_s} H_r^{k_1,k_2,\dots,k_s}(x)\right) \\ &\quad\times \left(\sum_{p_1+p_2+\ldots+p_s=n-r} t_1^{p_1}t_2^{p_2}\ldots t_s^{p_s} R_{n-r}^{p_1,p_2,\dots,p_s}(x)\right) \\ &= \sum_{\substack{k_1+k_2+\ldots+k_s=r\\ p_1+p_2+\ldots+p_s=n-r}} \frac{r!}{k_1!k_2!\ldots k_s!} t_1^{k_1+p_1}t_2^{k_2+p_2}\ldots t_s^{k_s+p_s} H_r^{k_1,k_2,\dots,k_s}(x) R_{n-r}^{p_1,p_2,\dots,p_s}(x). \end{aligned}$$

If we substitute this equalities and equalities (6), (7) into (5) and equate multipliers at the all powers of $t_i, i = 1, \dots, s$ we obtain the required formula. $\square$

Note that equation (4) is invertible and so we have

$$\begin{aligned} \frac{n!}{k_1!\ldots k_s!} H_n^{k_1,\dots,k_s} &= \sum_{\substack{|\mathbf{q}|=n-1\\ k_r\geq q_r}} \frac{(n-1)!}{q_1!\ldots q_s!} H_{n-1}^{q_1,\dots,q_s} R_1^{k_1-q_1,\dots,k_s-q_s} + \ldots \\ &\quad + (-1)^{n-1} \sum_{\substack{|\mathbf{q}|=2\\ k_r\geq q_r}} \frac{2!}{q_1!\ldots q_s!} H_2^{q_1,\dots,q_s} R_{n-2}^{k_1-q_1,\dots,k_s-q_s} \\ &\quad + (-1)^n \sum_{\substack{|\mathbf{q}|=1\\ k_r\geq q_r}} H_1^{q_1,\dots,q_s} R_{n-1}^{k_1-q_1,\dots,k_s-q_s} + (-1)^{n+1} n R_n^{k_1,\dots,k_s}. \end{aligned}$$

Let us rewrite formula (4) using multi-index notations. We denote by $\mathbf{k}\,! = k_1!k_2!\dots k_s!$ and by $\mathbf{k}-\mathbf{q} = (k_1-q_1, k_2-q_2, \dots k_s-q_s)$. Also, we say that $\mathbf{k} \geq \mathbf{q}$ if and only if $k_1 \geq q_1$, $k_2 \geq q_2, \dots, k_s \geq q_S$. Then (4) can be expressed by

$$\begin{aligned} nR_n^{\mathbf{k}} &= \sum_{\substack{|\mathbf{q}|=1 \\ \mathbf{k}\geq\mathbf{q}}} H_1^{\mathbf{q}} R_{n-1}^{\mathbf{k}-\mathbf{q}} - \sum_{\substack{|\mathbf{q}|=2 \\ \mathbf{k}\geq\mathbf{q}}} \frac{|\mathbf{q}|!}{\mathbf{q}\,!} H_2^{\mathbf{q}} R_{n-2}^{\mathbf{k}-\mathbf{q}} + \dots + (-1)^{n-2} \sum_{\substack{|\mathbf{q}|=n-1 \\ \mathbf{k}\geq\mathbf{q}}} \frac{|\mathbf{q}|!}{\mathbf{q}\,!} H_{n-1}^{\mathbf{q}} R_1^{\mathbf{k}-\mathbf{q}} \\ &+ (-1)^{n-1}\frac{n!}{\mathbf{k}\,!} H_n^{\mathbf{k}} = \sum_{j=1}^{n} (-1)^{j-1} \sum_{\substack{|\mathbf{q}|=j \\ \mathbf{k}\geq\mathbf{q}}} \frac{|\mathbf{q}|!}{\mathbf{q}\,!} H_j^{\mathbf{q}} R_{n-j}^{\mathbf{k}-\mathbf{q}}, \quad \text{where} \quad n = |\mathbf{k}|. \end{aligned} \tag{8}$$

Comparing formula (8) with the classical Newton formula we can see that their are coincide if $s = 1$.

Let us turn out to the space $\ell_p(\mathbb{C}^s)$. Taking into account formula (2) we can see that by definition, $H_m^{\mathbf{k}} = 0$ in $\mathcal{P}_{vs}(\ell_p(\mathbb{C}^s))$ if $|\mathbf{k}| < \lceil p \rceil$. So, using (8), we can define *elementary block-symmetric polynomials on* $\ell_p(\mathbb{C}^s)$ by

$$nR_n^{\mathbf{k}} = \sum_{j=\lceil p\rceil}^{n-\lceil p\rceil} (-1)^{j-1} \sum_{\substack{|\mathbf{q}|=j \\ \mathbf{k}\geq\mathbf{q}}} \frac{|\mathbf{q}|!}{\mathbf{q}\,!} H_j^{\mathbf{q}} R_{n-j}^{\mathbf{k}-\mathbf{q}}, \quad \text{where} \quad n = |\mathbf{k}| \geq \lceil p \rceil. \tag{9}$$

Theorem 2. *Elementary block-symmetric polynomials on* $\ell_p(\mathbb{C}^s)$ *defined by* (9) *form an algebraic basis of* n*-homogeneous polynomials* $n \geq \lceil p \rceil$ *in* $\mathcal{P}_{vs}(\ell_p(\mathbb{C}^s))$.

Proof. It is easy to see that equation (9) is invertible. So we have a bijection between polynomials $H_n^{\mathbf{q}}$ and $R_n^{\mathbf{q}}$. Since $\{H_n^{\mathbf{q}}\}_{n\geq\lceil p\rceil}$ is an algebraic basis in $\mathcal{P}_{vs}(\ell_p(\mathbb{C}^s))$, so the set $\{R_n^{\mathbf{q}}\}_{n\geq\lceil p\rceil}$ is an algebraic basis in $\mathcal{P}_{vs}(\ell_p(\mathbb{C}^s))$ too. □

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Кравців В.В. *Аналог формули Ньютона для блочно-симетричних поліномів на* $\ell_p(\mathbb{C}^n)$ // Карпатські матем. публ. — 2020. — Т.12, №1. — С. 17–22.

Класичні формули Ньютона задає рекурентні співвідношення між алгебраїчними базисами симетричних поліномів. Ці формули залишаються правильними і для симетричних поліномів на нескінченновимірних банахових просторах послідовностей.

В цій статті ми розглядаємо блочно-симетричні поліноми (або симетричні полінома Макмахона) на банахових просторах $\ell_p(\mathbb{C}^s)$, $1 \le p \le \infty$. Ми доводимо аналог формули Ньютона для блочно-симетричних поліномів у випадку $p = 1$. У випадку $1 < p$ немає класичних елементарних блочно-симетричних поліномів. Проте ми продовжили отриману формулу типу Ньютона для $\ell_1(\mathbb{C}^s)$ на випадок $\ell_p(\mathbb{C}^s)$, $1 < p \le \infty$, і, в такий спосіб, запропонували природнє означення елементарних блочно-симетричних поліномів на $\ell_p(\mathbb{C}^s)$.

Ключові слова і фрази: симетричні поліноми, блочно-симетричні поліноми, алгебраїчний базис, формула Ньютона.

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NONLOCAL INVERSE BOUNDARY-VALUE PROBLEM FOR A 2D PARABOLIC EQUATION WITH INTEGRAL OVERDETERMINATION CONDITION

This article studies a nonlocal inverse boundary-value problem for a two-dimensional second-order parabolic equation in a rectangular domain. The purpose of the article is to determine the unknown coefficient and the solution of the considered problem. To investigate the solvability of the inverse problem, we transform the original problem into some auxiliary problem with trivial boundary conditions. Using the contraction mappings principle, existence and uniqueness of the solution of an equivalent problem are proved. Further, using the equivalency, the existence and uniqueness theorem of the classical solution of the original problem is obtained.

Key words and phrases: inverse problem, two-dimensional parabolic equation, Fourier method, classical solution, overdetermination condition.

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1 INTRODUCTION AND FORMULATION OF THE INVERSE PROBLEM

In the present paper, we consider an inverse boundary-value problem for a two-dimensional parabolic equation in a rectangular domain. The main goal of this article is to prove the existence and uniqueness of a classical solution of an inverse boundary-value problem.

The inverse problems arise in many different areas of mathematical modeling types, such as mineral exploration, biology, medicine, seismology, desalination of seawater, movement of liquid in a porous medium, financial market behavior, etc. Fundamentals of the theory and practice of research of inverse problems were established and developed in the pioneering works of Tikhonov [18], Lavrent'ev [15], Ivanov [10], Romanov [17], Denisov [3, 4]. Recently, there have been many studies of inverse problems for 1D parabolic and other types of equations. A more detailed bibliography and a classification of problems may be found in [1, 2, 6, 7, 11, 12].

Problems of the solvability of inverse problems for a two-dimensional heat equation is extensively studied by many authors, see, for example, Ismailov [5], Ivanchov [8, 9], Kabanikhin [13], Kinash [14], Zaynullov [19], and others. But the statement of the problem and the proof techniques used in this study are different from representations in these papers.

Motivated by these works, we study in this paper the existence and uniqueness of a classical solution for the following inverse problem: in the domain $D_T = \bar{Q}_{xy} \times [0, T]$, where $Q_{xy} = \{(x, y) : 0 < x < 1, 0 < y < 1\}$, consider a two-dimensional parabolic equation

$$u_t(x, y, t) - c(t)(u_{xx}(x, y, t) + u_{yy}(x, y, t)) = a(t)u(x, y, t) + f(x, y, t),\ (x, y, t) \in D_T, \quad (1)$$

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with the nonlocal condition

$$u(x,y,0) + \delta u(x,y,T) = \varphi(x,y), \qquad (x,y) \in \bar{Q}_{xy}, \tag{2}$$

the boundary conditions

$$u(0,y,t) = u_x(1,y,t) = 0, \qquad 0 \le y \le 1,\ 0 \le t \le T, \tag{3}$$

$$u_y(x,0,t) = u(x,1,t) = 0, \qquad 0 \le x \le 1,\ 0 \le t \le T, \tag{4}$$

and the overdetermination condition

$$u(x_0,y_0,t) + \int_0^1\int_0^1 K(x,y)u(x,y,t)dxdy = h(t), \qquad 0 \le t \le T,\ 0 < x_0, y_0 < 1, \tag{5}$$

where $\delta \ge 0$ is known number, $(x_0,y_0) \in Q_{xy}$ is some fixed point, $0 < c(t), f(x,y,t), \varphi(x,y)$, $h(t)$ are given functions, $u(x,y,t)$ and $a(t)$ are unknown functions.

Definition 1. *The pair* $\{u(x,y,t), a(t)\}$ *is said to be a classical solution of the problem* (1)–(5), *if the functions* $u(x,y,t) \in C^{2,2,1}(D_T)$ *and* $a(t) \in C[0,T]$ *satisfy equation* (1) *in* D_T, *and the conditions* (2)–(5) *in the classical (usual) sense.*

To investigate the existence and uniqueness of the classical solution of problem (1)–(5), we prove the following theorem.

Theorem 1. *Suppose that* $\delta \ge 0$, $f(x,y,t) \in C(D_T)$, $\varphi(x,y) \in C(\bar{Q}_{xy})$, $K(x,y) \in L_1(Q_{xy})$, $h(t) \in C^1[0,T]$, $h(t) \ne 0$, $0 \le t \le T$ *and the compatibility condition*

$$\varphi(x_0,y_0) + \int_0^1\int_0^1 K(x,y)\varphi(x,y)\,dx\,dy = h(0) + \delta h(T), \tag{6}$$

holds true. Then the problem of finding a classical solution of (1)–(5) *is equivalent to the problem of determining the functions* $u(x,y,t) \in C^{2,2,1}(D_T)$ *and* $a(t) \in C[0,T]$, *satisfying* (1)–(4), *and the condition*

$$\begin{aligned} h'(t) - c(t)\left(u_{xx}(x_0,y_0,t) + u_{yy}(x_0,y_0,t) + \int_0^1\int_0^1 K(x,y)(u_{xx}(x,y,t) + u_{yy}(x,y,t))\,dx\,dy\right) \\ = a(t)h(t) + f(x_0,y_0,t) + \int_0^1\int_0^1 K(x,y)f(x,y,t)\,dx\,dy, \qquad 0 \le t \le T. \end{aligned} \tag{7}$$

Proof. Let $\{u(x,y,t), a(t)\}$ be the classical solution of problem (1)–(5). Then from equation (1), we have

$$\frac{d}{dt}\left(u(x_0,y_0,t) + \int_0^1\int_0^1 K(x,y)u(x,y,t)\,dx\,dy\right)$$
$$- c(t)\left(u_{xx}(x_0,y_0,t) + u_{yy}(x_0,y_0,t) + \int_0^1\int_0^1 K(x,y)(u_{xx}(x,y,t) + u_{yy}(x,y,t))\,dx\,dy\right)$$

$$
\begin{aligned}
&= a(t)\left(u(x_0,y_0,t)+\int_0^1\int_0^1 K(x,y)u(x,y,t)\,dx\,dy\right)\\
&+ f(x_0,y_0,t)+\int_0^1\int_0^1 K(x,y)f(x,y,t)\,dx\,dy, \qquad 0\le t\le T.
\end{aligned}
\tag{8}
$$

Differentiating both sides of (6) with respect to t gives

$$
\frac{d}{dt}\left(u(x_0,y_0,t)+\int_0^1\int_0^1 K(x,y)u(x,y,t)\,dx\,dy\right)=h'(t), \qquad 0\le t\le T. \tag{9}
$$

From (8), taking into account (5) and (9), we arrive at (7).

Now, assume that $\{u(x,y,t),a(t)\}$ is a solution to the problem (1)–(4), (7). Then from (7) and (8), we get

$$
\begin{aligned}
&\frac{d}{dt}\left(u(x_0,y_0,t)+\int_0^1\int_0^1 K(x,y)u(x,y,t)\,dx\,dy-h(t)\right)\\
&= a(t)\left(u(x_0,y_0,t)+\int_0^1\int_0^1 K(x,y)u(x,y,t)\,dx\,dy-h(t)\right), \qquad 0\le t\le T.
\end{aligned}
\tag{10}
$$

Using (2) and the compatibility condition (6), we obtain the following relation

$$
\begin{aligned}
&u(x_0,y_0,0)+\int_0^1\int_0^1 K(x,y)u(x,y,0)\,dx\,dy-h(0)\\
&+\delta\left(u(x_0,y_0,T)+\int_0^1\int_0^1 K(x,y)u(x,y,T)\,dx\,dy-h(T)\right)\\
&= u(x_0,y_0,0)+\delta u(x_0,y_0,T)+\int_0^1\int_0^1 K(x,y)(u(x,y,0)+u(x,y,T))\,dx\,dy\\
&-(h(0)+\delta h(T)) = \varphi(x_0,y_0)+\int_0^1\int_0^1 K(x,y)\varphi(x,y)\,dx\,dy-(h(0)+\delta h(T))=0.
\end{aligned}
\tag{11}
$$

It is clear that the general solution of equation (10) has the form

$$
u(x_0,y_0,t)+\int_0^1\int_0^1 K(x,y)u(x,y,t)\,dx\,dy-h(t)=ce^{\int_0^t a(\tau)d\tau}, \tag{12}
$$

where c is an arbitrary constant.

Hence, using (11), we find

$$
c\left(1+\delta e^{-\int_0^T a(\tau)d\tau}\right)=0. \tag{13}
$$

By virtue of $\delta \geq 0$, from (13), we obtain that $c = 0$. Setting $c = 0$ in (12), we conclude that

$$u(x_0, y_0, t) + \int_0^1 \int_0^1 K(x,y)u(x,y,t)\,dx\,dy - h(t) = 0, \qquad 0 \leq t \leq T.$$

Hence, the condition (5) is satisfied. The proof is complete. $\square$

2 Solvability of the inverse boundary-value problem

We seek the first component $u(x,y,t)$ of classical solution $\{u(x,y,t), a(t)\}$ of the problem (1)–(4), (7) in the form

$$\begin{gathered} u(x,y,t) = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} u_{k,n}(t) \sin \lambda_k x \cos \gamma_n y, \\ \lambda_k = \frac{\pi}{2}(2k-1), \;\; \gamma_n = \frac{\pi}{2}(2n-1), \;\; k,n = 1,2,\ldots, \end{gathered} \tag{14}$$

where

$$u_{k,n}(t) = 4 \int_0^1 \int_0^1 u(x,y,t) \sin \lambda_k x \cos \gamma_n y\,dx\,dy.$$

Applying the formal scheme of the Fourier method, from (1) and (2), we have

$$u'_{k,n}(t) + (\lambda_k^2 + \gamma_n^2)c(t)u_{k,n}(t) = F_{k,n}(t; u, a), \qquad 0 \leq t \leq T, \tag{15}$$

$$u_{k,n}(0) + \delta u_{k,n}(T) = \varphi_{k,n}, \qquad k,n = 1,2,\ldots, \tag{16}$$

where

$$F_{k,n}(t; u, a) = f_{k,n}(t) + a(t)u_{k,n}(t),$$

$$f_{k,n}(t) = 4 \int_0^1 \int_0^1 f(x,y,t) \sin \lambda_k x \cos \gamma_n y\,dx\,dy, \qquad \varphi_{k,n} = 4 \int_0^1 \int_0^1 \varphi(x,y) \sin \lambda_k x \cos \gamma_n y\,dx\,dy.$$

Solving problem (15), (16), we find

$$\begin{aligned} u_{k,n}(t) = {} & \frac{\varphi_{k,n} e^{-\int_0^t \mu_{k,n}^2 c(s)ds}}{1 + \delta e^{-\int_0^T \mu_{k,n}^2 c(s)ds}} + \int_0^t F_{k,n}(\tau; u, a) e^{-\int_\tau^t \mu_{k,n}^2 c(s)ds}\,d\tau \\ & - \frac{\delta e^{-\int_0^T \mu_{k,n}^2 c(s)ds}}{1 + \delta e^{-\int_0^T \mu_{k,n}^2 c(s)ds}} \int_0^T F_{k,n}(\tau; u, a) e^{-\int_\tau^t \mu_{k,n}^2 c(s)ds}\,d\tau, \end{aligned} \tag{17}$$

where

$$\mu_{k,n}^2 = \lambda_k^2 + \gamma_n^2.$$

Substituting the expressions $u_{k,n}(t)$ $(k,n=1,2,\dots)$ described by (17) into (14), to determine the first component of the solution (1)–(4), (7), we obtain

$$\begin{aligned} u(x,y,t)=\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\Bigg\{ & \frac{\varphi_{k,n}e^{-\int_0^t \mu_{k,n}^2 c(s)ds}}{1+\delta e^{-\int_0^T \mu_{k,n}^2 c(s)ds}}+\int_0^t F_{k,n}(\tau;u,a)e^{-\int_\tau^t \mu_{k,n}^2 c(s)ds}d\tau \\ & -\frac{\delta e^{-\int_0^T \mu_{k,n}^2 c(s)ds}}{1+\delta e^{-\int_0^T \mu_{k,n}^2 c(s)ds}}\int_0^T F_{k,n}(\tau;u,a)e^{-\int_\tau^t \mu_{k,n}^2 c(s)ds}d\tau\Bigg\}\sin\lambda_k x\cos\gamma_n y. \end{aligned} \tag{18}$$

Further from (7), taking into account $h(t)\neq 0$, we get

$$\begin{aligned} a(t)=[h(t)]^{-1}\Bigg\{h'(t)-\Bigg(f(x_0,y_0,t)+\int_0^1\int_0^1 K(x,y)f(x,y,t)\,dx\,dy\Bigg) \\ -c(t)\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}u_{k,n}(t)p_{k,n}\Bigg\}, \end{aligned} \tag{19}$$

where

$$p_{k,n}=\mu_{k,n}^2\Bigg(\sin\lambda_k x_0\cos\gamma_n y_0+\int_0^1\int_0^1 K(x,y)\sin\lambda_k x\cos\gamma_n y\,dx\,dy\Bigg).$$

Next, substituting the expressions $u_{k,n}(t)$ $(k,n=1,2,\dots)$ represented by (17) into (19), to find the second component of the solution (1)–(4), (7), we have

$$\begin{aligned} a(t)=[h(t)]^{-1}\Bigg\{h'(t)-\Bigg(f(x_0,y_0,t)+\int_0^1\int_0^1 K(x,y)f(x,y,t)\,dx\,dy\Bigg) \\ -c(t)\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\Bigg[& \frac{\varphi_{k,n}e^{-\int_0^t \mu_{k,n}^2 c(s)ds}}{1+\delta e^{-\int_0^T \mu_{k,n}^2 c(s)ds}}+\int_0^t F_{k,n}(\tau;u,a)e^{-\int_\tau^t \mu_{k,n}^2 c(s)ds}d\tau \\ & -\frac{\delta e^{-\int_0^T \mu_{k,n}^2 c(s)ds}}{1+\delta e^{-\int_0^T \mu_{k,n}^2 c(s)ds}}\int_0^T F_{k,n}(\tau;u,a)e^{-\int_\tau^t \mu_{k,n}^2 c(s)ds}d\tau\Bigg]p_{k,n}\Bigg\}. \end{aligned} \tag{20}$$

Thus, the solution of problem (1)–(4), (7) was reduced to the solution by systems (18), (20) with respect to unknown functions $u(x,y,t)$ and $a(t)$.

Proceeding from the definition of the solution of the problem (1)–(4), (7) the following statement is proved.

Lemma 1. *If* $\{u(x,y,t),a(t)\}$ *is any solution of* (1)–(4), (7) *then the functions*

$$u_{k,n}(t)=4\int_0^1\int_0^1 u(x,y,t)\sin\lambda_k x\cos\gamma_n y\,dx\,dy,\qquad k,n=1,2,\dots,$$

satisfy the system (17) *on the interval* $[0,T]$.

Proof. Let $\{u(x,y,t), a(t)\}$ be any solution of problem (1)–(4), (7). Multiplying both sides of the equation (1) by function $4\sin\lambda_k x\cos\gamma_n y\ \ (k,n=1,2,\dots)$ and integrating both sides with respect to x and y from 0 to 1, and using relationships

$$4\int_0^1\int_0^1 u_t(x,y,t)\sin\lambda_k x\cos\gamma_n y\,dx\,dy = \frac{d}{dt}\left(4\int_0^1\int_0^1 u(x,y,t)\sin\lambda_k x\cos\gamma_n y\,dx\,dy\right) = u'_{k,n}(t),$$

$$4\int_0^1\int_0^1 (u_{xx}(x,y,t)+u_{yy}(x,y,t))\sin\lambda_k x\cos\gamma_n y\,dx\,dy$$

$$= -(\lambda_k^2+\gamma_n^2)\left(4\int_0^1\int_0^1 u(x,y,t)\sin\lambda_k x\cos\gamma_n y\,dx\,dy\right) = -(\lambda_k^2+\gamma_n^2)u_{k,n}(t),\quad k,n=1,2,\dots,$$

we get that the equation (15) is satisfied.

Similarly, from (2) we obtain that condition (16) is satisfied. Thus, $u_{k,n}(t)$, $k,n=1,2,\dots$, is a solution to the problem (15), (16). Hence, it straightforward follows that the functions $u_{k,n}(t)$, $k,n=1,2,\dots$, satisfy on $[0,T]$ system (17). Thus the lemma is proved. □

Obviously, if

$$u_{k,n}(t) = 4\int_0^1\int_0^1 u(x,y,t)\sin\lambda_k x\cos\gamma_n y\,dx\,dy,\qquad k,n=1,2,\dots,$$

is a solution to system (17), then the functions

$$u(x,y,t) = \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} u_{k,n}(t)\sin\lambda_k x\cos\gamma_n y,$$

and $a(t)$ is a solution of system (18), (20).

From Lemma 1 it follows the next assertion.

Corollary 1. *Suppose that system* (18), (20) *has a unique solution. Then the problem* (1)–(4), (7)*, couldn't have more than one solution, in other words, if problem* (1)–(4), (7) *has a solution, then it is a unique.*

In order to study the problem (1)–(4), (7), we consider the following spaces. Let $B^3_{2,T}$ denote the set of all functions of the form

$$u(x,y,t) = \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} u_{k,n}(t)\sin\lambda_k x\cos\gamma_n y,\quad \lambda_k=\frac{\pi}{2}(2k-1),\quad \gamma_n=\frac{\pi}{2}(2n-1),\quad k,n=1,2,\dots,$$

considered in domain D_T, where the functions $u_{k,n}(t)$, $k,n=1,2,\dots$, are continuous on $[0,T]$, and satisfy the condition

$$\left\{\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}(\mu_{k,n}^3\,\|u_{k,n}(t)\|_{C[0,T]})^2\right\}^{\frac{1}{2}} < +\infty.$$

The norm in the space $B^3_{2,T}$ is defined as follows

$$\|u(x,y,t)\|_{B^3_{2,T}} = \left\{ \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \left(\mu^3_{k,n} \|u_{k,n}(t)\|_{C[0,T]} \right)^2 \right\}^{\frac{1}{2}}.$$

We denote by E^3_T the topological product of $B^3_{2,T} \times C[0,T]$. The norm of the element $z = \{u, a\}$ is determined by the formula

$$\|z\|_{E^3_T} = \|u(x,y,t)\|_{B^3_{2,T}} + \|a(t)\|_{C[0,T]}.$$

It is known [16] that the spaces $B^3_{2,T}$ and E^3_T are Banach spaces.

Now, consider the operator

$$\Phi(u,a) = \{\Phi_1(u,a), \Phi_2(u,a)\}$$

in the space E^3_T, where

$$\Phi_1(u,a) = \tilde{u}(x,y,t) = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \tilde{u}_{k,n}(t) \sin \lambda_k x \cos \gamma_n y,$$
$$\Phi_2(u,a) = \tilde{a}(t),$$

and the functions $\tilde{u}_{k,n}(t)$, $k,n = 1,2,\ldots$, $\tilde{a}(t)$ are equal to the right-hand sides of (17), (20) respectively.

It is easy to see that

$$1 + \delta e^{-\int\limits_0^T \frac{\lambda_k^2 \beta(s)\, ds}{1+\lambda_k^2 \alpha(s)}} \geq 1,$$
$$\mu^3_{k,n} \leq (\lambda_k^2 + \gamma_k^2)(\lambda_k + \gamma_n) = \lambda_k^3 + \lambda_k^2 \gamma_n + \gamma_k^2 \lambda_k + \gamma_n^3,$$
$$\int\limits_0^T |f_{k,n}(\tau)|\, d\tau \leq \sqrt{T} \left(\int\limits_0^T |f_{k,n}(\tau)|^2\, d\tau \right)^{\frac{1}{2}},$$
$$|p_{k,n}| = \left(1 + \int\limits_0^1 \int\limits_0^1 |K(x,y)|\, dx\, dy \right) \mu^2_{k,n} \equiv p \mu^2_{k,n}.$$

Taking into consideration these relations, we have

$$\begin{gathered}
\left\{ \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \left(\mu^3_{k,n} \|\tilde{u}_{k,n}(t)\|_{C[0,T]} \right)^2 \right\}^{\frac{1}{2}} \leq 3 \left(\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \left(\lambda_k^3 |\varphi_{k,n}| \right)^2 \right)^{\frac{1}{2}} \\
+ 3 \left(\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} (\lambda_k^2 \gamma_n |\varphi_{k,n}|)^2 \right)^{\frac{1}{2}} + 3 \left(\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} (\lambda_k \gamma_n^2 |\varphi_{k,n}|)^2 \right)^{\frac{1}{2}} + 3 \left(\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} (\gamma_n^3 |\varphi_{k,n}|)^2 \right)^{\frac{1}{2}} \\
+ 3(1+\delta) \left[\sqrt{T} \left(\left(\int\limits_0^T \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} (\lambda_k^3 |f_{k,n}(\tau)|)^2 d\tau \right)^{\frac{1}{2}} + \left(\int\limits_0^T \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} (\lambda_k^2 \gamma_n |f_{k,n}(\tau)|)^2 d\tau \right)^{\frac{1}{2}} \right. \right. \\
\left. + \left(\int\limits_0^T \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} (\lambda_k \gamma_n^2 |f_{k,n}(\tau)|)^2 d\tau \right)^{\frac{1}{2}} + \left(\int\limits_0^T \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} (\gamma_n^3 |f_{k,n}(\tau)|)^2 d\tau \right)^{\frac{1}{2}} \right) \\
\left. + T \|a(t)\|_{C[0,T]} \left(\sum_{k=1}^{\infty} \sum_{k=1}^{\infty} (\mu^3_{k,n} \|u_{k,n}(t)\|_{C[0,T]})^2 \right)^{\frac{1}{2}} \right],
\end{gathered} \tag{21}$$

$$
\begin{aligned}
&\|\tilde{a}(t)\|_{C[0,T]} \\
&\leq \left\| [h(t)]^{-1} \right\|_{C[0,T]} \left\{ \left\| h'(t) - \left(f(x_0, y_0, t) + \int_0^1 \int_0^1 K(x,y) f(x,y,t)\, dx\, dy \right) \right\|_{C[0,T]} \right. \\
&+ p \, \|c(t)\|_{C[0,T]} \left(\sum_{k=1}^{\infty} \sum_{k=1}^{\infty} \mu_k^{-2} \right)^{\frac{1}{2}} \left[\left(\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} (\lambda_k^3 \, |\varphi_{k,n}|)^2 \right)^{\frac{1}{2}} + \left(\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} (\lambda_k^2 \gamma_n \, |\varphi_{k,n}|)^2 \right)^{\frac{1}{2}} \right. \\
&+ \left(\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} (\lambda_k \gamma_n^2 \, |\varphi_{k,n}|)^2 \right)^{\frac{1}{2}} + \left(\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} (\gamma_n^3 \, |\varphi_{k,n}|)^2 \right)^{\frac{1}{2}} \\
&+ (1+\delta)\sqrt{T} \left(\left(\int_0^T \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} (\lambda_k^3 \, |f_{k,n}(\tau)|)^2 d\tau \right)^{\frac{1}{2}} + \left(\int_0^T \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} (\lambda_k^2 \gamma_n \, |f_{k,n}(\tau)|)^2 d\tau \right)^{\frac{1}{2}} \right. \\
&+ \left. \left(\int_0^T \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} (\lambda_k \gamma_n^2 \, |f_{k,n}(\tau)|)^2 d\tau \right)^{\frac{1}{2}} + \left(\int_0^T \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} (\gamma_n^3 \, |f_{k,n}(\tau)|)^2 d\tau \right)^{\frac{1}{2}} \right) \\
&+ \left. \left. (1+\delta) T \, \|a(t)\|_{C[0,T]} \left(\sum_{k=1}^{\infty} \sum_{k=1}^{\infty} (\mu_{k,n}^3 \, \|u_{k,n}(t)\|_{C[0,T]})^2 \right)^{\frac{1}{2}} \right] \right\}.
\end{aligned} \tag{22}
$$

Assume that the data for the problem (1)–(4), (7) satisfy the following conditions:

(A) $\varphi(x,y),\ \varphi_x(x,y),\ \varphi_{xx}(x,y),\ \varphi_y(x,y),\ \varphi_{xy}(x,y),\ \varphi_{yy}(x,y) \in C(\bar{Q}_{xy})$,
$\varphi_{xxy}(x,y),\ \varphi_{xyy}(x,y),\ \varphi_{xxx}(x,y),\ \varphi_{yyy}(x,y) \in L_2(Q_{xy})$,
$\varphi(0,y) = \varphi_x(1,y) = \varphi_{xx}(0,y) = 0,\ 0 \leq y \leq 1$,
$\varphi_y(x,0) = \varphi(x,1) = \varphi_{yy}(x,1) = 0,\ 0 \leq x \leq 1$;

(B) $f(x,y,t),\ f_x(x,y,t),\ f_{xx}(x,y,t),\ f_y(x,y,t),\ f_{xy}(x,y,t),\ f_{xy}(x,y,t) \in C(D_T)$,
$f_{xxy}(x,y,t),\ f_{xyy}(x,y,t),\ f_{xxx}(x,y,t),\ f_{yyy}(x,y,t) \in L_2(D_T)$,
$f(0,y,t) = f_x(1,y,t) = f_{xx}(0,y,t) = 0,\ 0 \leq y \leq 1,\ 0 \leq t \leq T$,
$f_y(x,0,t) = f(x,1,t) = f_{yy}(x,1,t) = 0,\ 0 \leq x \leq 1,\ 0 \leq t \leq T$;

(C) $\delta \geq 0,\ K(x,y) \in L_1(Q_{xy}),\ 0 < c(t) \in C[0,T],\ h(t) \in C^1[0,T],\ h(t) \neq 0,\ 0 \leq t \leq T$.

Then, from (21) and (22), respectively, we obtain

$$\|\tilde{u}(x,y,t)\|_{B^3_{2,T}} \leq A_1(T) + B_1(T) \, \|a(t)\|_{C[0,T]} \, \|u(x,y,t)\|_{B^3_{2,T}}, \tag{23}$$

$$\|\tilde{a}(t)\|_{C[0,T]} \leq A_2(T) + B_2(T) \, \|a(t)\|_{C[0,T]} \, \|u(x,y,t)\|_{B^3_{2,T}}, \tag{24}$$

where

$$
\begin{aligned}
A_1(T) = 3 \, \|\varphi_{xxx}(x,y)\|_{L_2(Q_{xy})} &+ 3 \left\| \varphi_{xyy}(x,y) \right\|_{L_2(Q_{xy})} + 3 \left\| \varphi_{xxy}(x,y) \right\|_{L_2(Q_{xy})} \\
&+ 3 \left\| \varphi_{yyy}(x,y) \right\|_{L_2(Q_{xy})} + (1+\delta)\sqrt{T} \, 3 \Big(\, \|f_{xxx}(x,y,t)\|_{L_2(D_T)} \\
&+ 3 \left\| f_{xyy}(x,y,t) \right\|_{L_2(D_T)} + 3 \left\| f_{xxy}(x,y,t) \right\|_{L_2(D_T)} \\
&+ 3 \, \|f_{xxx}(x,y,t)\|_{L_2(D_T)} + 3 \left\| f_{yyy}(x,y,t) \right\|_{L_2(D_T)} \Big), \\
B_1(T) = 3(1+\delta)T&,
\end{aligned}
$$

$$
\begin{aligned}
A_2(T) = \left\| [h(t)]^{-1} \right\|_{C[0,T]} \Bigg\{ \Bigg\| (h'(t) - \Bigg(f(x_0, y_0, t) + \int\limits_0^1 \int\limits_0^1 K(x,y) f(x,y,t)\, dx\, dy \Bigg) \Bigg\|_{C[0,T]} \\
+ p \left\| c(t) \right\|_{C[0,T]} \left(\sum_{k=1}^{\infty} \sum_{k=1}^{\infty} \mu_k^{-2} \right)^{\frac{1}{2}} \Bigg[\left\| \varphi_{xxx}(x,y) \right\|_{L_2(Q_{xy})} + \left\| \varphi_{xyy}(x,y) \right\|_{L_2(Q_{xy})} \\
+ \left\| \varphi_{xxy}(x,y) \right\|_{L_2(Q_{xy})} + \left\| \varphi_{yyy}(x,y) \right\|_{L_2(Q_{xy})} + (1+\delta)\sqrt{T} \Big(\left\| f_{xxx}(x,y,t) \right\|_{L_2(D_T)} \\
+ \left\| f_{xyy}(x,y,t) \right\|_{L_2(D_T)} + \left\| f_{xxy}(x,y,t) \right\|_{L_2(D_T)} \\
+ \left\| f_{xxx}(x,y,t) \right\|_{L_2(D_T)} + \left\| f_{yyy}(x,y,t) \right\|_{L_2(D_T)} \Big) \Bigg] \Bigg\}, \\
B_2(T) = \left\| [h(t)]^{-1} \right\|_{C[0,T]} p \left\| c(t) \right\|_{C[0,T]} \left(\sum_{k=1}^{\infty} \sum_{k=1}^{\infty} \mu_k^{-2} \right)^{\frac{1}{2}} (1+\delta) T.
\end{aligned}
$$

From inequalities (23) and (24) we conclude

$$
\left\| \tilde{u}(x,y,t) \right\|_{B^3_{2,T}} + \left\| \tilde{a}(t) \right\|_{C[0,T]} \leq A(T) + B(T) \left\| a(t) \right\|_{C[0,T]} \left\| u(x,t) \right\|_{B^3_{2,T}}, \tag{25}
$$

where

$$
A(T) = A_1(T) + A_2(T), \qquad B(T) = B_1(T) + B_2(T).
$$

Let K_R denote the closed ball of radius $R = A(T) + 2$ centered at zero in E^3_T.

Theorem 2. *Let the conditions (A)–(C) and the condition*

$$
B(T)(A(T)+2)^2 < 1 \tag{26}
$$

be fulfilled. Then problem (1)–(4), (7) *has a unique solution in the ball* K_R.

Proof. Let us consider in the space E^3_T the equation

$$
z = \Phi z, \tag{27}
$$

where $z = \{u, a\}$. The components $\Phi_i(u,a)$, $i = 1,2$, of operator $\Phi(u,a)$ defined by the right side of equations (18), (20), respectively. Now, consider the operator $\Phi(u,a)$ in the ball K_R of the space E^3_T.

Similar to (25) we obtain that for any $z, z_1, z_2 \in K_R$ the following inequalities hold

$$
\left\| \Phi z \right\|_{E^3_T} \leq A(T) + B(T) \left\| p(t) \right\|_{C[0,T]} \left\| u(x,y,t) \right\|_{B^3_{2,T}} \leq A(T) + B(T)(A(T)+2)^2, \tag{28}
$$

$$
\left\| \Phi z_1 - \Phi z_2 \right\|_{E^3_T} \leq B(T) R \Big(\left\| u_1(x,y,t) - u_2(x,y,t) \right\|_{B^3_{2,T}} + \left\| a_1(t) - a_2(t) \right\|_{C[0,T]} \Big). \tag{29}
$$

Then by (26), from estimates (28) and (29) it is clear that the operator Φz acts in a ball K_R and satisfy the conditions of the contraction mapping principle. Therefore the operator Φz has a unique fixed point $\{u, a\}$ in the ball K_R, which is a unique solution of equation (27), i.e., $\{u, a\}$ is a unique solution of the systems (18), (20) in the ball K_R.

The function $u(x,y,t)$ as an element of the space E^3_T is continuous and has continuous derivatives $u_x(x,y,t)$, $u_{xx}(x,y,t)$, $u_y(x,y,t)$, $u_{xy}(x,y,t)$, $u_{yy}(x,y,t)$ in D_T.

From the equation (15) it is clear that

$$u'_{k,n}(t) + (\lambda_k^2 + \gamma_n^2)c(t)u_{k,n}(t) = F_{k,n}(t; u, a), \quad 0 \le t \le T,$$

$$\left\{ \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \left(\mu_{k,n} \| u'_{k,n}(t) \|_{C[0,T]} \right)^2 \right\}^{\frac{1}{2}} \le \sqrt{2} \, \|c(t)\|_{C[0,T]} \, \|u(x,y,t)\|_{B^3_{2,T}}$$
$$+ \left\| \| f_x(x,y,t) + f_y(x,y,t) + p(t)(u_x(x,y,t) + u_y(x,y,t)) \|_{C[0,T]} \right\|_{L_2(Q_{xy})}.$$

Thus $u_t(x,y,t)$ is continuous in D_T.

It is not hard to verify that equation (1) and conditions (2)–(4), (7) are satisfied in the usual sense. Thus, the solution of the problem (1)–(4), (7) is a pair of functions $\{u(x,t), a(t)\}$. By virtue of the Lemma 1, it is unique in the ball K_R. Theorem has been proved. $\square$

Thus, by Theorem 1 and Theorem 2, we arrive at the following main result.

Theorem 3. *Assume that all conditions of Theorem 2 and compatibility condition*

$$\varphi(x_0, y_0) + \int_0^1 \int_0^1 K(x,y)\varphi(x,y)\, dx\, dy = h(0) + \delta h(T)$$

hold. Then problem (1)–(5) *has a unique classical solution in the ball* K_R *for sufficiently small values of* T.

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Азізбайов Е.І., Мехралієв Ю.Т. *Нелокальна обернена крайова задача для двовимірного параболічного рівняння з інтегральною переозначеною умовою* // Карпатські матем. публ. — 2020. — Т.12, №1. — С. 23–33.

В роботі досліджено нелокальну обернену крайову задачу для двовимірного параболічного рівняння другого порядку у прямокутній області. Метою цієї статті є визначення невідомого коефіцієнта та розв'язку вказаної задачі. Щоб дослідити розв'язність оберненої задачі, ми перетворюємо оригінальну задачу у деяку допоміжну задачу з тривіальними крайовими умовами. Використовуючи принцип стискаючих відображень, доведено існування і єдиність розв'язку для еквівалентної задачі. Використовуючи еквівалентність, отримано теорему про існування і єдиність класичного розв'язку оригінальної задачі.

Ключові слова і фрази: обернена задача, двовимірне параболічне рівняння, метод Фур'є, класичний розв'язок, переозначена умова.

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MERSENNE-HORADAM IDENTITIES USING GENERATING FUNCTIONS

The main object of the present paper is to reveal connections between Mersenne numbers $M_n = 2^n - 1$ and generalized Fibonacci (i.e., Horadam) numbers w_n defined by a second order linear recurrence $w_n = pw_{n-1} + qw_{n-2}$, $n \geq 2$, with $w_0 = a$ and $w_1 = b$, where a, b, $p > 0$ and $q \neq 0$ are integers. This is achieved by relating the respective (ordinary and exponential) generating functions to each other. Several explicit examples involving Fibonacci, Lucas, Pell, Jacobsthal and balancing numbers are stated to highlight the results.

Key words and phrases: Mersenne numbers, Horadam sequence, Fibonacci sequence, Lucas sequence, Pell sequence, generating function, binomial transform.

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INTRODUCTION

A *generalized Fibonacci sequence* $(w_n)_{n\geq 0} = (w_n(a, b; p, q))_{n\geq 0}$ is defined by a second order homogeneous linear recurrence

$$w_n = pw_{n-1} + qw_{n-2}, \qquad n \geq 2,$$

with $w_0 = a$ and $w_1 = b$, where a, b, p and q are integers with $p > 0$, $q \neq 0$. Since these numbers were first studied by A.F. Horadam (see, e.g., [11, 12]), they are often referred to as *Horadam numbers*. The Binet formula for w_n is given by [11]

$$w_n = \alpha r_1^n + \beta r_2^n, \qquad n \geq 0,$$

where $r_1 = \frac{p+\sqrt{p^2+4q}}{2}$ and $r_2 = \frac{p-\sqrt{p^2+4q}}{2}$ denote the distinct roots of the quadratic equation $x^2 - px - q = 0$,

$$\alpha = \frac{a}{2} + \frac{2b - ap}{2\sqrt{p^2 + 4q}} \quad \text{and} \quad \beta = \frac{a}{2} - \frac{2b - ap}{2\sqrt{p^2 + 4q}}.$$

It is worth noticing that an equivalent version of the Binet formula is given by

$$w_n = b\frac{r_1^n - r_2^n}{r_1 - r_2} + aq\frac{r_1^{n-1} - r_2^{n-1}}{r_1 - r_2}, \qquad n \geq 1.$$

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The sequence can be extended to negative subscripts according to

$$w_{-n} = -\frac{1}{q}(pw_{-n+1} - w_{-n+2}), \qquad n \geq 1.$$

Further results on Horadam sequences can be found in the survey paper [14]. In what follows, we will make frequent use of the generating functions of $(w_n)_{n\geq 0}$. We know [15] that the sequence w_n has the ordinary (non-exponential) generating function

$$w(z) = \sum_{n=0}^{\infty} w_n z^n = \frac{a + (b - ap)z}{1 - pz - qz^2}, \tag{1}$$

while for sequences w_{2n+1} and w_{2n}

$$w_1(z) = \sum_{n=0}^{\infty} w_{2n+1} z^n = \frac{a + (bp - qa - ap^2)z}{1 - (p^2 + 2q)z + q^2z^2}, \tag{2}$$

$$w_2(z) = \sum_{n=0}^{\infty} w_{2n} z^n = \frac{b + (apq - bq)z}{1 - (p^2 + 2q)z + q^2z^2}. \tag{3}$$

The Horadam sequence generalizes many other number and polynomial sequences, for instance, the *Fibonacci sequence* $F_n = w_n(0, 1; 1, 1)$, the *Lucas sequence* $L_n = w_n(2, 1; 1, 1)$, the *Pell sequence* $P_n = w_n(0, 1; 2, 1)$, the *Jacobsthal sequence* $J_n = w_n(0, 1; 1, 2)$, the *Mersenne sequence* $M_n = w_n(0, 1; 3, -2)$, the *balancing numbers* $B_n = w_n(0, 1; 6, -1)$, and so on. The first few terms of each sequence are stated below.

n	0	1	2	3	4	5	6	7	8	9	10	11
F_n	0	1	1	2	3	5	8	13	21	34	55	89
L_n	2	1	3	4	7	11	18	29	47	76	123	199
P_n	0	1	2	5	12	29	70	169	408	985	2378	5741
J_n	0	1	1	3	5	11	21	43	85	171	341	683
M_n	0	1	3	7	15	31	63	127	255	511	1023	2047
B_n	0	1	6	35	204	1189	6930	40391	235416	1372105	7997214	46611179

The sequences $(F_n)_{n\geq 0}$, $(L_n)_{n\geq 0}$, $(P_n)_{n\geq 0}$, $(J_n)_{n\geq 0}$, $(M_n)_{n\geq 0}$ and $(B_n)_{n\geq 0}$ are indexed in the On-Line Encyclopedia of Integer Sequences [19] (see entries A000045, A000032, A000129, A001045, A000225 and A001109, respectively).

In the present paper, we derive some connection formulas between Mersenne numbers and the Horadam sequence.

Recall that Mersenne numbers M_n belong to the Horadam sequence family. They are given by the explicit form

$$M_n = 2^n - 1, \quad n \geq 0.$$

Mersenne numbers are popular research objects because of their interesting properties. For instance, Mersenne numbers are numbers with the following representation in the binary system: $(1)_2$, $(11)_2$, $(111)_2$, $(1111)_2$, $(11111)_2$, ... Also, the Mersenne number sequence contains primes, the so called *Mersenne primes* of the form $2^n - 1$. A simple calculation shows that if M_n is a prime number, then n is a prime number, though not all M_n are prime. Mersenne primes are also connected to perfect numbers. The search for Mersenne primes is an active field of research (see [18], among others). More information about Mersenne numbers and its generalizations can be taken from the papers [1, 3–6, 9, 10, 13, 16, 20] and references contained therein.

We conclude this section with some generating functions, which will be needed in the proofs. Using (1)–(3) we easily obtain non-exponential generating functions of the sequences M_n, M_{2n+1} and M_{2n} as follows

$$m(z) = \sum_{n=0}^{\infty} M_n z^n = \frac{z}{1-3z+2z^2}, \tag{4}$$

$$m_1(z) = \sum_{n=0}^{\infty} M_{2n+1} z^n = \frac{1+2z}{1-5z+4z^2}, \tag{5}$$

$$m_2(z) = \sum_{n=0}^{\infty} M_{2n} z^n = \frac{3z}{1-5z+4z^2}. \tag{6}$$

Finally, the exponential generating functions of the sequences M_n, M_{2n+1} and M_{2n} can be derived as

$$\mu(z) = \sum_{n=0}^{\infty} M_n \frac{z^n}{n!} = 2e^{\frac{3z}{2}} \sinh\left(\frac{z}{2}\right), \tag{7}$$

$$\mu_1(z) = \sum_{n=0}^{\infty} M_{2n+1} \frac{z^n}{n!} = 2e^{\frac{5z}{2}} \sinh\left(\frac{3z}{2}\right) + e^{4z}, \quad \mu_2(z) = \sum_{n=0}^{\infty} M_{2n} \frac{z^n}{n!} = 2e^{\frac{5z}{2}} \sinh\left(\frac{3z}{2}\right).$$

1 MERSENNE-HORADAM IDENTITIES USING ORDINARY GENERATING FUNCTIONS

Our first result provides a relation between Mersenne and Horadam numbers using its ordinary generating functions. The method of proof is the same as in [7] and [8]. We note that, in what follows, we will used the standard convention that $\sum_{k=0}^{n} a_k = 0$ for $n < 0$.

Theorem 1. *For $n \geq 0$, the following formula holds*

$$w_n = a + (b-a)M_n + \sum_{k=1}^{n-1} \big((p-3)w_{n-k} + (q+2)w_{n-k-1}\big) M_k.$$

Proof. By (1) and (4), we get

$$\frac{z}{m(z)} = 1 - 3z + 2z^2 = (1 - pz - qz^2) + (pz + qz^2 - 3z + 2z^2) = \frac{a + (b-ap)z}{w(z)}$$
$$+ (p-3)z + (q+2)z^2 = \frac{a + (b-ap)z + (p-3)zw(z) + (q+2)z^2 w(z)}{w(z)},$$

and thus $zw(z) = am(z) + (b-ap)zm(z) + (p-3)zw(z)m(z) + (q+2)z^2 w(z)m(z)$.

Expanding both sides of the last equation as a power series in z yields

$$z \sum_{n=0}^{\infty} w_n z^n = a \sum_{n=0}^{\infty} M_n z^n + (b-ap) \sum_{n=0}^{\infty} M_n z^{n+1}$$
$$+ (p-3)z \sum_{n=0}^{\infty} w_n z^n \sum_{n=0}^{\infty} M_n z^n + (q+2)z^2 \sum_{n=0}^{\infty} w_n z^n \sum_{n=0}^{\infty} M_n z^n.$$

Using the formula for multiplication of two power series

$$\sum_{n=0}^{\infty} a_n z^n \sum_{n=0}^{\infty} b_n z^n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_k b_{n-k} z^n, \tag{8}$$

we then obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} w_n z^{n+1} &= a \sum_{n=0}^{\infty} M_n z^n + (b-ap) \sum_{n=1}^{\infty} M_{n-1} z^n \\
&\quad + (p-3) \sum_{n=0}^{\infty} \sum_{k=0}^{n} w_{n-k} M_k z^{n+1} + (q+2) \sum_{n=0}^{\infty} \sum_{k=0}^{n} w_{n-k} M_k z^{n+2}, \\
az + \sum_{n=2}^{\infty} w_{n-1} z^n &= az + a \sum_{n=2}^{\infty} M_n z^n + (b-ap) \sum_{n=2}^{\infty} M_{n-1} z^n \\
&\quad + (p-3) \sum_{n=2}^{\infty} \sum_{k=0}^{n-1} w_{n-k-1} M_k z^n + (q+2) \sum_{n=2}^{\infty} \sum_{k=0}^{n-2} w_{n-k-2} M_k z^n.
\end{aligned}$$

Comparing the coefficients on both sides, we obtain

$$\begin{aligned}
w_n &= aM_{n+1} + (b-ap)M_n + (p-3) \sum_{k=0}^{n} w_{n-k} M_k + (q+2) \sum_{k=0}^{n-1} w_{n-k-1} M_k \\
&= a(2M_n+1) + (b-ap)M_n + (p-3) \sum_{k=0}^{n-1} w_{n-k} M_k + (p-3)aM_n + (q+2) \sum_{k=0}^{n-1} w_{n-k-1} M_k \\
&= a + (b-a)M_n + \sum_{k=1}^{n-1} \big((p-3)w_{n-k} + (q+2)w_{n-k-1}\big) M_k,
\end{aligned}$$

as desired. □

Example 1. *By choosing suitable values on a, b, p and q, one can obtain the following identities valid for $n \geq 0$:*

$$\begin{aligned}
F_n &= M_n - \sum_{k=1}^{n-1} (2F_{n-k} - 3F_{n-k-1}) M_k, \quad L_n = 2 - M_n - \sum_{k=1}^{n-1} (2L_{n-k} - 3L_{n-k-1}) M_k, \\
P_n &= M_n - \sum_{k=1}^{n-1} (P_{n-k} - 3P_{n-k-1}) M_k, \quad J_n = M_n - 2 \sum_{k=1}^{n-1} (J_{n-k} - 2J_{n-k-1}) M_k, \\
B_n &= M_n + \sum_{k=1}^{n-1} (3B_{n-k} + B_{n-k-1}) M_k.
\end{aligned}$$

In a similar manner, we can use the generating functions (2), (5) and (3), (6), respectively, to prove two other relations between odd (even) indexed Horadam and Mersenne numbers. These relations are contained in the next two theorems, those proofs we leave to the reader.

Theorem 2. *For $n \geq 1$, the following formula hold*

$$\begin{aligned}
w_{2n+1} + 2w_{2n-1} &= 3b + (bp^2 + apq + bq - b)M_{2n-1} \\
&\quad + \sum_{k=1}^{n-1} \big((p^2 + 2q - 5)pw_{2(n-k)} + (q^2 + qp^2 - 5q + 4)w_{2(n-k)-1}\big) M_{2k-1}.
\end{aligned} \tag{9}$$

Example 2. *Formula* (9) *yields*

$$
\begin{aligned}
F_{2n+1}+2F_{2n-1} &= 3+M_{2n-1}-\sum_{k=1}^{n-1}\Big(2F_{2(n-k)}-F_{2(n-k)-1}\Big)M_{2k-1},\\
L_{2n+1}+2L_{2n-1} &= 3+3M_{2n-1}-\sum_{k=1}^{n-1}\Big(2L_{2(n-k)}-L_{2(n-k)-1}\Big)M_{2k-1},\\
P_{2n+1}+2P_{2n-1} &= 3+4M_{2n-1}+2\sum_{k=1}^{n-1}\Big(P_{2(n-k)}+2P_{2(n-k)-1}\Big)M_{2k-1},\\
J_{2n+1}+2J_{2n-1} &= 2+M_{2n},\\
B_{2n+1}+2B_{2n-1} &= 3+34M_{2n-1}+2\sum_{k=1}^{n-1}\Big(87B_{2(n-k)}-13B_{2(n-k)-1}\Big)M_{2k-1}.
\end{aligned}
$$

Theorem 3. *For $n \geq 0$, the following formulas hold*

$$
w_{2n}=a+\frac{pb+qa-a}{3}M_{2n}+\frac{1}{3}\sum_{k=1}^{n-1}\big((p^2+2q-5)w_{2(n-k)}+(4-q^2)w_{2(n-k-1)}\big)M_{2k}. \tag{10}
$$

Example 3. *It follows from* (10) *that*

$$
\begin{aligned}
F_{2n} &= \frac{1}{3}M_{2n}-\frac{1}{3}\sum_{k=1}^{n-1}\Big(2F_{2(n-k)}-3F_{2(n-k-1)}\Big)M_{2k},\\
L_{2n} &= 2+\frac{1}{3}M_{2n}-\frac{1}{3}\sum_{k=1}^{n-1}\Big(2L_{2(n-k)}-3L_{2(n-k-1)}\Big)M_{2k},\\
P_{2n} &= \frac{2}{3}M_{2n}+\frac{1}{3}\sum_{k=1}^{n-1}\Big(P_{2(n-k)}+3P_{2(n-k-1)}\Big)M_{2k},\\
J_{2n} &= \frac{1}{3}M_{2n}, \qquad (11)\\
B_{2n} &= 2M_{2n}+\frac{1}{3}\sum_{k=1}^{n-1}\Big(29B_{2(n-k)}+3B_{2(n-k-1)}\Big)M_{2k}.
\end{aligned}
$$

Note that formula (11) is known (see [4]).

We finally remark, that Theorems 1, 2 and 3 can be generalized to sums of certain products of w_n and M_n; see [7] and [8] for details.

2 MERSENNE-HORADAM IDENTITIES VIA EXPONENTIAL GENERATING FUNCTIONS

Let us first consider the fundamental Fibonacci sequence $u_n = w_n(0, b; p, q)$. In this section, we derive connection formulas between u_n and Mersenne numbers M_n involving binomial coefficients.

Let $u(z)$, $u_1(z)$ and $u_2(z)$ be the exponential generating function of the sequences u_n, u_{2n+1} and u_{2n}. Then we have

$$
\begin{aligned}
u(z) &= \sum_{n=0}^{\infty} u_n \frac{z^n}{n!} = \frac{2b}{\Delta} e^{\frac{pz}{2}} \sinh\left(\frac{\Delta z}{2}\right), \qquad (12)\\
u_1(z) &= \sum_{n=0}^{\infty} u_{2n+1} \frac{z^n}{n!} = \frac{2b(p^2+q)}{p\Delta} e^{\frac{p^2+q}{2}z} \sinh\left(\frac{p\Delta z}{2}\right)\\
&\quad - \frac{b}{2p\Delta}\left((p^2+2q-p\Delta)e^{\frac{p^2+2q+p\Delta}{2}z} - (p^2+2q+p\Delta)e^{\frac{p^2+2q-p\Delta}{2}z}\right),\\
u_2(z) &= \sum_{n=0}^{\infty} u_{2n} \frac{z^n}{n!} = \frac{2b}{\Delta} e^{\frac{p^2+2q}{2}z} \sinh\left(\frac{p\Delta z}{2}\right),
\end{aligned}
$$

where $\Delta = \sqrt{p^2+4q}$; see [15].

Theorem 4. *For $n \geq 0$, the following identity holds*

$$
u_n = \frac{b}{\Delta}\left(\frac{p-3\Delta}{2}\right)^n \sum_{k=0}^{n} \binom{n}{k} \left(\frac{2\Delta}{p-3\Delta}\right)^k M_k. \qquad (13)
$$

Proof. Using (7) and (12), we have

$$
u\left(\frac{z}{\Delta}\right) = \frac{b}{\Delta}\mu(z) e^{\left(\frac{p}{2\Delta}-\frac{3}{2}\right)z}.
$$

From the formula above we now obtain

$$
\begin{aligned}
\sum_{n=0}^{\infty} u_n \frac{z^n}{\Delta^n n!} &= \frac{b}{\Delta}\sum_{n=0}^{\infty} M_n \frac{z^n}{n!} \cdot \sum_{n=0}^{\infty}\left(\frac{p}{2\Delta}-\frac{3}{2}\right)^n \frac{z^n}{n!} = \frac{b}{\Delta}\sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{M_k}{k!}\left(\frac{p}{2\Delta}-\frac{3}{2}\right)^{n-k} \frac{z^n}{(n-k)!}\\
&= \frac{b}{\Delta}\sum_{n=0}^{\infty}\sum_{k=0}^{n} \binom{n}{k}\left(\frac{p}{2\Delta}-\frac{3}{2}\right)^{n-k} M_k \frac{z^n}{n!},
\end{aligned}
$$

and after simplification we have (13). □

Example 4. *Let $n \geq 0$. Then formula* (13) *gives*

$$
F_n = \frac{1}{\sqrt{5}}\left(\frac{1-3\sqrt{5}}{2}\right)^n \sum_{k=0}^{n}\binom{n}{k}\left(-\frac{15+\sqrt{5}}{22}\right)^k M_k, \quad J_n = \frac{(-4)^n}{3}\sum_{k=0}^{n}\binom{n}{k}\left(-\frac{3}{4}\right)^k M_k,
$$

$$
P_n = \frac{(1-3\sqrt{2})^n}{2\sqrt{2}}\sum_{k=0}^{n}\binom{n}{k}\left(-\frac{12+2\sqrt{2}}{17}\right)^k M_k, \quad B_n = \frac{(3-6\sqrt{2})^n}{4\sqrt{2}}\sum_{k=0}^{n}\binom{n}{k}\left(-\frac{16+4\sqrt{2}}{21}\right)^k M_k.
$$

Theorem 4 highlights the following issue. If we define the sequence a_n as

$$
a_n = \left(\frac{2\Delta}{p-3\Delta}\right)^n M_n,
$$

then the sequence

$$
b_n = \frac{\Delta}{b}\left(\frac{p-3\Delta}{2}\right)^{-n} u_n,
$$

is the binomial transform of a_n, where the binomial transform and its inverse transform are given by [2, 17]

$$
b_n = \sum_{k=0}^{n}\binom{n}{k} a_k \qquad \Leftrightarrow \qquad a_n = \sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k} b_k.
$$

The inverse relation immediately gives the next identity.

Theorem 5. *For $n \geq 0$, we have*

$$M_n = \frac{\Delta^{1-n}}{b}\left(\frac{3\Delta - p}{2}\right)^n \sum_{k=0}^{n}\binom{n}{k}\left(\frac{2}{3\Delta - p}\right)^k u_k. \tag{14}$$

Example 5. *Formula* (14) *yields*

$$M_n = \sqrt{5}\left(\frac{15-\sqrt{5}}{10}\right)^n \sum_{k=0}^{n}\binom{n}{k}\left(\frac{1+3\sqrt{5}}{22}\right)^k F_k, \tag{15}$$

$$M_n = 3\left(\frac{4}{3}\right)^n \sum_{k=1}^{n}\binom{n}{k}\frac{J_k}{4^k}, \quad M_n = \sqrt{8}\left(\frac{6-\sqrt{2}}{4}\right)^n \sum_{k=0}^{n}\binom{n}{k}\left(\frac{1+3\sqrt{2}}{17}\right)^k P_k,$$

$$M_n = 4\sqrt{2}\left(\frac{12-3\sqrt{2}}{8}\right)^n \sum_{k=0}^{n}\binom{n}{k}\left(\frac{1+2\sqrt{2}}{21}\right)^k B_k.$$

Note that formula (15) may be rewritten in terms of the golden ratio $\varphi = \frac{1+\sqrt{5}}{2}$ as follows

$$M_n = \left(\frac{\varphi^2+2}{(2\varphi-1)\varphi}\right)^n \sum_{k=1}^{n}\binom{n}{k}\frac{\varphi^{2k}-(-1)^k}{(\varphi^2+2)^k}.$$

We also have the following summation identity.

Theorem 6. *Let A and B be arbitrary complex numbers. Then for $n \geq 1$ it is true that*

$$\sum_{k=0}^{n}\binom{n}{k}A^k B^{n-k}u_k = \frac{b}{\Delta}\sum_{k=0}^{n}\binom{n}{k}(A\Delta)^k\left(\frac{Ap+2B-3\Delta A}{2}\right)^{n-k} M_k.$$

Proof. It is known [17] that if a_n is an arbitrary sequence of numbers with exponential generating function $F(z)$, then

$$S(z) = \sum_{n=0}^{\infty} S_n(A,B;a)\frac{z^n}{n!} = e^{Bz}F(Az),$$

where

$$S_n(A,B;a) = \sum_{k=0}^{n}\binom{n}{k}A^k B^{n-k}a_k.$$

Hence,

$$S_u(z) = \sum_{n=0}^{\infty} S_n(A,B;u)\frac{z^n}{n!} = \frac{b}{\Delta}2e^{\frac{Ap+2B}{2}z}\sinh\left(\frac{A\Delta z}{2}\right),$$

$$S_M(z) = \sum_{n=0}^{\infty} S_n(A,B;M)\frac{z^n}{n!} = 2e^{\frac{3A+2B}{2}z}\sinh\left(\frac{Az}{2}\right),$$

and finally

$$\sum_{n=0}^{\infty} S_n(A,B;u)\frac{z^n}{n!} = \frac{b}{\Delta}\sum_{n=0}^{\infty} S_n\left(A\Delta, \frac{Ap+2B-3\Delta A}{2}; M\right)\frac{z^n}{n!}.$$

□

We give some examples of the above summation identity. However, we restrict the list of examples to the pair (F_n, M_n). If $(A, B) = (1, 1)$, then

$$\sum_{k=0}^{n} \binom{n}{k} F_k = \frac{1}{\sqrt{5}} \sum_{k=0}^{n} \binom{n}{k} (\sqrt{5})^k (1-\sqrt{5})^{n-k} \left(\frac{3}{2}\right)^{n-k} M_k.$$

Since $\sum\limits_{k=1}^{n} \binom{n}{k} F_k = F_{2n}$, we can restate the identity as $(\eta = -1/\varphi)$

$$F_{2n} = \frac{(3\eta)^n}{\varphi-\eta} \sum_{k=0}^{n} \binom{n}{k} \left(\frac{\varphi-\eta}{3\eta}\right)^k M_k.$$

If $(A, B) = (-1, 1)$, then

$$F_n = \frac{(2\varphi-\eta)^n}{\varphi-\eta} \sum_{k=0}^{n} (-1)^{k+1} \binom{n}{k} \left(\frac{\varphi-\eta}{2\varphi-\eta}\right)^k M_k.$$

This identity may be compared with the one from Example 8, where we have shown that

$$F_n = \frac{(2\eta-\varphi)^n}{\varphi-\eta} \sum_{k=0}^{n} \binom{n}{k} \left(\frac{\varphi-\eta}{2\eta-\varphi}\right)^k M_k.$$

Our last example is $(A, B) = (1, -1/2)$. In this case, we get the relation

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} 2^{-(n-k)} F_k = 5^{\frac{n-1}{2}} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \left(\frac{3}{2}\right)^{n-k} M_k$$

or

$$\sum_{k=0}^{n} \binom{n}{k} (-2)^k F_k = 3^n 5^{\frac{n-1}{2}} \sum_{k=0}^{n} \binom{n}{k} \left(-\frac{2}{3}\right)^k M_k.$$

Theorem 7. *For $n \geq 0$ it holds that*

$$u_n = \frac{b}{\Delta} \left(\frac{3p-5\Delta}{6}\right)^n \sum_{k=0}^{n} \binom{n}{k} \left(\frac{2\Delta}{3p-5\Delta}\right)^k M_{2k}$$

and

$$M_{2n} = \frac{\Delta}{b} \left(\frac{5\Delta-3p}{2\Delta}\right)^n \sum_{k=0}^{n} \binom{n}{k} \left(\frac{6}{5\Delta-3p}\right)^k u_k.$$

Proof. The first formula follows from the relation $u\left(\frac{3z}{\Delta}\right) = \frac{b}{\Delta} \mu_2(z) e^{\left(\frac{3p}{2\Delta}-\frac{5}{2}\right)z}$ and an application of formula (8). Moreover, the first formula shows that $u_n \frac{\Delta}{b} \left(\frac{6}{3p-5\Delta}\right)^n$ is the binomial transform of $M_{2n} \left(\frac{2\Delta}{3p-5\Delta}\right)^n$.

The second formula is a rearrangement of the inverse binomial transform relation. □

Theorem 8. *For $n \geq 0$ it holds that*

$$u_n = \frac{b}{\Delta} \left(\frac{3p-5\Delta}{6}\right)^n \sum_{k=0}^{n} \binom{n}{k} \left(\frac{2\Delta}{3p-5\Delta}\right)^k M_{2k+1} - \frac{b}{\Delta} \left(\frac{p+\Delta}{2}\right)^n,$$

and

$$M_{2n+1} = \frac{\Delta}{b} \left(\frac{5\Delta-3p}{2\Delta}\right)^n \sum_{k=1}^{n} \binom{n}{k} \left(\frac{6}{5\Delta-3p}\right)^k u_k + 4^n.$$

Proof. To prove the first formula we use

$$u\left(\frac{3z}{\Delta}\right) = \frac{b}{\Delta}\left(\mu_1(z)e^{(\frac{3p}{2\Delta}-\frac{5}{2})z} - e^{(\frac{3p}{2\Delta}+\frac{3}{2})z}\right).$$

The second formula is once more an application of the inverse binomial transform, where we used that

$$\left(\frac{5\Delta-3p}{2\Delta}\right)^n \sum_{k=0}^{n}\binom{n}{k}\left(\frac{3\Delta+3p}{5\Delta-3p}\right)^k = \left(\frac{5\Delta-3p}{2\Delta}\right)^n\left(1+\frac{3\Delta+3p}{5\Delta-3p}\right)^n = 4^n.$$

□

Theorem 9. *For $n \geq 0$ we have*

$$u_{2n} = \frac{b}{\Delta}\left(\frac{3p^2+6q-5p\Delta}{6}\right)^n \sum_{k=1}^{n}\binom{n}{k}\left(\frac{2p\Delta}{3p^2+6q-5p\Delta}\right)^k M_{2k}$$

and

$$M_{2n} = \frac{\Delta}{b}\left(\frac{5p\Delta-3p^2-6q}{2p\Delta}\right)^n \sum_{k=1}^{n}\binom{n}{k}\left(\frac{6}{5p\Delta-3p^2-6q}\right)^k u_{2k}.$$

Proof. The first formula follows from the relation

$$u_2(z) = \frac{b}{\Delta}e^{(\frac{p^2+2q}{2}-\frac{5p\Delta}{b})z}\mu_2\left(\frac{p\Delta}{3}z\right)$$

and an application of formula (8). The second formula is a rearrangement of the inverse binomial transform relation. □

A proof comparable to the one given for Theorem 4 yields the following relation between numbers u_{2n+1} and M_{2n+1}. In this case we use the relations

$$\begin{aligned} 2p\Delta\cdot u_1(z) &= 2(p^2+q)e^{\frac{3p^2+6q-5p\Delta}{6}z}\mu_1\left(\frac{p\Delta z}{3}\right) \\ &\quad - \left(2(p^2+q)+b(p^2+2q-p\Delta)\right)e^{\frac{p^2+2q+p\Delta}{2}z} + b(p^2+2q+p\Delta)e^{\frac{p^2+2q-p\Delta}{2}z} \end{aligned}$$

and

$$\begin{aligned} 2(p^2+q)\mu_1(z) &= 2p\Delta u_1\left(\frac{3z}{p\Delta}\right)e^{\frac{-3p^2-6q+5p\Delta}{2p\Delta}z} \\ &\quad + (2(p^2+q)+b(p^2+2q-p\Delta))e^{4z} - b(p^2+2q+p\Delta)e^{z}. \end{aligned}$$

Theorem 10. *For $n \geq 0$*

$$\begin{aligned} u_{2n+1} &= b\,\frac{p^2+q}{p\Delta}\left(\frac{3p^2+6q-5p\Delta}{6}\right)^n \sum_{k=1}^{n}\binom{n}{k}\left(\frac{2p\Delta}{3p^2+6q-5p\Delta}\right)^k M_{2k+1} \\ &\quad - b\,\frac{3p^2+4q-p\Delta}{2p\Delta}\left(\frac{p^2+2q+p\Delta}{2}\right)^n + b\,\frac{p^2+2q+p\Delta}{2p\Delta}\left(\frac{p^2+2q-p\Delta}{2}\right)^n \end{aligned}$$

and

$$\begin{aligned} M_{2n+1} &= \frac{(p\Delta)^{1-n}}{b(p^2+q)}\left(\frac{5p\Delta-3p^2-6q}{2}\right)^n \sum_{k=1}^{n}\binom{n}{k}\left(\frac{6}{5p\Delta-3p^2-6q}\right)^k u_{2k+1} \\ &\quad + \frac{p^2+2q-p\Delta}{2(p^2+q)}\cdot 4^n - \frac{p^2+2q+p\Delta}{2(p^2+q)} + 4^n. \end{aligned}$$

3 MERSENNE-LUCAS IDENTITIES VIA EXPONENTIAL GENERATING FUNCTIONS

In this section we establish connections between the fundamental Lucas sequence $v_n = w_n(2, p; p, q)$ and Mersenne numbers M_n.

Theorem 11. *For $n \geq 0$*

$$M_n + 2 = \left(\frac{3\Delta - p}{2\Delta}\right)^n \sum_{k=0}^{n} \binom{n}{k} v_k \left(\frac{2}{3\Delta - p}\right)^k,$$

and

$$v_n = \left(\frac{p - 3\Delta}{2}\right)^n \sum_{k=0}^{n} \binom{n}{k} (M_k + 2) \left(\frac{2\Delta}{p - 3\Delta}\right)^k. \tag{16}$$

Proof. It is known that the exponential generating function of the sequence v_n can be given as

$$v(z) = \sum_{n=0}^{\infty} v_n \frac{z^n}{n!} = 2e^{\frac{p}{2}z} \cosh\left(\frac{\Delta}{2}z\right). \tag{17}$$

Using (7) and (17) we obtain

$$\begin{aligned}\mu'(z) &= \sum_{n=0}^{\infty} nM_n \frac{z^{n-1}}{n!} = \sum_{n=0}^{\infty} M_{n+1} \frac{z^n}{n!} = 3e^{\frac{3z}{2}} \sinh\left(\frac{z}{2}\right) + e^{\frac{3z}{2}} \cosh\left(\frac{z}{2}\right) \\ &= \frac{3}{2}\mu(z) + \frac{1}{2}v\left(\frac{z}{\Delta}\right) e^{\left(\frac{3}{2} - \frac{p}{2\Delta}\right)z}.\end{aligned}$$

Therefore,

$$\sum_{n=0}^{\infty} (2M_{n+1} - 3M_n) \frac{z^n}{n!} = v\left(\frac{z}{\Delta}\right) e^{\left(\frac{3\Delta - p}{2\Delta}\right)z}.$$

To complete the first part, observe that $2M_{n+1} - 3M_n = M_n + 2$. To get (16) we may apply the argument of the inverse binomial transform. $\square$

When $v_n = L_n$ is the Lucas sequence, then

$$M_n + 2 = \left(\frac{\varphi - 2\eta}{\varphi - \eta}\right)^n \sum_{k=0}^{n} \binom{n}{k} (\varphi - 2\eta)^{-k} L_k,$$

and

$$L_n = (2\eta - \varphi)^n \sum_{k=0}^{n} \binom{n}{k} \left(\frac{\varphi - \eta}{2\eta - \varphi}\right)^k (M_k + 2).$$

In view of

$$\sum_{k=0}^{n} \binom{n}{k} \left(\frac{\varphi - \eta}{2\eta - \varphi}\right)^k = \left(\frac{\eta}{2\eta - \varphi}\right)^n,$$

we observe that an equivalent version of the last identity is

$$L_n = 2\eta^n + (2\eta - \varphi)^n \sum_{k=1}^{n} \binom{n}{k} \left(\frac{\varphi - \eta}{2\eta - \varphi}\right)^k M_k. \tag{18}$$

We also have the following summation identity.

Theorem 12. *Let A and B be arbitrary complex numbers. Then for $n \geq 0$ it is true that*

$$\sum_{k=0}^{n} \binom{n}{k} A^k B^{n-k} v_k = \sum_{k=0}^{n} \binom{n}{k} (A\Delta)^k \left(\frac{Ap + 2B - 3\Delta A}{2} \right)^{n-k} (M_k + 2).$$

Proof. The proof is very similar to that one given in the last section and omitted. □

As examples, we will state the companion results for $v_n = L_n$ from the previous section. If $(A, B) = (1, 1)$, then

$$\sum_{k=0}^{n} \binom{n}{k} L_k = \sum_{k=0}^{n} \binom{n}{k} (\sqrt{5})^k (1 - \sqrt{5})^{n-k} \left(\frac{3}{2} \right)^{n-k} (M_k + 2).$$

This gives the identity

$$L_{2n} = 2\eta^{2n} + (3\eta)^n \sum_{k=1}^{n} \binom{n}{k} \left(\frac{\varphi - \eta}{3\eta} \right)^k M_k.$$

If $(A, B) = (-1, 1)$, then the result is

$$L_n = 2\varphi^n + (2\varphi - \eta)^n \sum_{k=1}^{n} \binom{n}{k} (-1)^k \left(\frac{\varphi - \eta}{2\varphi - \eta} \right)^k M_k,$$

which should be compared with (18).

Finally, for $(A, B) = (1, -1/2)$ we get the relation

$$\sum_{k=0}^{n} \binom{n}{k} 2^{-(n-k)} L_k = (-1)^n 2^{1-n} 5^{\frac{n}{2}} + 5^{\frac{n}{2}} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \left(\frac{3}{2} \right)^{n-k} M_k$$

or

$$\sum_{k=0}^{n} \binom{n}{k} (-2)^k L_k = 2 \cdot 5^{\frac{n}{2}} + 3^n 5^{\frac{n}{2}} \sum_{k=1}^{n} \binom{n}{k} \left(-\frac{2}{3} \right)^k M_k.$$

The results of this section also highlight some other hidden relations, since ([4], Proposition 2.4)

$$M_n + 2 = \begin{cases} j_n, & \text{if } n \text{ is even}, \\ 3J_n, & \text{if } n \text{ is odd}, \end{cases}$$

where $(J_n)_{n \geq 0}$ is the Jacobsthal and $(j_n)_{n \geq 0}$ is the Jacobsthal-Lucas sequence.

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Фрончак Р., Гой Т. *Тотожності Мерсенна-Горадама з використанням генератрис* // Карпатські матем. публ. — 2020. — Т.12, №1. — С. 34–45.

У роботі встановлені формули зв'язку між числами Мерсенна $M_n = 2^n - 1$ та узагальненими числами Фібоначчі (числами Горадама) w_n, які задовольняють лінійне рекурентне співвідношення другого порядку $w_n = pw_{n-1} + qw_{n-2}$, де $n \geq 2$, $w_0 = a$, $w_1 = b$, числа a, b, $p > 0$ і $q \neq 0$ є цілими. При цьому ми використовуємо відповідні співвідношення між звичайними та експоненційними генератрисами обох числових послідовностей. Зокрема, наведені приклади, які стосуються чисел Фібоначчі, Люка, Пелля, Якобсталя та збалансованих чисел.

Ключові слова і фрази: Числа Мерсенна, послідовність Горадама, послідовність Фібоначчі, послідовність Люка, послідовність Пелля, генератриса, біноміальне перетворення.

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KHATS' R.V.

SUFFICIENT CONDITIONS FOR THE IMPROVED REGULAR GROWTH OF ENTIRE FUNCTIONS IN TERMS OF THEIR AVERAGING

Let f be an entire function of order $\rho \in (0,+\infty)$ with zeros on a finite system of rays $\{z : \arg z = \psi_j\}$, $j \in \{1,\dots,m\}$, $0 \le \psi_1 < \psi_2 < \ldots < \psi_m < 2\pi$ and $h(\varphi)$ be its indicator. In 2011, the author of the article has been proved that if f is of improved regular growth (an entire function f is called a function of improved regular growth if for some $\rho \in (0,+\infty)$, $\rho_1 \in (0,\rho)$, and a 2π-periodic ρ-trigonometrically convex function $h(\varphi) \not\equiv -\infty$ there exists a set $U \subset \mathbb{C}$ contained in the union of disks with finite sum of radii and such that $\log|f(z)| = |z|^\rho h(\varphi) + o(|z|^{\rho_1})$, $U \not\ni z = re^{i\varphi} \to \infty$), then for some $\rho_3 \in (0,\rho)$ the relation

$$\int_1^r \frac{\log|f(te^{i\varphi})|}{t}\,dt = \frac{r^\rho}{\rho}h(\varphi) + o(r^{\rho_3}), \quad r \to +\infty,$$

holds uniformly in $\varphi \in [0,2\pi]$. In the present paper, using the Fourier coefficients method, we establish the converse statement, that is, if for some $\rho_3 \in (0,\rho)$ the last asymptotic relation holds uniformly in $\varphi \in [0,2\pi]$, then f is a function of improved regular growth. It complements similar results on functions of completely regular growth due to B. Levin, A. Grishin, A. Kondratyuk, Ya. Vasyl'kiv and Yu. Lapenko.

Key words and phrases: entire function of completely regular growth, entire function of improved regular growth, indicator, Fourier coefficients, averaging, finite system of rays.

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1 INTRODUCTION

It is well known ([13, p. 24]) that an entire function f of order $\rho \in (0,+\infty)$ may be represented in the form

$$f(z) = z^\lambda e^{Q(z)} \prod_{n=1}^{\infty} E\left(\frac{z}{\lambda_n}, p\right),$$

where λ_n are all nonzero roots of the function $f(z)$, $\lambda \in \mathbb{Z}_+$ is the multiplicity of the root at the origin, $Q(z) = \sum_{k=1}^{\nu} Q_k z^k$ is a polynomial of degree $\nu \le \rho$, $p \le \rho$ is the smallest integer for which $\sum_{n=1}^{\infty} |\lambda_n|^{-p-1} < +\infty$ and $E(w,p) = (1-w)\exp(w + w^2/2 + \cdots + w^p/p)$ is the Weierstrass primary factor.

Let f be an entire function of order $\rho \in (0,+\infty)$. The function

$$h(\varphi) = h_f(\varphi) = \limsup_{r\to\infty} \frac{\log|f(re^{i\varphi})|}{r^\rho}, \quad \varphi \in [0,2\pi],$$

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is called the *indicator* of f ([13, p. 51]). The indicator is a continuous 2π-periodic ρ-trigonometrically convex function (see [13, pp. 53–54]). A set $C \subset \mathbb{C}$ is called a C^0-*set* ([13, p. 90]) if it can be covered by a system of disks $\{z : |z - a_k| < s_k\}$, $k \in \mathbb{N}$, satisfying $\sum_{|a_k| \le r} s_k = o(r)$ as $r \to +\infty$.

An entire function f of order $\rho \in (0, +\infty)$ with the indicator $h(\varphi)$ is said to be of *completely regular growth* in the sense of Levin and Pfluger ([13, p. 139]) if there exists a C^0-set such that $\log |f(re^{i\varphi})| = r^\rho h(\varphi) + o(r^\rho)$, $C^0 \not\ni re^{i\varphi} \to \infty$, uniformly in $\varphi \in [0, 2\pi)$. In the theory of entire functions of completely regular growth (see [13, pp. 139–167]) the following theorem is valid.

Theorem A ([13, p. 150]). *In order that an entire function f of order $\rho \in (0, +\infty)$ with the indicator $h(\varphi)$ be of completely regular growth, it is necessary and sufficient that uniformly in $\varphi \in [0, 2\pi]$ one of the following relations hold:*

$$J_f^r(\varphi) := \int_1^r \frac{\log |f(te^{i\varphi})|}{t}\, dt = \frac{r^\rho}{\rho} h(\varphi) + o(r^\rho), \quad r \to +\infty,$$
$$I_f^r(\varphi) := \int_1^r J_f^t(\varphi) \frac{dt}{t} = \frac{r^\rho}{\rho^2} h(\varphi) + o(r^\rho), \quad r \to +\infty.$$

Similar results for entire functions of ρ-regular growth were obtained by A. Grishin [2] and for meromorphic functions of completely regular growth of finite λ-type ([11, p. 75]) by A. Kondratyuk [11, p. 112] and Ya. Vasyl'kiv [14] (see also Yu. Lapenko [12]).

In [5, 16] the notion of entire function of improved regular growth was introduced, and a criterion for this regularity was obtained in terms of the distribution of zeros under the condition that they are located on a finite system of rays.

An entire function f is called a function of *improved regular growth* ([5, 16]) if for some $\rho \in (0, +\infty)$ and $\rho_1 \in (0, \rho)$, and a 2π-periodic ρ-trigonometrically convex function $h(\varphi) \not\equiv -\infty$ there exists a set $U \subset \mathbb{C}$ contained in the union of disks with finite sum of radii and such that $\log |f(z)| = |z|^\rho h(\varphi) + o(|z|^{\rho_1})$, $U \not\ni z = re^{i\varphi} \to \infty$. If an entire function f is of improved regular growth, then it has the order ρ and indicator $h(\varphi)$ ([16]). In the case when zeros of an entire function f of improved regular growth are situated on a finite system of rays $\{z : \arg z = \psi_j\}$, $j \in \{1, \ldots, m\}$, $0 \le \psi_1 < \psi_2 < \ldots < \psi_m < 2\pi$, the indicator h has the form (see [16])

$$h(\varphi) = \sum_{j=1}^m h_j(\varphi), \quad \rho \in (0, +\infty) \setminus \mathbb{N}, \tag{1}$$

where $h_j(\varphi)$ is a 2π-periodic function such that on $[\psi_j, \psi_j + 2\pi)$

$$h_j(\varphi) = \frac{\pi \Delta_j}{\sin \pi\rho} \cos \rho(\varphi - \psi_j - \pi), \quad \Delta_j \in [0, +\infty).$$

In the case $\rho \in \mathbb{N}$, the indicator h is defined by the formula ([5])

$$h(\varphi) = \begin{cases} \tau_f \cos(\rho\varphi + \theta_f) + \sum_{j=1}^m h_j(\varphi), & p = \rho, \\ Q_\rho \cos \rho\varphi, & p = \rho - 1, \end{cases} \tag{2}$$

where $\delta_f \in \mathbb{C}$, $\tau_f = |\delta_f/\rho + Q_\rho|$, $\theta_f = \arg(\delta_f/\rho + Q_\rho)$ and $h_j(\varphi)$ is a 2π-periodic function such that on $[\psi_j, \psi_j + 2\pi)$

$$h_j(\varphi) = \Delta_j(\pi - \varphi + \psi_j) \sin \rho(\varphi - \psi_j) - \frac{\Delta_j}{\rho} \cos \rho(\varphi - \psi_j).$$

At present, many different conditions are known that are necessary and sufficient for the improved regular growth of entire functions (see [1,3–10,15–17]). In view of this, it is natural to establish an analog of Theorem A for the class of entire functions of improved regular growth. In this direction, the following results were obtained in [6,8].

Theorem B ([8]). *If an entire function f of order $\rho \in (0, +\infty)$ is of improved regular growth, then for some $\rho_2 \in (0, \rho)$, one has*

$$I_f^r(\varphi) = \frac{r^\rho}{\rho^2} h(\varphi) + O(r^{\rho_2}), \quad r \to +\infty,$$

uniformly in $\varphi \in [0, 2\pi]$.

Theorem C ([6]). *If an entire function f of order $\rho \in (0, +\infty)$ with zeros on a finite system of rays $\{z : \arg z = \psi_j\}$, $j \in \{1, \dots, m\}$, $0 \le \psi_1 < \psi_2 < \dots < \psi_m < 2\pi$, is of improved regular growth, then for some $\rho_3 \in (0, \rho)$ the relation*

$$J_f^r(\varphi) = \frac{r^\rho}{\rho} h(\varphi) + o(r^{\rho_3}), \quad r \to +\infty, \tag{3}$$

holds uniformly in $\varphi \in [0, 2\pi]$, where $h(\varphi)$ be defined by (1) *and* (2).

However, the problem of finding the converse of Theorems B and C remained open. The aim of the present paper is to prove the converse of Theorem C. Our principal result is the following theorem.

Theorem 1. *Let f be an entire function of order $\rho \in (0, +\infty)$ with zeros on a finite system of rays $\{z : \arg z = \psi_j\}$, $j \in \{1, \dots, m\}$, $0 \le \psi_1 < \psi_2 < \dots < \psi_m < 2\pi$ and $h(\varphi)$ be its indicator. If for some $\rho_3 \in (0, \rho)$ the relation* (3) *holds uniformly in $\varphi \in [0, 2\pi]$ with $h(\varphi)$ defined by* (1) *and* (2)*, then f is a function of improved regular growth.*

2 PRELIMINARIES

Let f be an entire function with $f(0) = 1$ and $(\lambda_n)_{n \in \mathbb{N}}$ be the sequence of its zeros. For $k \in \mathbb{Z}$ and $r > 0$, we set

$$n_k(r, f) := \sum_{|\lambda_n| \le r} e^{-ik \arg \lambda_n}, \qquad N_k(r, f) := \int_0^r \frac{n_k(t, f)}{t}\, dt,$$

$$N_k^*(r, f) := \int_0^r \frac{N_k(t, f)}{t}\, dt, \qquad n(r, \psi; f) := \sum_{|\lambda_n| \le r, \quad \arg \lambda_n = \psi} 1,$$

$$N(r, \psi; f) := \int_0^r \frac{n(t, \psi; f)}{t}\, dt, \qquad N^*(r, \psi; f) := \int_0^r \frac{N(t, \psi; f)}{t}\, dt,$$

$$c_k(r, \log|f|) := \frac{1}{2\pi} \int_0^{2\pi} e^{-ik\varphi} \log|f(re^{i\varphi})|\, d\varphi, \qquad c_k(r, J_f^r) := \frac{1}{2\pi} \int_0^{2\pi} e^{-ik\varphi} J_f^r(\varphi)\, d\varphi.$$

In the proof of Theorem 1, we use the following auxiliary statements.

Lemma 1 ([5, 16]). *An entire function f of order $\rho \in (0, +\infty)$ with zeros on a finite system of rays $\{z : \arg z = \psi_j\}$, $j \in \{1, \dots, m\}$, $0 \le \psi_1 < \psi_2 < \dots < \psi_m < 2\pi$, is a function of improved regular growth if and only if for some $\rho_4 \in (0, \rho)$ and each $j \in \{1, \dots, m\}$*

$$n(t, \psi_j; f) = \Delta_j t^\rho + o(t^{\rho_4}), \quad t \to +\infty, \quad \Delta_j \in [0, +\infty), \tag{4}$$

and, in addition, for $\rho \in \mathbb{N}$ and some $\rho_5 \in (0,\rho)$ and $\delta_f \in \mathbb{C}$, one has

$$\sum_{0<|\lambda_n|\le r} \lambda_n^{-\rho} = \delta_f + o(r^{\rho_5-\rho}), \quad r \to +\infty. \tag{5}$$

In this case, the indicator $h(\varphi)$ be defined by formulas (1) *and* (2).

We remark that, for $\rho = p+1$ equality (4) holds with $\Delta_j = 0$, because $\sum_{n\in\mathbb{N}} |\lambda_n|^{-p-1} < +\infty$ (see [5, p. 19]).

Lemma 2. *If an entire function f of order $\rho \in (0,+\infty)$ satisfies the conditions of Theorem 1, then for some $\rho_3 \in (0,\rho)$ and each $k \in \mathbb{Z}$, one has*

$$c_k(r, J_f^r) = c_k \frac{r^\rho}{\rho} + o(r^{\rho_3}), \quad r \to +\infty, \tag{6}$$

$$N_k^*(r,f) = c_k \left(1 - \frac{k^2}{\rho^2}\right) \frac{r^\rho}{\rho} + o(r^{\rho_3}), \quad r \to +\infty, \tag{7}$$

where

$$c_k := \frac{1}{2\pi} \int_0^{2\pi} e^{-ik\varphi} h(\varphi)\, d\varphi = \frac{\rho}{\rho^2 - k^2} \sum_{j=1}^m \Delta_j e^{-ik\psi_j}, \quad \Delta_j \in [0,+\infty), \tag{8}$$

if $\rho \in (0,+\infty) \setminus \mathbb{N}$, and

$$c_k = \begin{cases} \dfrac{\rho}{\rho^2 - k^2} \displaystyle\sum_{j=1}^m \Delta_j e^{-ik\psi_j}, & |k| \neq \rho = p, \\ \dfrac{\tau_f e^{i\theta_f}}{2} - \dfrac{1}{4\rho} \displaystyle\sum_{j=1}^m \Delta_j e^{-i\rho\psi_j}, & k = \rho = p, \\ 0, & |k| \neq \rho = p+1, \\ \dfrac{Q_\rho}{2}, & k = \rho = p+1, \end{cases} \tag{9}$$

if $\rho \in \mathbb{N}$.

Proof. Under the conditions of the lemma, by using (3), for some $\rho_3 \in (0,\rho)$ and each $k \in \mathbb{Z}$, we get

$$c_k(r, J_f^r) = \frac{1}{2\pi} \int_0^{2\pi} e^{-ik\varphi} \left(\frac{r^\rho}{\rho} h(\varphi) + o(r^{\rho_3}) \right) d\varphi = c_k \frac{r^\rho}{\rho} + o(r^{\rho_3}), \quad r \to +\infty,$$

where c_k is defined by formulas (8) and (9) (see [6, 7, 9, 10]). Thus, relation (6) holds. Let us prove relation (7). Using relations (see [14, pp. 39, 43], [11, pp. 107, 112], [6, p. 13])

$$c_k(r, J_f^r) = \int_0^r \frac{c_k(t, \log|f|)}{t}\, dt,$$

$$N_k(r,f) = c_k(r, \log|f|) - k^2 \int_0^r \frac{dt}{t} \int_0^t \frac{c_k(u, \log|f|)}{u}\, du, \quad k \in \mathbb{Z}, \quad r > 0,$$

we obtain

$$N_k^*(r,f) = \int_0^r \frac{N_k(t,f)}{t}\, dt = c_k(r, J_f^r) - k^2 \int_0^r \frac{dt}{t} \int_0^t \frac{c_k(u, J_f^u)}{u}\, du, \quad k \in \mathbb{Z}, \quad r > 0.$$

Then, using (6) and passing to the limit as $r \to +\infty$, we get

$$N_k^*(r,f) = c_k \frac{r^\rho}{\rho} + o(r^{\rho_3}) - k^2 \int_0^r \frac{dt}{t} \int_0^t \left(c_k \frac{u^{\rho-1}}{\rho} + o(u^{\rho_3 - 1}) \right) du = c_k \left(1 - \frac{k^2}{\rho^2}\right) \frac{r^\rho}{\rho} + o(r^{\rho_3}).$$

Lemma 2 is proved. □

Lemma 3. *Let f be an entire function of order $\rho \in (0, +\infty)$ with zeros on a finite system of rays $\{z : \arg z = \psi_j\}$, $j \in \{1, \dots, m\}$, $0 \le \psi_1 < \psi_2 < \ldots < \psi_m < 2\pi$. In order that the equality*

$$N^*(r, \psi_j; f) = \frac{\Delta_j}{\rho^2} r^\rho + o(r^{\rho_3}), \quad r \to +\infty, \quad \Delta_j \in [0, +\infty), \tag{10}$$

holds for some $\rho_3 \in (0, \rho)$ and each $j \in \{1, \dots, m\}$, it is necessary and sufficient that, for some $\rho_3 \in (0, \rho)$ and $k_0 \in \mathbb{Z}$ and each $k \in \{k_0, k_0 + 1, \dots, k_0 + m - 1\}$, relation (7) *with c_k, defined by* (8) *and* (9) *be true. Besides, we have $\sum\limits_{j=1}^{m} \Delta_j e^{-i\rho\psi_j} = 0$, if $\rho \in \mathbb{N}$.*

Proof. Necessity. Since (see [11, p. 127])

$$n_k(r, f) = \sum_{j=1}^{m} e^{-ik\psi_j} n(r, \psi_j; f), \quad k \in \mathbb{Z},$$

then

$$N_k(r, f) = \sum_{j=1}^{m} e^{-ik\psi_j} \int_0^r \frac{n(t, \psi_j; f)}{t} \, dt = \sum_{j=1}^{m} e^{-ik\psi_j} N(r, \psi_j; f),$$
$$N_k^*(r, f) = \sum_{j=1}^{m} e^{-ik\psi_j} N^*(r, \psi_j; f), \quad k \in \mathbb{Z}.$$

Using (10), for some $\rho_3 \in (0, \rho)$ and each $k \in \mathbb{Z}$ we obtain relation (7) with c_k, defined by (8) and (9). In this case, $\sum\limits_{j=1}^{m} \Delta_j e^{-i\rho\psi_j} = 0$, if $\rho \in \mathbb{N}$.

Let us prove the *sufficiency*. Without loss of generality, we can assume that $k_0 = 0$. Then, by analogy with [7, p. 1957] (see also [10, p. 118], [11, p. 127]), for $k \in \{0, 1, \dots, m-1\}$ we get

$$\begin{aligned}
N_0^*(r, f) &= N^*(r, \psi_1; f) + N^*(r, \psi_2; f) + \ldots + N^*(r, \psi_m; f), \\
N_1^*(r, f) &= e^{-i\psi_1} N^*(r, \psi_1; f) + e^{-i\psi_2} N^*(r, \psi_2; f) + \ldots + e^{-i\psi_m} N^*(r, \psi_m; f), \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
N_{m-1}^*(r, f) &= e^{-i(m-1)\psi_1} N^*(r, \psi_1; f) + e^{-i(m-1)\psi_2} N^*(r, \psi_2; f) + \ldots + e^{-i(m-1)\psi_m} N^*(r, \psi_m; f).
\end{aligned}$$

This is a system of linear equations for the unknowns $N^*(r, \psi_j; f)$, $j \in \{1, \dots, m\}$. Its determinant is the nonzero Vandermonde determinant

$$D = \begin{vmatrix} 1 & 1 & \dots & 1 \\ e^{-i\psi_1} & e^{-i\psi_2} & \dots & e^{-i\psi_m} \\ \dots & \dots & \dots & \dots \\ e^{-i(m-1)\psi_1} & e^{-i(m-1)\psi_2} & \dots & e^{-i(m-1)\psi_m} \end{vmatrix} \ne 0.$$

Therefore, the functions $N^*(r, \psi_j; f)$, $j \in \{1, \dots, m\}$, can be represented as linear combinations of the functions $N_k^*(r, f)$, $k \in \{0, 1, \dots, m-1\}$. Using (7), we obtain relation (10), where by the Cramer's rule $\Delta_j = \rho^2 D_j / D$, $j \in \{1, \dots, m\}$, and D_j is the determinant formed from the determinant D by replacing the j-column with the corresponding column $(\widetilde{c}_0, \widetilde{c}_1, \dots, \widetilde{c}_{m-1})$, $\widetilde{c}_k := \frac{c_k}{\rho}\big(1 - \frac{k^2}{\rho^2}\big)$, $k \in \{0, 1, \dots, m-1\}$. Lemma 3 is proved. $\square$

Remark 1. *Let* $\rho \in (0,+\infty) \setminus \mathbb{N}$, $\mu_n = \left(n + \frac{n}{\log n}\right)^{1/\rho}$, $\{\lambda_n : n \in \mathbb{N}\setminus\{1\}\} := \bigcup_{j=1}^{m}\{\mu_n e^{i\frac{2\pi(j-1)}{m}} : n \in \mathbb{N}\setminus\{1\}\}$, $m \in \mathbb{N} \setminus \{1\}$ *and ([7, p. 1958])*

$$f(z) = \prod_{n=1}^{\infty}\left(1 - \frac{z}{\lambda_n}\right)\exp\left(\sum_{\zeta=1}^{p}\frac{1}{\zeta}\left(\frac{z}{\lambda_n}\right)^{\zeta}\right), \quad p = [\rho].$$

Then for each $j \in \{1,\dots,m\}$, *we obtain (see [7, p. 1959])*

$$N^*\left(r, \frac{2\pi(j-1)}{m}; f\right) = \frac{r^\rho}{\rho^2} + O\left(\frac{r^\rho}{\log r}\right), \quad r \to +\infty.$$

Therefore, relation (10) *is not true for any* $\rho_3 \in (0,\rho)$. *Furthermore,*

$$N_0^*(r,f) = \sum_{j=1}^{m} N^*\left(r, \frac{2\pi(j-1)}{m}; f\right) = \frac{m}{\rho^2}r^\rho + O\left(\frac{r^\rho}{\log r}\right), \quad r \to +\infty.$$

Thus, relation (7) *is not true for* $k = 0$. *Moreover, since*

$$\sum_{j=1}^{m} e^{-ik\frac{2\pi(j-1)}{m}} = \frac{1 - e^{-2\pi k i}}{1 - e^{-i\frac{2\pi k}{m}}} = 0, \quad k \in \{1,\dots,m-1\},$$

we conclude that

$$n_k(r,f) = \sum_{\mu_n \le r}\sum_{j=1}^{m} e^{-ik\frac{2\pi(j-1)}{m}} = 0,$$

for each $k \in \{1,\dots,m-1\}$ *and all* $r > 0$. *Therefore, relation* (7) *holds for any* $\rho_3 \in (0,\rho)$ *and each* $k \in \{1,\dots,m-1\}$.

Lemma 4. *Let* f *be an entire function of order* $\rho \in (0,+\infty)$ *with zeros on a finite system of rays* $\{z : \arg z = \psi_j\}$, $j \in \{1,\dots,m\}$, $0 \le \psi_1 < \psi_2 < \ldots < \psi_m < 2\pi$. *In order that the equality* (4) *holds for some* $\rho_4 \in (0,\rho)$ *and each* $j \in \{1,\dots,m\}$, *it is necessary and sufficient that for some* $\rho_3 \in (0,\rho)$ *and each* $j \in \{1,\dots,m\}$ *relation* (10) *be true.*

Proof. Indeed, using Lemma 3 from [15, p. 143] twice, we obtain the required statement. ☐

3 PROOF OF THEOREM 1

Let the conditions of Theorem 1 be satisfied. Then, by Lemmas 2–4, the relations (6), (7) and (4) hold. Let us prove the equality (5) for $\rho \in \mathbb{N}$. Since (see the proof of Lemmas 2 and 3)

$$c_k(r, \log|f|) = N_k(r,f) + k^2\int_0^r \frac{c_k(t, J_f^t)}{t}\,dt, \quad N_k(r,f) = \sum_{j=1}^{m} e^{-ik\psi_j}N(r,\psi_j;f), \quad k \in \mathbb{Z},$$

and ([4, p. 101])

$$c_\rho(r, \log|f|) = \frac{1}{2}Q_\rho r^\rho + \frac{1}{2\rho}\sum_{0<|\lambda_n|\le r}\left(\left(\frac{r}{\lambda_n}\right)^\rho - \left(\frac{\overline{\lambda}_n}{r}\right)^\rho\right), \quad k = \rho = p \in \mathbb{N},$$

then, using formulas (4), (6), (7), (9) and the identity $\sum\limits_{j=1}^{m} \Delta_j e^{-i\rho\psi_j} = 0$, $\rho = p \in \mathbb{N}$, for some $\rho_5 \in (0,\rho)$ we get

$$\begin{aligned}
\sum_{0<|\lambda_n|\le r} \lambda_n^{-\rho} =& 2\rho r^{-\rho} c_\rho(r, \log|f|) - \rho Q_\rho + r^{-\rho} \sum_{0<|\lambda_n|\le r} \left(\frac{\overline{\lambda}_n}{r}\right)^{\rho} \\
=& 2\rho r^{-\rho}\left(N_\rho(r,f) + \rho^2 \int_0^r \frac{c_\rho(t, J_f^t)}{t}\,dt\right) - \rho Q_\rho + r^{-2\rho}\sum_{j=1}^{m} e^{-i\rho\psi_j}\int_0^r t^\rho\,dn(t,\psi_j;f) \\
=& 2\rho r^{-\rho}\left(\sum_{j=1}^{m} e^{-i\rho\psi_j}\int_0^r \frac{n(t,\psi_j;f)}{t}\,dt + \rho^2\int_0^r \frac{c_\rho(t,J_f^t)}{t}\,dt\right) - \rho Q_\rho \\
&+ r^{-2\rho}\sum_{j=1}^{m} e^{-i\rho\psi_j}\left(r^\rho n(r,\psi_j;f) - \rho\int_0^r t^{\rho-1} n(t,\psi_j;f)\,dt\right) \\
=& 2\rho r^{-\rho}\left(\sum_{j=1}^{m} e^{-i\rho\psi_j}\int_0^r (\Delta_j t^{\rho-1} + o(t^{\rho_4-1}))\,dt + \rho^2\int_0^r \left(\frac{c_\rho}{\rho}t^{\rho-1} + o(t^{\rho_3-1})\right)dt\right) \\
&- \rho Q_\rho + r^{-2\rho}\sum_{j=1}^{m} e^{-i\rho\psi_j}\left(\Delta_j r^{2\rho} + o(r^{\rho_4+\rho}) - \rho\int_0^r (\Delta_j t^{2\rho-1} + o(t^{\rho_4+\rho-1}))\,dt\right) \\
=& 2\rho r^{-\rho}\left(\frac{r^\rho}{\rho}\sum_{j=1}^{m}\Delta_j e^{-i\rho\psi_j} + c_\rho r^\rho + o(r^{\rho_4}) + o(r^{\rho_3})\right) - \rho Q_\rho \\
&+ r^{-2\rho}\sum_{j=1}^{m} e^{-i\rho\psi_j}\left(\frac{\Delta_j}{2}r^{2\rho} + o(r^{\rho_4+\rho})\right) \\
=& \rho(\tau_f e^{i\theta_f} - Q_\rho) + o(r^{\rho_4-\rho}) + o(r^{\rho_3-\rho}) = \delta_f + o(r^{\rho_5-\rho}), \quad r \to +\infty.
\end{aligned}$$

Hence, equality (5) holds for $\rho = p$ with $\delta_f = \rho(\tau_f e^{i\theta_f} - Q_\rho)$. In the case $\rho = p+1$, condition (5) follows from (4) (see [5, p. 23, Remark 2]). Thus, according to Lemma 1, the entire function f is a function of improved regular growth. This completes the proof of Theorem 1.

Combining Theorem 1 with Theorem C, we obtain the following theorem.

Theorem 2. *In order that an entire function f of order $\rho \in (0,+\infty)$ with zeros on a finite system of rays $\{z : \arg z = \psi_j\}$, $j \in \{1,\dots,m\}$, $0 \le \psi_1 < \psi_2 < \dots < \psi_m < 2\pi$, be of improved regular growth with the indicator $h(\varphi)$ defined by* (1) *and* (2)*, it is necessary and sufficient that for some $\rho_3 \in (0,\rho)$ the relation* (3) *holds uniformly in $\varphi \in [0,2\pi]$.*

Remark 2. *For each $m \in \mathbb{N} \setminus \{1;2\}$ there exists an entire function f of order $\rho \in (0,+\infty) \setminus \mathbb{N}$ with zeros on a finite system of rays $\{z : \arg z = \psi_j\}$, $\psi_j := \frac{2\pi(j-1)}{m}$, $j \in \{1,\dots,m\}$, such that uniformly in $\varphi \in [0,2\pi]$ the relation* (3) *is not true for any $\rho_3 \in (0,\rho)$ and f is not a function of improved regular growth.*

Indeed, let f be an entire function of order $\rho \in (0,+\infty) \setminus \mathbb{N}$, defined as in Remark 1. Then (see [7, p. 1959])

$$n\left(t, \frac{2\pi(j-1)}{m}; f\right) = t^\rho - \frac{t^\rho}{\rho\log t} + o\left(\frac{t^\rho}{\log t}\right), \quad t \to +\infty,$$

for each $j \in \{1,\dots,m\}$. Thus, relation (4) *is not true for any $\rho_4 \in (0,\rho)$, and, according to Lemma 1, the entire function f is not a function of improved regular growth. Further, for each*

$j \in \{1, \dots, m\}$, we obtain ([7, p. 1959])

$$c_0(r, \log|f|) = \sum_{j=1}^{m} N\left(r, \frac{2\pi(j-1)}{m}; f\right) = \frac{m}{\rho} r^{\rho} + O\left(\frac{r^{\rho}}{\log r}\right), \quad r \to +\infty.$$

Furthermore, (see [6, p. 11], [7, p. 1959])

$$c_k(r, \log|f|) = \overline{c_{-k}(r, \log|f|)}, \quad k \le -1,$$
$$c_k(r, \log|f|) = \frac{1}{2k} \sum_{\mu_n \le r} \left[\left(\frac{r}{\mu_n}\right)^k - \left(\frac{\mu_n}{r}\right)^k\right] \sum_{j=1}^{m} e^{-ik\frac{2\pi(j-1)}{m}}, \quad 1 \le k \le p,$$

and

$$c_k(r, \log|f|) = -\frac{1}{2k} \left\{ \sum_{\mu_n > r} \left(\frac{r}{\mu_n}\right)^k + \sum_{\mu_n \le r} \left(\frac{\mu_n}{r}\right)^k \right\} \sum_{j=1}^{m} e^{-ik\frac{2\pi(j-1)}{m}}, \quad k \ge p+1,$$

where (see Remark 1)

$$\sum_{j=1}^{m} e^{-ik\frac{2\pi(j-1)}{m}} = \begin{cases} 0, & k \in \mathbb{N}, \quad k \ne m, \\ m, & k = m. \end{cases}$$

In view of this, since

$$c_k(r, J_f^r) = \int_0^r \frac{c_k(t, \log|f|)}{t} \, dt, \quad k \in \mathbb{Z}, \quad r > 0,$$
$$c_0(r, J_f^r) = \frac{m}{\rho^2} r^{\rho} + O\left(\frac{r^{\rho}}{\log r}\right), \quad r \to +\infty,$$
$$J_f^r(\varphi) = \sum_{k \in \mathbb{Z}} c_k(r, J_f^r) e^{ik\varphi} = c_0(r, J_f^r) + \sum_{k \in \mathbb{Z} \setminus \{0\}} c_k(r, J_f^r) e^{ik\varphi}, \quad \varphi \in [0, 2\pi],$$

we conclude that the relation (3) *is not true for any* $\rho_3 \in (0, \rho)$.

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Хаць Р.В. *Достатні умови покращеного регулярного зростання цілих функцій в термінах їх усереднення* // Карпатські матем. публ. — 2020. — Т.12, №1. — С. 46–54.

Нехай f — ціла функція порядку $\rho \in (0, +\infty)$ з нулями на скінченній системі променів $\{z : \arg z = \psi_j\}$, $j \in \{1, \ldots, m\}$, $0 \le \psi_1 < \psi_2 < \ldots < \psi_m < 2\pi$ і $h(\varphi)$ — її індикатор. У 2011 році автор цієї статті довів, що якщо f є функцією покращеного регулярного зростання (ціла функція f називається функцією покращеного регулярного зростання, якщо для деяких $\rho \in (0, +\infty)$, $\rho_1 \in (0, \rho)$ і 2π-періодичної ρ-тригонометрично опуклої функції $h(\varphi) \not\equiv -\infty$ існує множина $U \subset \mathbb{C}$, яка міститься в об'єднанні кругів із скінченною сумою радіусів, така, що $\log|f(z)| = |z|^\rho h(\varphi) + o(|z|^{\rho_1})$, $U \not\ni z = re^{i\varphi} \to \infty$), то для деякого $\rho_3 \in (0, \rho)$ співвідношення

$$\int_1^r \frac{\log|f(te^{i\varphi})|}{t}\, dt = \frac{r^\rho}{\rho} h(\varphi) + o(r^{\rho_3}), \quad r \to +\infty,$$

виконується рівномірно за $\varphi \in [0, 2\pi]$. В даній роботі, використовуючи метод коефіцієнтів Фур'є, ми встановлюємо обернене твердження, а саме, якщо для деякого $\rho_3 \in (0, \rho)$ останнє асимптотичне співвідношення виконується рівномірно за $\varphi \in [0, 2\pi]$, то f є функцією покращеного регулярного зростання. Це доповнює аналогічні результати Б. Левіна, А. Гришина, А. Кондратюка, Я. Васильківа та Ю. Лапенка про функції цілком регулярного зростання.

Ключові слова і фрази: ціла функція цілком регулярного зростання, ціла функція покращеного регулярного зростання, індикатор, коефіцієнти Фур'є, усереднення, скінченна система променів.

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ZERO PRODUCT PRESERVING BILINEAR OPERATORS ACTING IN SEQUENCE SPACES

Consider a couple of sequence spaces and a product function — a canonical bilinear map associated to the pointwise product — acting in it. We analyze the class of "zero product preserving" bilinear operators associated with this product, that are defined as the ones that are zero valued in the couples in which the product equals zero. The bilinear operators belonging to this class have been studied already in the context of Banach algebras, and allow a characterization in terms of factorizations through $\ell^r(\mathbb{N})$ spaces. Using this, we show the main properties of these maps such as compactness and summability.

Key words and phrases: sequence spaces, bilinear operators, factorization, zero product preserving map, product.

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1 INTRODUCTION

Let us fix a couple of Banach spaces having a characteristic operation involving couples of vectors for giving an element in other Banach space. For example, the pointwise product of functions from L^p and $L^{p'}$ for obtaining an element of L^1, or the internal product in a Banach algebra. Let us call "product" this bilinear map. Bilinear maps factoring through such a product preserve some of its good properties, and so it is interesting to know which bilinear operators satisfy such a factorization. This general philosophy is in the root of some current developments in mathematical analysis, mainly in the Banach algebras and vector lattices setting (see for example [1,5,7,12] and references therein).

In this paper we analyze the class of bilinear maps factoring through a product in a different context. We study the main characterizations and properties when the operators act in couples of classical Banach sequence spaces ($\ell^p(\mathbb{N})$-spaces). The essential result (Theorem 1) shows that the factorization is equivalent to a certain "zero product preservation" property. Concretely, bilinear maps satisfying this property are the ones that are 0-valued for couples of elements whose products are equal to zero.

Let us explain the relation of our class of maps with some notions and results that can be found in the current literature. Alaminos J. et al have studied zero product preserving bilinear maps defined on a product of Banach algebras and C^*-algebras to get a characterization for (weighted) homomorphisms and derivations. They have obtained a class of Banach algebras A that satisfy the equality $\varphi(ab,c) = \varphi(a,bc)$, $a,b,c \in A$, for every continuous zero product preserving bilinear map $\varphi : A \times A \to B$. By adding some conditions to the algebra, they have

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proved that $\varphi(ab,c) = \varphi(a,bc)$ gives a factorization for the bilinear operator φ as $\varphi(a,b) := P(ab)$ for a certain linear map $P : A \to B$ [1]. Recently, Alaminos J. et al have shown in [2] that there are some Banach algebras that do not satisfy the equality $\varphi(ab,c) = \varphi(a,bc)$ such as the algebra $C^1[0,1]$ of continuously differentiable functions from $[0,1]$ to $\mathbb{C}$, although the operator φ is zero product preserving map. In particular, this shows that any bilinear operator cannot be factored through the product.

In the meantime, some authors have studied the zero product preserving property for the bilinear maps acting in vector lattices and function spaces with the name *orthosymmetry*. This term is firstly used by Buskes G. and van Rooij A. to give a factorization for bilinear maps defined on vector lattices and they obtained the powers of vector lattices by orthosymmetric maps, see [6,7]. Recently, Ben Amor F. has studied the commutators of orthosymmetric maps in [4] and investigated an expanded class of orthosymmetric bilinear maps that are related to symmetric operators given by Buskes G. and van Rooij A. The interested reader can see the reference [5] for a detailed information about the orthosymmetric maps acting in vector lattices.

In a different direction, factorization of zero product preserving bilinear maps for the convolution product acting in function spaces has been studied by Erdoğan E. et al (see [10]). Recently, Erdoğan E. and Gök Ö. have studied a class of bilinear operators acting in a product of Banach algebras of integrable functions and showed a zero product preserving bilinear operator defined on the product of Banach algebras that factors through a subalgebra of absolutely integrable functions by convolution product (see [11]). Moreover, Erdoğan E. et al have obtained a class of zero product preserving bilinear operators acting in pairs of Banach function spaces that factor through the pointwise product and they have given characterizations by means of norm inequalities for these bilinear maps [12].

The aim of this paper is to give a new version of the factorization results given in the mentioned studies for the zero product preserving bilinear operators defined on the product of sequence spaces. We center our attention on bilinear operators B defined on the product of Banach spaces E and F satisfying the zero product preserving property

$$x \circledast y = 0 \text{ implies } B(x,y) = 0, \quad (x,y) \in E \times F,$$

where $\circledast$ is defined using the pointwise product of sequences, showing that they are exactly the ones that factors through $\circledast$.

This paper is organised as follows: Section 2 is devoted to giving some preliminary results on products and factorization through them. In Section 3, the main result of the paper on factorization of zero product preserving on sequence spaces is proved (Theorem 1). Using it, compactness and summability properties of product factorable operators are investigated and some applications are given.

2 Preliminaries: products and bilinear maps

We use standard notations and notions from Banach space theory. The sets of natural numbers and integers are denoted by $\mathbb{N}$ and $\mathbb{Z}$, respectively. For a Banach space E, B_E will denote the unit ball of E. We write χ_A for the characteristic function of a set A. *Operator* (linear or multilinear) indicates *continuous operator*. The space of all linear operators between Banach spaces X, Y is denoted by $L(X, Y)$, and we write $\mathcal{B}(X \times Y, Z)$ for the vector space of all bilinear Z-valued operators, where Z is also a Banach space.

For a positive real number p, $\ell^p(\mathbb{N})$ is the space of all complex valued absolutely p-summable sequences. It is a Banach space with the norm $\|(x_i)\|_p = \Big(\sum\limits_{i=1}^{\infty} |x_i|^p\Big)^{1/p}$ for $p \geq 1$, and $\ell^\infty(\mathbb{N})$ shows the Banach space of all bounded sequences endowed with the norm $\|(x_i)\|_\infty = \sup_{i\in\mathbb{N}} |x_i|$.

If μ is a measure and $1 \leq p < \infty$, we write $L^p(\mu)$ for the Lebesgue space of classes of μ-a.e. equal p-integrable functions.

We call a continuous operator *(weakly) compact* if it maps the closed unit ball to a relatively (weakly) compact set.

A Banach space E has Dunford-Pettis property if every weakly compact linear operator $T : E \to F$ is completely continuous (that is, it maps every weakly compact set $A \in E$ into a compact set with respect to the norm topology of the Banach space F).

A linear operator $T : X \to Y$ is said to be *(p,q)-summing* ($T \in \Pi_{p,q}(X,Y)$) if there is a constant $k > 0$ such that for every $x_1, \ldots, x_n \in X$ and for all positive integers n

$$\Big(\sum_{i=1}^{n} \|T(x_i)\|_Y^p\Big)^{1/p} \leq k \sup_{x' \in B_{X'}} \Big(\sum_{i=1}^{n} |\langle x_i, x'\rangle|^q\Big)^{1/q}.$$

For the summing operators we refer the reader to [9].

Throughout the paper we will use the term *product* for a specific bilinear map, typically with some special properties and being canonical in some sense. However, the only assumption on such a product is that it is a continuous bilinear map. We will need stronger properties for the products that are presented in [12] by Erdoğan E. et al.

Definition 1. *Consider a bilinear operator $\circledast : X \times Y \to Z$, $(x,y) \rightsquigarrow \circledast(x,y) =: x \circledast y$, where X, Y, Z are Banach spaces. We say that the bilinear operator $\circledast$ is a norm preserving product (n.p. product for short) if it satisfies the inclusion $B_Z \subseteq \circledast(B_X \times B_Y)$ and*

$$\|x \circledast y\|_Z = \inf\big\{\|x'\|_X \|y'\|_Y : x' \in X,\ y' \in Y,\ x \circledast y = x' \circledast y'\big\},$$

for every $(x,y) \in X \times Y$.

Now let us give some examples of bilinear operators that are n.p. product or not.

Example 1. *Let (Ω, Σ, μ) be a complete σ-finite measure space and let $(E, \|.\|_E)$ be a Banach function space over μ. (For the definition of Banach function space we refer to [14, Def 1.b.17]). We will write $E^{(p)}$, $p \geq 1$, for the p-convexification of the Banach lattice E in the sense of [14, Ch. 1.d] (see also the equivalent notion of pth power in [17, Ch.2] for a more explicit description). In the case that E is a Banach function space, $E^{(p)}$ is also a Banach function space with the norm $\|f\|_{E^{(p)}} = \||f|^p\|_E^{1/p}$ for $f \in E$ (see [16, Prop.1]).*

Let us consider the bilinear operator defined by the (μ-a.e.) pointwise product $\odot : E^{(p)} \times E^{(q)} \to E^{(r)}$, $(f,g) \rightsquigarrow f \cdot g$, where $\dfrac{1}{p} + \dfrac{1}{q} = \dfrac{1}{r}$ for $1 \leq r < p, q < \infty$. We claim that this bilinear map is a norm preserving product. Indeed, consider $f \in B_{E^{(r)}}$, $h := |f|^{r/p} sgn f \in E^{(p)}$ and $g := |f|^{r/q} \in E^{(q)}$, where $sgn f$ denotes the sign function of f. By the definition of the norm of *the p-convexification, it follows that* $\|h\|_{E^{(p)}} = \left\| \left| |f|^{r/p} sgnf \right|^{p} \right\|_{E}^{1/p} = \| |f|^{r} \|_{E}^{1/p} = \|f\|_{E^{(r)}}^{r/p} \leq 1$. *Similarly,* $\|g\|_{E^{(q)}} = \|f\|_{E^{(r)}}^{r/q} \leq 1$. *Therefore,* $B_{E^{(r)}} \subseteq \odot(B_{E^{(p)}} \times B_{E^{(q)}})$ *is obtained.*

Let us show now that $\|h \cdot g\|_{E^{(r)}} = \inf\{\|h'\|_{E^{(p)}} \|g'\|_{E^{(q)}} : h' \in E^{(p)}, g' \in E^{(q)}, h \cdot g = h' \cdot g'\}$ *for* $h \in E^{(p)}$ *and* $g \in E^{(q)}$. *Indeed, by the generalized Hölder's inequality we have that* $h \cdot g \in E^{(r)}$ *and* $\|h \cdot g\|_{E^{(r)}} \leq \|h\|_{E^{(p)}} \|g\|_{E^{(q)}}$ *(see [16, Lemma 1]). Since this inequality holds for all couples* (h', g') *such that* $f = h \cdot g = h' \cdot g'$, *we obtain* $\|h \cdot g\|_{E^{(r)}} \leq \inf\{\|h'\| \|g'\| : h \cdot g = h' \cdot g'\}$. *Conversely, consider an arbitrary element* $f \in E^{(r)}$. *Then* f *has the following factorization:* $h = |f|^{r/p} sgnf \in E^{(p)}$, $g = |f|^{r/q} \in E^{(q)}$ *and* $h \cdot g \in E^{(r)}$. *Moreover,* $\|h\|_{E^{(p)}} = \|f\|_{E^{(r)}}^{r/p}$ *and* $\|g\|_{E^{(q)}} = \|f\|_{E^{(r)}}^{r/q}$. *Therefore* $\|h\|_{E^{(p)}} \|g\|_{E^{(q)}} = \|f\|_{E^{(r)}}^{r/p} \|f\|_{E^{(r)}}^{r/q} = \|f\|_{E^{(r)}}$. *This proves*

$$\|f\|_{E^{(r)}} = \|h \cdot g\|_{E^{(r)}} \geq \inf\left\{\|h'\|_{E^{(p)}} \|g'\|_{E^{(q)}} : h \cdot g = h' \cdot g'\right\},$$

and so $\odot$ *is an n.p. product.*

Note that if we consider $E = L^1(\mu)$ *we obtain that the pointwise product is an n.p. product from* $L^p(\mu) \times L^q(\mu)$ *to* $L^r(\mu)$. *In particular, if* μ *is the counting measure on* $\mathbb{N}$, *the pointwise product* $\odot : \ell^p(\mathbb{N}) \times \ell^q(\mathbb{N}) \to \ell^r(\mathbb{N})$ *is an n.p. product (for a more detailed information see [16, Lemma 1] or [17, Lemma 2.21(i)]).*

Example 2. *Let* E, F *be normed spaces and* $E \otimes F$ *denotes their algebraic tensor product. Projective norm* π *and injective norm* ε *on* $E \otimes F$ *are calculated by* $\pi(z) = \inf\left\{ \sum_{i=1}^{n} \|x_i\| \|y_i\| : z = \sum_{i=1}^{n} x_i \otimes y_i \right\}$, *and* $\varepsilon(z) = \sup\left\{ \langle x' \otimes y', z \rangle : x' \in B_{E'}, y' \in B_{F'} \right\}$, *respectively (see [8, Section 2,3]). It is well-known that any reasonable tensor norm* α *on the tensor product* $E \otimes F$ *satisfies the inequality* $\varepsilon \leq \alpha \leq \pi$. *For every* $(x, y) \in E \times F$, *it is seen that by the definitions of these norms*

$$\varepsilon(x \otimes y) \leq \alpha(x \otimes y) \leq \pi(x \otimes y) \leq \inf\{\|x'\| \|y'\| : x' \otimes y' = x \otimes y\}.$$

Besides, for every simple tensor $x \otimes y$ *it is known that for any reasonable tensor norm* α *we have* $\alpha(x \otimes y) = \|x\|_E \|y\|_F$ *(see [8, §12.1]). Then, any reasonable tensor norm satisfies the equality involving the norm in Definition 1. But the tensor product does not satisfy the inclusion, since clearly it is not surjective. So, it is not a norm preserving product.*

Example 3. *Let us define the following seminorm on* $X \otimes L(X, Y)$. *If* $z = \sum_{j=1}^{n} x_j \otimes T_j$ *is such that* $\sum_{j=1}^{n} T_j(x_j) = y_z \in Y$, *we define*

$$\pi_{\bullet}(z) = \inf\left\{ \pi(z') : z' = \sum_{j=1}^{m} x_j' \otimes T_j', \text{ such that } \sum_{j=1}^{m} T_j'(x_j') = y_z \right\}.$$

That is, $\pi_{\bullet}$ *is the quotient norm given by the tensor contraction* $c : X \hat{\otimes}_{\pi} L(X, Y) \to Y$ *defined as* $c(z) = c\big(\sum_{j=1}^{n} \sum_{j=1}^{n} x_j \otimes T_j \big) = \sum_{j=1}^{n} \bullet(x_j, T_j) = \sum_{j=1}^{n} T_j(x_j)$ *associated to the following factorization.*

$$\begin{tikzcd} X \times L(X,Y) \arrow[d, "\otimes"'] \arrow[dr, dotted, "\bullet"] & \\ X \hat{\otimes}_{\pi} L(X,Y) \arrow[r, "c"] & Y. \end{tikzcd}$$

The description of this seminorm can be found in [18]. It defines a norm if we construct a quotient space $X\hat{\otimes}_{\pi_\bullet}L(X,Y)$ by identifying the equivalence classes of the projective tensor product $X\hat{\otimes}_{\pi}L(X,Y)$ with the range of c in Y, i.e. $c(X\hat{\otimes}_{\pi}L(X,Y)) \subset Y$. Thus, for $z = \sum_{j=1}^{n} x_j \otimes T_j$ and $z' = \sum_{j=1}^{m} x'_j \otimes T'_j$, $z \sim z'$ if and only if $\sum_{j=1}^{m} T_j(x'_j) = \sum_{j=1}^{n} T_j(x_j)$. The norm of a class $[z] = \{z' : z \sim z'\}$, for $z = \sum_{j=1}^{n} x_j \otimes T_j$, is given by

$$\pi_\bullet(z) = \inf\{\pi(z') : z \sim z'\}.$$

Let us show that $\bullet$ is a norm preserving product.

Fix $T \in L(X,Y)$ and $x \in X$ and consider $y_z = T(x)$; clearly the inequality $\|y_z\| \leq \|T\|\|x\|$ holds. Now, consider another tensor $z = \sum_{i=1}^{n} x_i \otimes T_i$ such that $y_z = \sum_{i=1}^{n} T_i(x_i)$. Since $\|y_z\| = \|\sum_{i=1}^{n} T_i(x_i)\| \leq \sum_{i=1}^{n} \|T_i\|\|x_i\|$, we obtain that $\|x \bullet T\| = \|y_z\| \leq \pi_\bullet(z)$.

In the opposite direction, for $y \in Y$ there are elements $T_0 \in L(X,Y)$ and $x_0 \in X$ such that $T_0(x_0) = y$ and $\|y\| = \|T_0\|\|x_0\|$. To see this, just take a couple (x_0, x'_0) of norm one elements $x_0 \in X$ and $x'_0 \in X'$ such that $\langle x_0, x'_0 \rangle = 1$. Now define $T_0(x) := \langle x, x'_0 \rangle\, y$, $x \in X$, and note that $\|T_0\| = \|y\|$. Therefore, if $z = x_0 \otimes T_0$, we have that $y = y_z$. So, this gives in particular that $B_Y \subseteq \bullet(B_X \times B_{L(X,Y)})$, since $\pi_\bullet(z) \leq \|y_z\|$. Together with the inequality in the previous paragraph this also gives $\|x_0 \bullet T_0\| = \|y_z\| = \pi_\bullet(z)$. More precisely, we have proven that

$$\|x \bullet T\|_Y = \inf\left\{\|x_0\|_X \|T_0\|_{L(X,Y)} : x_0 \in X,\ T_0 \in L(X,Y),\ x \bullet T = x_0 \bullet T_0\right\}$$

for all $T \in L(X,Y)$ and $x \in X$. Thus, $\bullet$ is a norm preserving product.

Since to find the factors of a Banach space is a current problem in the mathematical literature, there are found more examples of the norm preserving products including the Banach function spaces (see [13, 15, 19]).

Let X, Y, Z be Banach spaces. A bilinear operator $B : X \times Y \to Z$ is called *$\circledast$-factorable* for the Banach valued n.p. product $\circledast : X \times Y \to G$ if there exists a linear continuous map $T : G \to Z$ such that B factors through T and $\circledast$ (see [12, Definition 1]).

In this case, the following triangular diagram

$$\begin{tikzcd} X \times Y \arrow[r, "B"] \arrow[dr, dotted, "\circledast"] & Z \\ & G \arrow[u, "T"'] \end{tikzcd}$$

holds. In the paper [12], Erdogan E. et al have proved a necessary and sufficint condition for $\odot$-factorability by a summability requirement as follows.

Lemma 1 (Lemma 1, [12]). *The bilinear operator $B : X \times Y \to Z$ is $\circledast$-factorable for the n.p. product $\circledast$ if and only if there exists a constant K such that for all $x_1, \ldots, x_n \in X$ and $y_1, \ldots, y_n \in Y$ we have*

$$\Big\|\sum_{i=1}^{n} B(x_i, y_i)\Big\|_Z \leq K \Big\|\sum_{i=1}^{n} x_i \circledast y_i\Big\|_G.$$

Example 4. *Consider a bilinear continuous operator $B : \ell^2(\mathbb{N}) \times \ell^2(\mathbb{N}) \to \ell^1(\mathbb{N})$. Let us use the result above for characterizing when B is $\odot$-factorable with respect to the pointwise product. It was shown in the first example that the pointwise product $\odot$ from $\ell^2(\mathbb{N}) \times \ell^2(\mathbb{N})$*

to $\ell^1(\mathbb{N})$ is an n.p. product. Let $(a,b) = \left(\sum_{k=1}^{\infty} \alpha_k \chi_{\{k\}}, \sum_{m=1}^{\infty} \beta_m \chi_{\{m\}}\right) \in \ell^2(\mathbb{N}) \times \ell^2(\mathbb{N})$. Then the image of this element under pointwise product is

$$a \odot b = \sum_{k=1}^{\infty} \alpha_k \sum_{m=1}^{\infty} \beta_m (\chi_{\{k\}} \odot \chi_{\{m\}}) = \sum_{k=1}^{\infty} \alpha_k \beta_k \chi_{\{k\}}.$$

Thus, for the finite sets of sequences $a_1, \ldots, a_n$, $b_1, \ldots, b_n$ we have

$$\sum_{i=1}^{n} a_i \odot b_i = \sum_{i=1}^{n} \sum_{k=1}^{\infty} \alpha_{ik} \beta_{ik} \chi_{\{k\}} = \sum_{k=1}^{\infty} \Big(\sum_{i=1}^{n} \alpha_{ik} \beta_{ik} \Big) \chi_{\{k\}}.$$

The $\ell^1(\mathbb{N})$ norm of this sequence is $\|(z_k)\|_{\ell^1(\mathbb{N})} = \sum_{k=1}^{\infty} |\sum_{i=1}^{n} \alpha_{ik}\beta_{ik}|$. By Lemma 1, we obtain that the bilinear operator B factors through the pointwise product if and only if there is a constant K for all finite sequences $(a_i)_{i=1}^n$, $(b_i)_{i=1}^n \subset \ell^2(\mathbb{N})$ such that

$$\Big\| \sum_{i=1}^{n} B(a_i, b_i) \Big\|_1 \leq K \sum_{k=1}^{\infty} \Big| \sum_{i=1}^{n} \alpha_{ik} \beta_{ik} \Big|.$$

Let us consider now a more specific bilinear operator $B : \ell^2(\mathbb{N}) \times \ell^2(\mathbb{N}) \to \ell^1(\mathbb{N})$: a diagonal multilinear operator. Recall that a bilinear operator $B \in \mathcal{B}(\ell^2(\mathbb{N}) \times \ell^2(\mathbb{N}), \ell^1(\mathbb{N}))$ is called bilinear diagonal if there is a bounded sequence $\xi = (\xi_k)_k$ such that $B(a,b) = \sum_{k=1}^{\infty} \xi_k \alpha_k \beta_k \chi_{\{k\}}$. By Hölder inequality, it is easily seen that $B \in \mathcal{B}(\ell^2(\mathbb{N}) \times \ell^2(\mathbb{N}), \ell^1(\mathbb{N}))$ if and only if $\xi \in \ell^\infty(\mathbb{N})$. For arbitrary finite sequences $(a_i)_{i=1}^n$, $(b_i)_{i=1}^n \subset \ell^2(\mathbb{N})$, we obtain

$$\Big\| \sum_{i=1}^{n} B(a_i, b_i) \Big\|_1 = \Big\| \sum_{i=1}^{n} \sum_{k=1}^{\infty} \xi_k \alpha_{ik} \beta_{ik} \chi_{\{k\}} \Big\|_1 \leq \|\xi_k\|_\infty \sum_{k=1}^{\infty} \Big| \sum_{i=1}^{n} \alpha_{ik} \beta_{ik} \Big| = K \sum_{k=1}^{\infty} \Big| \sum_{i=1}^{n} \alpha_{ik} \beta_{ik} \Big|.$$

Therefore, it is seen that every bilinear diagonal operator is factorable through $\odot$. Remark that a bilinear diagonal operator satisfies that $B(a,b) = 0$ whenever $a \odot b = 0$. We will prove in what follows that this is also a sufficient condition for factorability of bilinear operators defined on the topological product of sequence spaces.

3 The pointwise product in sequence spaces

Let us center our attention in this section in a particular product that is important in mathematical analysis. It is given by the pointwise product of sequences, functions and generalized sequences belonging to Banach lattices. In order to give a full generality to our results, we will consider several extensions of the bilinear map given by the pointwise products.

In the case of sequences, we will consider the following notion. The reference product is the pointwise product of sequences, that is $\odot : \ell^p(\mathbb{N}) \times \ell^q(\mathbb{N}) \to \ell^r(\mathbb{N})$, $a \odot b = a \cdot b = (a_i b_i)_{i=1}^{\infty} \in \ell^r(\mathbb{N})$, that is well-defined and continuous by Hölder's inequality. This is clearly an n.p. product, as have been explained in the previous section. Also, it has commutativity and associativity properties.

The following notion is crucial in this paper.

Let X, Y, Z be Banach spaces. We say that a bilinear continuous operator $B : X \times Y \to Z$ is *zero "product"-preserving* if it is 0-valued for couples of elements whose product equals 0.

Theorem 1. *Let $\dfrac{1}{p} + \dfrac{1}{q} = \dfrac{1}{r}$ for $1 \leq r < p, q < \infty$. Consider a bilinear operator $B : \ell^p(\mathbb{N}) \times \ell^q(\mathbb{N}) \to Y$. The following assertions are equivalent.*

(1) The operator B is zero $\odot$-preserving, i.e. $B(x,y)=0$ whenever $x\odot y=0$.

(2) The operator B is $\odot$-factorable. That is, there is a linear and continuous operator $T:\ell^r(\mathbb{N})\to Y$ such that $B=T\circ\odot$, and so we have the factorization

$$\begin{tikzcd} \ell^p(\mathbb{N})\times\ell^q(\mathbb{N}) \arrow[r, "B"] \arrow[dr, dotted, "\odot"'] & Y \\ & \ell^r(\mathbb{N}). \arrow[u, dotted, "T"'] \end{tikzcd}$$

Proof. Let us show that there is a linear continuous operator T such that $B:=T\circ\odot$ whenever the operator B is a zero $\odot$-preserving. Define the map $T_n:\ell^p(\mathbb{N})\odot\ell^q(\mathbb{N})\to Y$, $T_n(z):=B(z\odot\chi_{\{1,2,\dots,n\}},\chi_{\{1,2,\dots,n\}})$ for all $n\in\mathbb{N}$, where $z\in\ell^p(\mathbb{N})\odot\ell^q(\mathbb{N})$; note that $z\odot\chi_{\{1,2,\dots,n\}}\in\ell^p(\mathbb{N})$, and $\chi_{\{1,2,\dots,n\}}\in\ell^q(\mathbb{N})$, and so T_n is well defined for each $n\in\mathbb{N}$. The linearity of T_n is a consequence of the linearity of the bilinear operator B in the first variable. To show the boundedness of the map T_n, we give an equivalent formula for this operator. Since $\chi_{\{1,2,\dots,n\}}=\sum_{i=1}^n\chi_{\{i\}}$ by the properties of characteristic function, we have

$$T_n(a\odot b)=B(a\odot b\odot\chi_{\{1,2,\dots,n\}},\chi_{\{1,2,\dots,n\}})=\sum_{i=1}^n B(a\odot b\odot\chi_{\{1,2,\dots,n\}},\chi_{\{i\}}).$$

The pointwise product of $a=(\alpha_k)_{k=1}^\infty\in\ell^p(\mathbb{N})$ and $b=(\beta_k)_{k=1}^\infty\in\ell^q(\mathbb{N})$ is $a\odot b=(\alpha_k\beta_k)_{k=1}^\infty=\sum_{k=1}^\infty\alpha_k\beta_k\chi_{\{k\}}$. By the continuity of B, the image of the couple $(a,b)\in\ell^p(\mathbb{N})\times\ell^q(\mathbb{N})$ under the bilinear operator B is

$$B(a,b)=B\Big(\sum_{k=1}^\infty\alpha_k\chi_{\{k\}},\sum_{m=1}^\infty\beta_m\chi_{\{m\}}\Big)=\sum_{k=1}^\infty\alpha_k\sum_{m=1}^\infty\beta_m B(\chi_{\{k\}},\chi_{\{m\}}).$$

Since $\chi_{\{k\}}\odot\chi_{\{m\}}=0\,(k\neq m)$ and by the zero $\odot$-preservation of the operator B, we have $B(a,b)=\sum_{k=1}^\infty\alpha_k\beta_k B(\chi_{\{k\}},\chi_{\{k\}})$. Thus,

$$\begin{aligned}T_n(a\odot b)=\sum_{i=1}^n B(a\odot b\odot\chi_{\{1,2,\dots,n\}},\chi_{\{i\}})&=\sum_{i=1}^n B\Big(\sum_{k=1}^\infty\alpha_k\beta_k\chi_{\{k\}}\odot\chi_{\{1,2,\dots,n\}},\chi_{\{i\}}\Big)\\&=\sum_{i=1}^n\sum_{k=1}^n\alpha_k\beta_k B(\chi_{\{k\}},\chi_{\{i\}}).\end{aligned}$$

Using the zero $\odot$-preservation property once again, we obtain

$$T_n(a\odot b)=\sum_{i=1}^n\alpha_i\beta_i B(\chi_{\{i\}},\chi_{\{i\}})=B\Big(\sum_{i=1}^n\alpha_i\beta_i\chi_{\{i\}},\sum_{i=1}^n\chi_{\{i\}}\Big)=B\Big(\sum_{i=1}^n\alpha_i\chi_{\{i\}},\sum_{i=1}^n\beta_i\chi_{\{i\}}\Big).$$

By the boundedness of the bilinear operator B, it follows that

$$\begin{aligned}\sup_{z\in B_{\ell^r(\mathbb{N})}}\|T_nz\|_Y&=\sup_{\substack{(a,b)\in B_{\ell^p(\mathbb{N})}\times B_{\ell^q(\mathbb{N})}\\ z=a\odot b}}\Big\|B\Big(\sum_{i=1}^n\alpha_i\chi_{\{i\}},\sum_{i=1}^n\beta_i\chi_{\{i\}}\Big)\Big\|_Y\\&\leq\sup_{\substack{(a,b)\in B_{\ell^p(\mathbb{N})}\times B_{\ell^q(\mathbb{N})}\\ z=a\odot b}}\sum_{i=1}^n|\alpha_i\beta_i|\|B(\chi_{\{i\}},\chi_{\{i\}})\|_Y<\infty.\end{aligned}$$

This shows that T_n is (uniformly) bounded, $n \in \mathbb{N}$, and therefore $(T_n)_{n=1}^{\infty}$ is a bounded sequence of linear operators acting in $\ell^r(\mathbb{N})$, since $\ell^r(\mathbb{N}) = \ell^p(\mathbb{N}) \odot \ell^q(\mathbb{N})$. Indeed, note that since $\odot$ is an n.p. product, we have that it is surjective and preserves the norm, and so for every $z \in \ell^r(\mathbb{N})$ we find adequate $a \in \ell^p(\mathbb{N})$ and $b \in \ell^q(\mathbb{N})$ such that $z = a \odot b$.

The sequence $\{T_n(a \odot b)\}_{n=1}^{\infty}$ is a Cauchy sequence for every $a \in \ell^p(\mathbb{N})$ and $b \in \ell^q(\mathbb{N})$, and it is convergent by completeness of the Banach space Y. Indeed, since $a \odot b \in \ell^r(\mathbb{N})$, then for every $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that

$$\Big\| \sum_{i=n}^{\infty} \alpha_i \chi_{\{i\}} \Big\|_{\ell^p(\mathbb{N})} \Big\| \sum_{i=n}^{\infty} \beta_i \chi_{\{i\}} \Big\|_{\ell^q(\mathbb{N})} < \frac{\varepsilon}{\|B\|} \quad \forall n > N.$$

Using again that $B(\chi_{\{i\}}, \chi_{\{j\}}) = 0$ if $i \neq j$, we obtain

$$\begin{aligned} \|T_m(a \odot b) - T_n(a \odot b)\|_Y &= \Big\| B\Big(\sum_{i=n+1}^{m} \alpha_i \beta_i \chi_{\{i\}}, \sum_{i=n+1}^{m} \chi_{\{i\}} \Big) \Big\|_Y \\ &\leq \|B\| \Big\| \sum_{i=n+1}^{m} \alpha_i \chi_{\{i\}} \Big\|_{\ell^p(\mathbb{N})} \Big\| \sum_{i=n+1}^{m} \beta_i \chi_{\{i\}} \Big\|_{\ell^q(\mathbb{N})} < \varepsilon \quad \forall m > n > N. \end{aligned}$$

Let us define now the limit operator $T : \ell^r(\mathbb{N}) \to Y$ of the operator sequence $\{T_n\}$, that is $T(a \odot b) = \lim_{n\to\infty} T_n(a \odot b)$. It is easily seen that T is well defined and linear. Since $T_n(a \odot b)$ converges for every $a \odot b \in \ell^r(\mathbb{N})$, then it is bounded for every $a \odot b$. By the Uniform Boundedness Theorem, it follows that T is continuous. Therefore, we obtain

$$B(a, b) = \lim_{n\to\infty} \sum_{i=1}^{n} \alpha_i \beta_i B(\chi_{\{i\}}, \chi_{\{i\}}) = \lim_{n\to\infty} T_n(a \odot b) = T(a \odot b).$$

Besides, the image of an element is independent from its representation. Indeed, for the element $x = a_1 \odot b_1 = a_2 \odot b_2$, we obtain

$$\begin{aligned} T(a_1 \odot b_1) &= \lim_{n\to\infty} B(a_1 \odot b_1 \odot \chi_{\{1,2,\ldots,n\}}, \chi_{\{1,2,\ldots,n\}}) \\ &= \lim_{n\to\infty} B(a_2 \odot b_2 \odot \chi_{\{1,2,\ldots,n\}}, \chi_{\{1,2,\ldots,n\}}) = T(a_2 \odot b_2). \end{aligned}$$

Hence we obtain the factorization of the bilinear operator B through the pointwise product as $B = T \circ \odot$.

For the converse, assume that the map B is $\odot$-factorable. Then, by Lemma 1 given in [12] (see also page 59) it is obtained that there is a positive real number K such that, for all $x_1, \ldots, x_n \in \ell^p(\mathbb{N})$ and $y_1, \ldots, y_n \in \ell^q(\mathbb{N})$, the following inequality holds

$$\Big\| \sum_{i=1}^{n} B(x_i, y_i) \Big\|_Y \leq K \Big\| \sum_{i=1}^{n} x_i \odot y_i \Big\|_{\ell^r(\mathbb{N})}.$$

Clearly, this inequality implies zero $\odot$-preservation of the bilinear map B. This finishes the proof. ☐

Now we will give a generalization of our results. Consider two Banach spaces E and F that are isomorphic -as Banach spaces- to $\ell^p(\mathbb{N})$ and $\ell^q(\mathbb{N})$, respectively, and the isomorphisms are given by the operators $P : E \to \ell^p(\mathbb{N})$ and $Q : F \to \ell^q(\mathbb{N})$. We define the product $\odot_{P\times Q} : E \times F \to \ell^r(\mathbb{N})$ by

$$\odot_{P\times Q}(x, y) = P(x) \odot Q(y), \quad x \in E, \quad y \in F.$$

To make this definition more understandable, let us illustrate it by the following diagram

$$\begin{tikzcd} E \times F \arrow[r, "\odot_{P\times Q}"] \arrow[d, dotted, "P\times Q"'] & \ell^r(\mathbb{N}). \\ \ell^p(\mathbb{N}) \times \ell^q(\mathbb{N}) \arrow[ur, dotted, "\odot"'] & \end{tikzcd}$$

In this situation considered above of the product $\odot_{P\times Q} = P(.) \odot Q(.)$, a bilinear map $B : E \times F \to Y$ is zero $\odot_{P\times Q}$-preserving if

$$\odot_{P\times Q}(x, y) = 0 \quad \text{implies} \quad B(x, y) = 0$$

for all $x \in E$ and $y \in F$. Namely, the map B is said to be zero $\odot_{P\times Q}$-preserving if $B(x, y) = 0$ whenever $P(x) \odot Q(y) = 0$.

Theorem 2. *Let $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ for $1 \leq r < p, q < \infty$. Let the Banach spaces E and F be isomorphic to $\ell^p(\mathbb{N})$ and $\ell^q(\mathbb{N})$ by means of the isomorphisms P and Q, respectively. Consider a Banach valued bilinear operator $B : E \times F \to Y$. The following assertions imply each other.*

(1) The operator B is $\odot_{P\times Q}$-factorable. That is, there exists a linear continuous operator $T : \ell^r(\mathbb{N}) \to Y$ such that $B = T \circ \odot_{P\times Q}$, and the following diagram commutes.

$$\begin{tikzcd} E \times F \arrow[r, "B"] \arrow[d, dotted, "P\times Q"'] & Y \\ \ell^p(\mathbb{N}) \times \ell^q(\mathbb{N}) \arrow[r, dotted, "\odot"] & \ell^r(\mathbb{N}). \arrow[u, dotted, "T"'] \end{tikzcd}$$

(2) There is a positive real number K such that, for every finite set of elements $\{x_i\}_{i=1}^n \in E$ and $\{y_i\}_{i=1}^n \in F$, the following inequality holds

$$\Big\| \sum_{i=1}^n B(x_i, y_i) \Big\|_Y \leq K \Big\| \sum_{i=1}^n P(x_i) \odot Q(y_i) \Big\|_{\ell^r(\mathbb{N})}.$$

(3) The operator B is zero $\odot_{P\times Q}$-preserving, that is, $x \odot_{P\times Q} y = 0$ implies $B(x, y) = 0$.

Proof. Let us prove that (3) implies (1). Under the conditions of the theorem, consider the bilinear map $\overline{B} = B \circ (P^{-1} \times Q^{-1}) : \ell^p(\mathbb{N}) \times \ell^q(\mathbb{N}) \to Y$. We have that for all $x \in E$ and $y \in F$, $x \odot_{P\times Q} y = P(x) \odot Q(y) = 0$ implies that $0 = B(x, y) = \overline{B}(P(x), Q(y)) = 0$. That is, since P and Q are isomorphisms, we have that for all $a \in \ell^p(\mathbb{N})$ and $b \in \ell^q(\mathbb{N})$, $a \odot b = 0$ implies that $\overline{B}(a, b) = 0$.

We are in situation of using Theorem 1 for $\overline{B}$. So we have that there is a linear operator $T : \ell^r(\mathbb{N}) \to Y$ such that $\overline{B} = T \circ \odot$. By the definition of $\overline{B}$, we obtain $B = \overline{B} \circ (P \times Q) = T \circ \odot \circ (P \times Q)$, the required factorization.

The equivalences among the three statements of the theorem follow directly using Lemma 1 in [12] and this factorization. $\square$

We will say a bilinear map $B : X \times X \to Y$ is *symmetric* if $B(f,g) = B(g,f)$ for every couple $(f,g) \in X \times X$.

It is easily seen that any $\odot$-factorable bilinear map $B : \ell^p(\mathbb{N}) \times \ell^p(\mathbb{N}) \to Y$ factorized through $\ell^r(\mathbb{N})$ for $2r = p$ is symmetric, since $B(a_n, b_n) = T(a_n \odot b_n) = T(b_n \odot a_n) = B(b_n, a_n)$ holds for all $(a_n)_{n=1}^{\infty}$, $(b_n)_{n=1}^{\infty} \in \ell^p(\mathbb{N})$ by the commutativity of the pointwise product.

Now, we will give symmetry condition for general product.

Corollary 1. *Let the Banach space X be isomorphic to $\ell^p(\mathbb{N})$ for $p \geq 2$. Then any zero $\odot_{P\times P}$-preserving bilinear map $B : X \times X \to Y$ satisfies the symetry condition, that is $B(x,y) = B(y,x)$ for all x, $y \in X$.*

Proof. Since the map B is zero $\odot_{P\times P}$-preserving, it is $\odot_{P\times P}$-factorable. Then, for $r = p/2$ there is a linear continuous map $T : \ell^r(\mathbb{N}) \to Y$ defined by $B(x,y) = T \circ \odot \circ (P \times P)(x,y) = T(P(x) \odot P(y))$. By the commutativity of the pointwise product we get the symmetry

$$B(x,y) = T(P(x) \odot P(y)) = T(P(y) \odot P(x)) = B(y,x).$$

□

Remark 1. *The extension of the result given in Theorem 1 from the case of $\odot$ to the case of $\odot_{P\times Q}$ products implicitly shows a fundamental fact about factorization through the pointwise product. The requirement "$a \odot b = 0$ implies $B(a,b) = 0$" can be understood as a lattice-type property: indeed, note that for sequences a and b in the corresponding spaces, $a \odot b = 0$ if and only if a and b are disjoint, and so we can rewrite the requirement of being zero $\odot$-preserving as "if $|a| \wedge |b| = 0$, then $B(a,b) = 0$". Since P and Q are just (Banach space) isomorphisms, we have shown that the property is primarely related to the pointwise product, and not to the lattice properties. The result is particularly meaningful if we consider P and Q to be the isomorphisms associated to changes of unconditional basis of $\ell^p(\mathbb{N})$ and $\ell^q(\mathbb{N})$ whose elements are not in general disjoint.*

Remark 2. *Consider the bilinear map $B : E \times E' \to Y$, where E' denotes the topological dual of E. This bilinear map can only be $\odot_{P\times Q}$-factorable through the sequence space $\ell^1(\mathbb{N})$. Indeed, let P denote the isomorphism between E and $\ell^p(\mathbb{N})$ $(p \geq 1)$. Since the duals of isomorphic spaces are isomorphic, it follows that E' is isomorphic to $(\ell^p(\mathbb{N}))' = \ell^{p'}(\mathbb{N})$ for $\frac{1}{p} + \frac{1}{p'} = 1$ by the isomorphism P' that is adjoint map of P. Therefore B can only be $\odot_{P\times P'}$-factorable and in this case it is factorized through $\ell^1(\mathbb{N})$.*

3.1 Compactness properties of zero $\odot_{P\times Q}$-preserving bilinear maps

Theorem 2 provides a useful tool to obtain the main properties of zero $\odot_{P\times Q}$-preserving bilinear maps. It is already clear that (weakly) compactness of the factorization map T is necessary and sufficient condition for the (weakly) compactness of the zero $\odot_{P\times Q}$-preserving map B by the definition of the norm preserving product. Indeed, for a zero $\odot_{P\times Q}$-preserving map B,

$$\begin{aligned} B \text{ is (weakly) compact} &\iff B(U_X \times U_Y) \text{ is relatively (weakly) compact} \\ &\iff \overline{B}(U_{\ell^p(\mathbb{N})} \times U_{\ell^q(\mathbb{N})}) \text{ is relatively (weakly) compact} \\ &\iff T(U_{\ell^r(\mathbb{N})}) \text{ is relatively (weakly) compact} \\ &\iff T \text{ is (weakly) compact.} \end{aligned}$$

Now, we will give more specific situations.

Proposition 1. *Let* $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ *for* $1 \leq r < p, q < \infty$. *Suppose that there are isomorphisms* P *and* Q *such that the bilinear operator* $B : E \times F \to Y$ *is zero* $\odot_{P\times Q}$*-preserving. Then*

(i) $B(E \times F)$ *is a linear space;*

(ii) if P *and* Q *are isometries, then* $B(B_E \times B_F)$ *is convex;*

(iii) if $r = 1$ *and* Y *is reflexive, then* $B(B_E \times y)$ *is a relatively compact set for every* $y \in F$ *as well as* $B(x \times B_F)$ *is relatively compact for every* $x \in E$*;*

(iv) if $r > 1$, *then* $B(B_E \times B_F)$ *is relatively weakly compact;*

(v) if $1 \leq s < r < \infty$ *and* $Y = \ell^s(\mathbb{N})$, *then* $B(B_E \times B_F)$ *is relatively compact.*

Proof. Consider the factorization for B given by $B = T \circ (P \odot Q)$.

(i) Since $\odot$ is an n.p. product and B factors through it by Theorem 2, we have that $B(E \times F) = T(\ell^p(\mathbb{N}) \odot \ell^q(\mathbb{N})) = T(\ell^r(\mathbb{N}))$, that is, the range of a linear map. So it is a linear space.

(ii) Clearly, $A = P \odot Q(B_E \times B_F) = B_{\ell^p(\mathbb{N})} \odot B_{\ell^q(\mathbb{N})} = B_{\ell^r(\mathbb{N})}$ is a convex set, and so $T(A)$ is also convex.

(iii) Note that there is a sequence $b = Q(y)$ such that $A = P \odot Q(B_E, y)$ is equivalent to $B_{\ell^p(\mathbb{N})} \odot b \subset \ell^1(\mathbb{N})$. Recall that $1 < p, q < \infty$. Note also that $T : \ell^1(\mathbb{N}) \to Y$ is weakly compact by the reflexivity of the range space Y. Since A is a weakly compact set in $\ell^1(\mathbb{N})$ we have that $T(A)$ is relatively compact by the Dunford-Pettis property of $\ell^1(\mathbb{N})$.

(iv) Since $B(B_E \times B_F) = T(P(B_E) \odot Q(B_F))$, and $P(B_E) \odot Q(B_F)$ is equivalent to the unit ball of the reflexive space $\ell^r(\mathbb{N})$, we get the result.

(v) Recall that by Pitt's Theorem (see [9, Ch. 12]), every bounded linear operator from $\ell^r(\mathbb{N})$ into $\ell^s(\mathbb{N})$ is compact whenever $1 \leq s < r < \infty$. The factorization gives directly the result. ☐

3.2 Zero $\odot_{P\times Q}$-preserving bilinear operators among Hilbert spaces

In this section, assume that E, F and Y are separable Hilbert spaces. Our first result shows a summability property of zero product preserving bilinear maps, and is a direct consequence of Grothendieck's Theorem. It also provides an integral domination for B. The second corollary is obtained as a result of the Schur's property of $\ell^1(\mathbb{N})$ (recall that a Banach space has the Schur's property if weakly convergent sequences and norm convergent sequences are the same) and it is again an application of the compactness properties of the bounded subsets of $\ell^1(\mathbb{N})$.

Corollary 2. *Let* H_1, H_2 *and* H_3 *be separable Hilbert spaces. Let* $B : H_1 \times H_2 \to H_3$ *be a zero* $\odot_{P\times Q}$*-preserving bilinear operator. Then*

(i) for every $x_1, \ldots, x_n \in H_1, y_1, \ldots, y_n \in H_2$ *there is a constant* $K > 0$ *such that*

$$\sum_{i=1}^{n} \left\| B(x_i, y_i) \right\| \leq K \sup_{z' \in B_{\ell^\infty(\mathbb{N})}} \sum_{i=1}^{n} \left| \left\langle P(x_i) \odot Q(y_i), z' \right\rangle \right|,$$

(ii) and there is a regular Borel measure η *over* $B_{\ell^\infty(\mathbb{N})}$ *such that*

$$\|B(x, y)\| \leq K \int_{B_{\ell^\infty(\mathbb{N})}} |\langle P(x) \odot Q(y), z' \rangle| \, d\eta(z'), \quad x \in H_1, \; y \in H_2.$$

Proof. Let us consider the zero $\odot_{P\times Q}$-preserving bilinear map $B : H_1 \times H_2 \to H_3$. Since any separable Hilbert space is isomorphic to the sequence space $\ell^2(\mathbb{N})$, we can define a bilinear map $\overline{B} = B(P^{-1} \times Q^{-1}) : \ell^2(\mathbb{N}) \times \ell^2(\mathbb{N}) \to H_3$. The zero $\odot_{P\times Q}$-preserving property of B implies the $\odot$-preserving property of the map $\overline{B}$. Therefore, by Theorem 2 we have the factorization $\overline{B} := T \circ \odot$, where $T : \ell^1(\mathbb{N}) \to H_3$. One of the results of Grothendieck's Theorem states that every linear operator from $\ell^1(\mathbb{N})$ to a Hilbert space is 1-summing. It follows that, for every $x_1, \ldots, x_n \in H_1$, $y_1, \ldots, y_n \in H_2$ there is a constant $K > 0$ such that

$$\sum_{i=1}^{n} \left\| B(x_i, y_i) \right\| = \sum_{i=1}^{n} \left\| \overline{B}(P(x_i), Q(y_i)) \right\| \le K \sup_{z' \in B_{\ell^\infty(\mathbb{N})}} \sum_{i=1}^{n} \left| \left\langle P(x_i) \odot Q(y_i), z' \right\rangle \right|.$$

The second inequality of the corollary given above is clearly seen by Pietsch Domination Theorem (see [9, Theorem 2.12]). This theorem states that every 1-summable operator has such a regular Borel measure. Thus, we get a regular Borel measure η over $B_{\ell^\infty(\mathbb{N})}$ satisfying

$$\|B(x,y)\| = \|\overline{B}(P(x), Q(y))\| \le K \int_{B_{\ell^\infty(\mathbb{N})}} |\langle P(x) \odot Q(y), z' \rangle| \, d\eta(z')$$

for $x \in H_1, \; y \in H_2$. □

Corollary 3. *Let H_1, H_2 and H_3 be separable Hilbert spaces. Let $B : H_1 \times H_2 \to H_3$ be a zero $\odot_{P\times Q}$-preserving bilinear operator. Then*

(i) for every couple of sequences $(x_i)_{i=1}^{\infty}$ in H_1 and $(y_i)_{i=1}^{\infty}$ in H_2 such that $(P(x_i) \odot Q(y_i))_{i=1}^{\infty}$ is weakly convergent, we have that $(B(x_i, y_i))_{i=1}^{\infty}$ converges in the norm;

(ii) for $S_1 \subseteq H_1$ and $S_2 \subseteq H_2$ such that $P(S_1) \odot Q(S_2) \subseteq \ell^1(\mathbb{N})$ is relatively weakly compact, we have that $B(S_1 \times S_2)$ is relatively compact.

We can obtain some (weaker) summability results if we consider the range space Y with some cotype-related properties. It is known that a Banach space has the Orlicz property, if it is of cotype 2 (see [8, 8.9]). Recall that a Banach space is said to have the Orlicz property if the identity map in it is (2,1)-summing. It follows that for any zero $\odot_{P\times Q}$-preserving bilinear map $B : E \times F \to Y$ whose range space Y has the Orlicz property, we get a domination as follows: for $f_1, \ldots, f_n \in E$ and $g_1, \ldots, g_n \in F$,

$$\Big(\sum_{i=1}^{n} \|B(f_i, g_i)\|_Y^2 \Big)^{1/2} \le k \sup_{\varepsilon_i \in \{-1,1\}} \Big\| \sum_{i=1}^{n} \varepsilon_i \left(P(f_i) \odot Q(g_i) \right) \Big\|_{\ell^r(\mathbb{N})}.$$

Let us finish the paper with an application by using convolution maps defined on sequence spaces and function spaces.

3.3 Application: convolution maps

Consider any bilinear map $B : L^2[0,2\pi] \times L^2[0,2\pi] \to Y$ such that $B(f,g) = 0$ whenever $f, g \in L^2[0,2\pi]$ are such that $f \odot_{\widehat{\ } \times \widehat{\ }} g = \widehat{f} \odot \widehat{g} = 0$, where $\widehat{\ }$ denotes the Fourier transform. Plancherel's well-known theorem states that the Banach space $L^2[0,2\pi]$ is isometrically isomorphic to $\ell^2(\mathbb{Z})$ by the Fourier transform. Therefore, the bilinear map B is zero

$\odot_{\widehat{}\times\widehat{}}$-preserving. The class of these bilinear maps was investigated by Erdoğan E. et al in [10] by the term $*$-continuous map and they gave a factorization for B such that

$$B = T \circ \check{} \circ \odot \circ (\widehat{}\times\widehat{}) = T \circ *,$$

where $\check{}$ is the inverse Fourier transform.

Now, we will give a more specific example. $\mathcal{H}$ and $\mathcal{H}^2$ stand for the holomorphic functions on the unit disc $\mathbb{D}$ and Hardy space of the functions, respectively. Recall that Hardy space $\mathcal{H}^2$ consists of the functions whose all Fourier coefficients are zero with negative index, besides, it is closed subspace of $L^2[0,2\pi]$ which is isomorphically isomorphic to the sequence space $\ell^2(\mathbb{N})$ by Fourier transform. It is possible to represent any holomorphic function $f \in \mathcal{H}$ as a Taylor polynomial $f(z) = \sum_{n=0}^{\infty} a_n z^n$. This representation is given by the Fourier coefficients for the elements of $\mathcal{H}^2$ whenever $f \in \mathcal{H}^2$.

Arregui and Blasco defined the u-convolution of the holomorphic functions f and g in $\mathcal{H}$ given by $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ as $f *_u g(z) = \sum_{n=0}^{\infty} u(a_n, b_n) z^n$, where $u : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is a bilinear continuous map (see [3, Definition 1.1.]). If we consider the bilinear map u defined as $u(a_n, b_n) = a_n \odot b_n$, then we get $f *_u g(z) = \sum_{n=0}^{\infty} (a_n \odot b_n) z^n$. Therefore, it is seen that u-convolution defined on $\mathcal{H}^2 \times \mathcal{H}^2$ to $\mathcal{H}^2$ is a zero $\odot_{\widehat{}\times\widehat{}}$-preserving, since $f \odot_{\widehat{}\times\widehat{}} g = \widehat{f}(n) \odot \widehat{g}(n) = 0$ implies $f *_u g = 0$ for all $f, g \in \mathcal{H}^2$. By Theorem 2, it follows that there is a linear map $T : \ell^1(\mathbb{N}) \to \mathcal{H}^2$ such that $f *_u g = T(\widehat{f}(n) \odot \widehat{g}(n)) = \sum_{n=0}^{\infty} x_n z^n$, where $(x_n)_{n=1}^{\infty}$ is the sequence in $\ell^1(\mathbb{N})$ obtained by the pointwise product $\widehat{f}(n) \odot \widehat{g}(n)$. Also, by Corollary 1 it is obtained that u-convolution is a symmetric map.

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Ердоган Е. *Білінійні оператори, що зберігають нульовий добуток, на просторах послідовностей* // Карпатські матем. публ. — 2020. — Т.12, №1. — С. 55–68.

Розглянемо пару просторів послідовностей і функцію добутку (канонічне білінійне відображення, асоційоване з поточковим множенням), що діє на ньому. Ми аналізуємо клас білінійних операторів, що "зберігають нульовий добуток", асоційований з цим добутком, визначених таким чином, що вони дорівнюють нулю на парах, в яких добуток дорівнює нулю. Білінійні оператори, що належать цьому класу, вже досліджувалися в контексті банахових алгебр, вони можуть бути охарактеризовані в термінах факторизації $\ell^r(\mathbb{N})$ просторів. Використовуючи це, ми демонструємо основні властивості цих відображень, такі як компактність і сумовність.

Ключові слова і фрази: простори послідовностей, білінійні оператори, факторизація, зберігаюче нульовий добуток відображення, добуток.

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LEGENDRIAN NORMALLY FLAT SUBMANIFOLS OF $\mathcal{S}$-SPACE FORMS

In the present study, we consider a Legendrian normally flat submanifold M of $(2n+s)$-dimensional $\mathcal{S}$-space form $\widetilde{M}^{2n+s}(c)$ of constant φ-sectional curvature c. We have shown that if M is pseudo-parallel then M is semi-parallel or totally geodesic.

We also prove that if M is Ricci generalized pseudo-parallel, then either it is minimal or $L = \frac{1}{n-1}$, when $c \neq -3s$.

Key words and phrases: $\mathcal{S}$-space form, Legendrian submanifold, normally flat submanifold, pseudo-parallel submanifold, Ricci generalized pseudo-parallel submanifold.

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INTRODUCTION

An n-dimensional submanifold M in an m-dimensional Riemannian manifold $\widetilde{M}$ is *pseudo-parallel* [1,2], if its second fundamental form σ satisfies the following condition

$$\widetilde{R} \,.\, \sigma = L\, Q(g, \sigma), \tag{1}$$

where $\widetilde{R}$ is the curvature operator with respect to the Van der Waerden-Bortolotti connection $\widetilde{\nabla}$ of $\widetilde{M}$, L is some smooth function on M and $Q(g,\sigma)$ is a $(0,4)$ tensor on M determined by $Q(g,\sigma)(Z,W;X,Y) = ((X \wedge_g Y)\,.\,\sigma)(Z,W)$. Recall that the $(0,k+2)$-tensor $Q(B,T)$ associated with any $(0,k)$-tensor field T, $k \geq 1$, and $(0,2)$-tensor field B, is defined by

$$\begin{aligned} Q(B,T)(X_1, X_2, \dots, X_k; X, Y) &= ((X \wedge_B Y)\,.\,T)(X_1, X_2, \dots, X_k) \\ &= -T((X \wedge_B Y)X_1, X_2, \dots, X_k) - \dots - T(X_1, X_2, \dots, X_{k-1}, (X \wedge_B Y)X_k), \end{aligned} \tag{2}$$

where $X \wedge_B Y$ is defined by

$$(X \wedge_B Y)Z = B(Y,Z)X - B(X,Z)Y. \tag{3}$$

In particular, if $L = 0$, M is called a *semi-parallel* submanifold. Pseudo-parallel submanifolds were introduced in [1,2] as naturel extension of semi-parallel submanifolds and as the extrinsic analogues of pseudo-symmetric Riemannian manifolds in the sense of Deszcz [7], which generalize semi-symmetric Riemannian manifolds. On the other hand, Murathan et al. [11] defined submanifolds satisfying the condition

$$\widetilde{R} \,.\, \sigma = L\, Q(S, \sigma), \tag{4}$$

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where S is the Ricci tensor of M. The kind of submanifolds are called *Ricci generalized pseudo-parallel*. Recently, many authors studied pseudo-parallel and Ricci generalized pseudo-parallel submanifolds on various spaces, where the ambient manifold $\widetilde{M}$ has constant sectional curvature, we refer for example to [2, 5, 10–12, 14]. An integral submanifold of maximal dimension M^n of an $\mathcal{S}$-manifold $\widetilde{M}^{2n+s}$ is called *Legendrian* and it plays an important role in contact geometry. The study of Legendrian submanifolds of Sasakian manifolds from the Riemannian geometry point of view was initiated in 1970's. Legendrian submanifolds like their analogues in symplectic geometry, i.e. Lagrangian submanifolds. In [12], authors showed that a pseudo-parallel integral minimal submanifold M^n of an $\mathcal{S}$-space form $\widetilde{M}^{2n+s}(c)$ is totally geodesic if $Ln - \frac{1}{4}(n(c+3s)+c-s) \geq 0$.

In this work, we mainly prove that if a Legendrian normally flat submanifold M of an $\mathcal{S}$-space form $\widetilde{M}^{2n+s}(c)$ is pseudo-parallel (resp. Ricci generalized pseudo-parallel) then it is semi-parallel or totally geodesic (resp. minimal or $L = \frac{1}{n-1}$).

1 PRELIMINARIES

We remember some necessary useful notions and results for our next considerations. Let $\widetilde{M}^n$ be an n-dimensional Riemannian manifold and M^m an m-dimensional submanifold of $\widetilde{M}^n$. Let g be the metric tensor field on $\widetilde{M}^n$ as well as the metric induced on M^m. We denote by $\widetilde{\nabla}$ the covariant differentiation in $\widetilde{M}^n$ and by ∇ the covariant differentiation in M^m. Let $T\widetilde{M}$ (resp. TM) be the Lie algebra of vector fields on $\widetilde{M}^n$ (resp. on M^m) and $T^\perp M$ the set of all vector fields normal to M^m. The Gauss-Weingarten formulas are given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y), \quad \widetilde{\nabla}_X V = -A_V X + \nabla^\perp_X V,$$

$X, Y \in TM$, $V \in T^\perp M$, where $\nabla^\perp$ is the connection in the normal bundle, σ is the second fundamental form of M^m and A_V is the Weingarten endomorphism associated with V. A_V and σ are related by $g(A_V X, Y) = g(\sigma(X,Y), V) = g(X, A_V Y)$.

The submanifold M^m is said to be *totally geodesic* in $\widetilde{M}^n$ if its second fundamental form is identically zero and it is said to be *minimal* if $H \equiv 0$, where H is the mean curvature vector defined by $H = \frac{1}{m} trace(\sigma)$ [13].

We denote by $\widetilde{R}$ and R the curvature tensors associated with $\widetilde{\nabla}$, ∇ and $\nabla^\perp$ respectively.

The basic equations of Gauss and Ricci are

$$g(\widetilde{R}(X,Y)Z,W) = g(R(X,Y)Z,W) + g(\sigma(X,Z),\sigma(Y,W)) - g(\sigma(X,W),\sigma(Y,Z)),$$

$$g(\widetilde{R}(X,Y)N,V) = g(R^\perp(X,Y)N,V) - g([A_N, A_V]X,Y),$$

respectively, $X,Y,Z,W \in TM$, $N,V \in T^\perp M$.

The covariant derivative $\widetilde{\nabla}\sigma$ of the second fundamental form σ is given by

$$\widetilde{\nabla}_X \sigma(Y,Z) = \nabla^\perp_X(\sigma(Y,Z)) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z).$$

The operators $\widetilde{R}(X,Y)$ from the curvature of $\widetilde{\nabla}$ and $X \wedge Y$ can be extended as derivations of tensor fields in the usual way, so

$$(\widetilde{R}(X,Y).\sigma)(Z,W) = R^\perp(X,Y)(\sigma(Z,W)) - \sigma(R(X,Y)Z,W) - \sigma(Z,R(X,Y)W). \tag{5}$$

Putting $B = g$, $T = \sigma$ in (2) and (3), we get

$$\begin{aligned} Q(g,\sigma)(Z,W;X,Y) &= ((X \wedge Y)\,.\,\sigma)(Z,W) = -\sigma((X \wedge Y)Z,W) - \sigma(Z,(X \wedge Y)W) \\ &= -g(Y,Z)\sigma(X,W) + g(X,Z)\sigma(Y,W) - g(Y,W)\sigma(Z,X) + g(X,W)\sigma(Z,Y). \end{aligned} \tag{6}$$

Let $\widetilde{M}^{2n+s}$ be a $(2n+s)$-dimensional Riemannian manifold endowed with an φ-structure (that is a tensor field of type $(1,1)$ and rank $2n$ satisfying $\varphi^3 + \varphi = 0$). If moreover there exist on $\widetilde{M}^{2n+s}$ global vector fields $\xi_1, \ldots, \xi_s$ (called structure vector fields), and their duals 1-forms $\eta_1, \ldots, \eta_s$ such that for all $X, Y \in T\widetilde{M}$ and $\alpha, \beta \in \{1, \ldots, s\}$ (see [8])

$$\eta_\alpha(\xi_\beta) = \delta_{\alpha\beta},\ \ \varphi\xi_\alpha = 0,\ \ \eta_\alpha(\varphi X) = 0,\ \ \varphi^2 X = -X + \textstyle\sum_{\alpha=1}^{s} \eta_\alpha(X)\xi_\alpha\,, \tag{7}$$

then there exists on $\widetilde{M}$ a Riemannian metric g satisfying

$$g(X,Y) = g(\varphi X, \varphi Y) + \textstyle\sum_{\alpha=1}^{s} \eta_\alpha(X)\eta_\alpha(Y), \tag{8}$$

and

$$\eta_\alpha(X) = g(X,\xi_\alpha),\ \ g(\varphi X, Y) = -g(X, \varphi Y), \tag{9}$$

for all $\alpha \in \{1, \ldots, s\}$, $\widetilde{M}$ is then said to be a metric φ-manifold. The φ-structure is normal if $N_\varphi + 2\sum_{\alpha=1}^{s} \xi_\alpha \otimes d\eta_\alpha = 0$, where N_φ is the Nijenhuis torsion of φ.

Let Φ be the fundamental 2-form on M defined for all vector fields X, Y on $\widetilde{M}$ by $\Phi(X,Y) = g(X, \varphi Y)$. A normal metric φ-structure with closed fundamental 2-form will be called K-structure and $\widetilde{M}^{2n+s}$ called K-manifold. Finally, if $d\eta_1 = \ldots = d\eta_s = \Phi$, the K-structure is called $\mathcal{S}$-structure and $\widetilde{M}$ is called *$\mathcal{S}$-manifold*.

The Riemannian connection $\widetilde{\nabla}$ of an $\mathcal{S}$-manifold satisfies [3]

$$\widetilde{\nabla}_X \xi_\alpha = -\varphi X,\ \alpha \in \{1, \ldots, s\},$$

$$(\widetilde{\nabla}_X \varphi)Y = \textstyle\sum_{\alpha=1}^{s} (g(\varphi X, \varphi Y)\xi_\alpha + \eta_\alpha(Y)\varphi^2 X), \quad X, Y \in T\widetilde{M},$$

where $\widetilde{\nabla}$ is the Levi-Civita connection of g.

A plane section π is called an *φ-section* if it is determined by a unit vector X, normal to the structure vector fields and φX. The sectional curvature of π is called an *φ-sectional curvature*. An $\mathcal{S}$-manifold is said to be an *$\mathcal{S}$-space form* if it has constant φ-sectional curvature c and then, it is denoted by $\widetilde{M}^{2n+s}(c)$ $(n > 1)$ and its curvature tensor has the form [9]

$$\begin{aligned} \widetilde{R}(X,Y)Z = {} & \frac{c+3s}{4}\Big\{ g(\varphi X, \varphi Z)\varphi^2 Y - g(\varphi Y, \varphi Z)\varphi^2 X \Big\} \\ & + \frac{c-s}{4}\Big\{ g(\varphi Y, Z)\varphi X - g(\varphi X, Z)\varphi Y + 2g(X, \varphi Y)\varphi Z \Big\} \\ & + \sum_{\alpha,\beta=1}^{s} \Big\{ \eta_\alpha(X)\eta_\beta(Z)\varphi^2 Y - \eta_\alpha(Y)\eta_\beta(Z)\varphi^2 X \\ & \qquad + g(\varphi Y, \varphi Z)\eta_\alpha(X)\xi_\beta - g(\varphi X, \varphi Z)\eta_\alpha(Y)\xi_\beta \Big\}, \end{aligned} \tag{10}$$

for all $X, Y, Z \in T\widetilde{M}$.

When $s = 1$, an $\mathcal{S}$-space form $\widetilde{M}(c)$ reduces to a Sasakian space form $\widetilde{M}(c)$ and $s = 0$ becomes a complex space form.

2 PSEUDO-PARALLEL LEGENDRIAN SUBMANIFOLDS OF AN $\mathcal{S}$-SPACE FORM

Let M^n be an n-dimensional submanifold of an $\mathcal{S}$-space form $\widetilde{M}^{2n+s}(c)$. If $\eta_\alpha(X) = 0$, $\alpha \in \{1, \ldots, s\}$, for every tangent vector X to M, then we say M is a *Legendrian submanifold*. Recall that a submanifold M of $\widetilde{M}$ is an anti-invariant submanifold if $\varphi(TM) \subseteq T^{\perp}M$. So, a Legendrian submanifold is identical with an anti-invariant submanifold normal to the structure vector fields $\xi_1, \ldots, \xi_s$. Actually, a Legendrian submanifold is special an integral submanifold. Therefore, from (8) and (9) we obtain

$$g(\varphi X, \varphi Y) = g(X, Y), \ \eta_\alpha(X) = g(X, \xi_\alpha) = 0,$$

for any $X, Y \in TM$ and $\alpha \in \{1, \ldots, s\}$. Then we have the following known Lemma (see [4]).

Lemma 1. *Let M^n be a Legendrian submanifold of an $\mathcal{S}$-manifold, then*

$$A_{\xi_\alpha} = 0,$$
$$A_{\varphi X}Y = A_{\varphi Y}X, \tag{11}$$

for all $\alpha \in \{1, \ldots, s\}$ and $X, Y \in TM$.

The previous Lemma implies immediately the following result.

Lemma 2. *For a Legendrian submanifold M^n of an $\mathcal{S}$-manifold $\widetilde{M}^{2n+s}$, the following equations*

$$g(\sigma(X, Y), \varphi Z) = g(\sigma(X, Z), \varphi Y), \tag{12}$$

$$A_{\varphi X}Y = -\varphi\sigma(X, Y) = A_{\varphi Y}X \tag{13}$$

hold for all $X, Y, Z \in TM$.

Moreover, from (7) and (13) we obtain

$$\varphi A_{\varphi X}Y = \sigma(X, Y) = \varphi A_{\varphi Y}X. \tag{14}$$

Using (14), (9) and the Gauss equation, we have

$$\widetilde{R}(X, Y) = R(X, Y) - [A_{\varphi X}, A_{\varphi Y}]. \tag{15}$$

We recall that the submanifold M is said to have *flat normal connection* (or trivial normal connection) if $R^{\perp} = 0$. If M has normal connection flat then we call it to be *normally flat*.

Then, making use of (14), (5) and (6), if M is normally flat, the pseudo-parallelity condition (1) turns into

$$\begin{aligned} -A_{\varphi W}R(X, Y)Z - A_{\varphi Z}R(X, Y)W = L\big\{ &- g(Y, Z)A_{\varphi X}W + g(X, Z)A_{\varphi Y}W \\ &- g(Y, W)A_{\varphi X}Z + g(X, W)A_{\varphi Y}Z\big\}. \end{aligned} \tag{16}$$

So, a Legendrian normally flat submanifold M^n of an $\mathcal{S}$-space form $\widetilde{M}^{2n+s}(c)$ is pseudo-parallel if and only if the equation (16) holds.

In particular, if $L = 0$ in (16) the M is said to be semi-parallel.

As a parallel submanifold, $\widetilde{\nabla}\sigma = 0$ (in particular, totally geodesic submanifold $\sigma = 0$) is semi-parallel it is obvious that also is a pseudo-parallel submanifold.

The following two propositions are the analogous results to [5, Prop. 3.1, Prop. 3.2] in case of pseudo-parallel Legendrian submanifold of an $\mathcal{S}$-space form, respectively.

Proposition 1. *Let M^n be a pseudo-parallel Legendrian submanifold of an $\mathcal{S}$-space form $\widetilde{M}^{2n+s}(c)$. If there is another smooth function L' satisfying* (1)*, then $L = L'$ at least on $M - K$, where $K = \{p \in M/\sigma_p = 0\}$.*

Proof. If L and L' are two functions that satisfy (1), we get $(L - L')\, Q(g, \sigma) = 0$. Choose an orthonormal basis $\{e_1, \ldots, e_n\}$ of T_pM, $p \in M$. We have

$$\begin{aligned}(L - L')\, Q(g,\sigma)(e_k, e_l; e_i, e_j) &= (L - L')[(e_i \wedge e_j).\sigma](e_k, e_l)\\ &= (L - L')\big\{ -g(e_j, e_k)\sigma(e_i, e_l) + g(e_i, e_k)\sigma(e_j, e_l)\\ &\qquad - g(e_j, e_l)\sigma(e_k, e_i) + g(e_i, e_l)\sigma(e_k, e_j)\big\}\\ &= (L - L')\big\{ -\delta_{jk}\sigma(e_i, e_l) + \delta_{ik}\sigma(e_j, e_l)\\ &\qquad - \delta_{jl}\sigma(e_k, e_i) + \delta_{il}\sigma(e_k, e_j)\big\} = 0.\end{aligned}$$

For $i = k \neq j = l$ we get

$$(L - L')\{\sigma(e_j, e_j) - \sigma(e_i, e_i)\} = 0.$$

For $i = k = l \neq j$ we get

$$(L - L')\sigma(e_i, e_j) = 0.$$

If $L(p) \neq L'(p)$, $p \in M$, then

$$\sigma(e_i, e_j) = 0, \quad \sigma(e_i, e_i) = \sigma(e_j, e_j), \qquad \forall i, j \in \{1, \ldots, n\}.$$

Moreover, since $i \neq j$ and from (12)

$$g(\sigma(e_i, e_i), \varphi e_j) = g(\sigma(e_i, e_j), \varphi e_i) = 0,$$

$$g(\sigma(e_i, e_i), \varphi e_i) = g(\sigma(e_j, e_j), \varphi e_i) = g(\sigma(e_i, e_j), \varphi e_j) = 0,$$
$$g(\sigma(e_i, e_i), \xi_\alpha) = g(\varphi A_{\varphi e_i} e_i, \xi_\alpha) = 0, \qquad \forall \alpha \in \{1, \ldots, s\}.$$

So, we obtain $g(\sigma(e_i, e_i), N) = 0\, \forall i \in \{1, \ldots, n\}, \forall N \in T^\perp M$ and since $\{\varphi e_1, \ldots, \varphi e_n, \xi_1, \ldots, \xi_s\}$ is a basis of $T^\perp M$ for a Legendrian submanifold M, then $\sigma = 0$. Consequently

$$\{p \in M, L(p) \neq L'(p)\} \subseteq K.$$

This proves the proposition. □

Proposition 2. *Let M^n be a pseudo-parallel Legendrian normally flat submanifold of an $\mathcal{S}$-space form $\widetilde{M}^{2n+s}(c)$, then for any vector fields $X, Y \in TM$ we have*

$$R(X, Y)\varphi H = L\{g(\varphi H, X)Y - g(\varphi H, Y)X\},$$

where H is a mean curvature vector.

Proof. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of TM and Z unit vector field of T_pM for $p \in M$. $\forall U \in TM$, (16) can be rewritten as

$$\begin{aligned}g(R(X,Y)Z, A_{\varphi W}U) + g(R(X,Y)W, A_{\varphi Z}U) = L\big\{&g(Y,Z)g(A_{\varphi X}W, U)\\ - g(X,Z)g(A_{\varphi Y}W, U) + g(Y,W)g(A_{\varphi X}Z, U) &- g(X,W)g(A_{\varphi Y}Z, U)\big\}.\end{aligned} \tag{17}$$

If we put $W = U = e_i$ in (17), we obtain

$$g(R(X,Y)Z, A_{\varphi e_i} e_i) + g(R(X,Y)e_i, A_{\varphi Z} e_i) = L\{g(Y,Z)g(A_{\varphi X} e_i, e_i) - g(X,Z)g(A_{\varphi Y} e_i, e_i) + g(Y,e_i)g(A_{\varphi X} Z, e_i) - g(X,e_i)g(A_{\varphi Y} Z, e_i)\}.$$

Assuming that $\{\lambda_1, \ldots, \lambda_n\}$ are the eigenvalues of $A_{\varphi Z}$ corresponding to frame $\{e_1, \ldots, e_n\}$. Using (11) in the above equation, we have

$$\begin{aligned} -g(R(X,Y)A_{\varphi e_i} e_i, Z) + \lambda_i g(R(X,Y)e_i, e_i) &= L\{g(Y,Z)g(A_{\varphi e_i} e_i, X) - g(X,Z)g(A_{\varphi e_i} e_i, Y) \\ &\quad + g(Y,e_i)g(A_{\varphi Z} e_i, X) - g(X,e_i)g(A_{\varphi Z} e_i, Y)\} \\ &= L\{g(Y,Z)g(A_{\varphi e_i} e_i, X) - g(X,Z)g(A_{\varphi e_i} e_i, Y) \\ &\quad + \lambda_i g(Y,e_i)g(e_i, X) - \lambda_i g(X,e_i)g(e_i, Y)\}. \end{aligned}$$

So that

$$-g(R(X,Y)A_{\varphi e_i} e_i, Z) = L\{g(Y,Z)g(A_{\varphi e_i} e_i, X) - g(X,Z)g(A_{\varphi e_i} e_i, Y)\}.$$

From (13), we get

$$g(R(X,Y)\varphi H, Z) = -\frac{1}{n}\sum_{i=1}^{n} g(R(X,Y)A_{\varphi e_i} e_i, Z) = L\{g(Y,Z)g(\varphi H, X) - g(X,Z)g(\varphi H, Y)\}.$$

□

3 MAIN RESULTS

Theorem 1. *Let M^n be a Legendrian normally flat submanifold of an $\mathcal{S}$-space form $\widetilde{M}^{2n+s}(c)$ with $c \leq s$, then M^n is pseudo-parallel if and only if it is semi-parallel or totally geodesic.*

Proof. Since M^n is a Legendrian submanifold and from (10) we have

$$\widetilde{R}(X,Y)Z = \frac{c+3s}{4}\Big\{g(Y,Z)X - g(X,Z)Y\Big\}, \tag{18}$$

for any $X, Y, Z \in TM$, so that

$$\widetilde{R}(X,Y)\varphi H = \frac{c+3s}{4}\Big\{g(Y,\varphi H)X - g(X,\varphi H)Y\Big\},$$

where H is the mean curvature vector. As $R^{\perp} = 0$ and from (18), the Ricci equation reduces to $[A_{\varphi X}, A_{\varphi Y}] = 0$, so from (15) we get $\widetilde{R}(X,Y)\varphi H = R(X,Y)\varphi H$, thus

$$R(X,Y)\varphi H = \frac{c+3s}{4}\Big\{g(Y,\varphi H)X - g(X,\varphi H)Y\Big\}.$$

Using the above equation and Proposition 2, we obtain

$$\Big(\frac{c+3s}{4} + L\Big)\Big\{g(Y,\varphi H)X - g(X,\varphi H)Y\Big\} = 0, \tag{19}$$

this implies that $L = -\frac{c+3s}{4}$ or $H = 0$.

When $L = -\frac{c+3s}{4}$, if $c = -3s$, i.e. $L = 0$, that is, M is semi-parallel. If $c \neq -3s$, so $L \neq 0$, then from (16), (10) and (11) we have

$$-g(Y,Z)A_{\varphi X} W + g(X,Z)A_{\varphi Y} W - g(Y,W)A_{\varphi X} Z + g(X,W)A_{\varphi Y} Z = 0. \tag{20}$$

Thus by using (20) and Proposition 1, we have $\sigma = 0$, i.e. M is totally geodesic.

Now, assuming that $L \neq -\frac{c+3s}{4}$, then from (19), $H = 0$. By substituting (18) into (16) we obtain

$$\Big(L - \frac{c+3s}{4}\Big)\Big\{ - g(Y,Z)A_{\varphi X}W + g(X,Z)A_{\varphi Y}W - g(Y,W)A_{\varphi X}Z + g(X,W)A_{\varphi Y}Z\Big\} = 0.$$

Putting $X = W = e_i$ and summing over $i = 1, \dots, n$, as $H = 0$ we get $L = \frac{c+3s}{4}$ or $A_{\varphi Y}Z = 0$ (i.e. M is totally geodesic), for all $Y, Z \in TM$.

On the other hand, if we suppose that $L = \frac{c+3s}{4}$. Notice that in [12] the authors gave a necessary condition for a minimal pseudo-parallel integral submanifold M^n (of an $\mathcal{S}$-space form $\widetilde{M}^{2n+s}(c)$) to be totally geodesic is $Ln - \frac{1}{4}[n(c+3s) + c - s] \geq 0$. Hence, in this case M is totally geodesic. Conversely, if M is semi-parallel or totally geodesic obviously it is trivial pseudo-parallel. □

From (19), we easily prove the following result.

Corollary 1. *Let M^n $(n > 1)$ be a Legendrian normally flat submanifold of an $\mathcal{S}$-space form $\widetilde{M}^{2n+s}(c)$, with $c \neq -3s$. If M^n is semi-parallel then it is minimal.*

In [12], the authors have shown that for a minimal Legendrian submanifold M^n of an $\mathcal{S}$-space form $\widetilde{M}^{2n+s}(c)$, if it is semi-parallel and satisfies $n(c+3s) + c - s \leq 0$, then it is totally geodesic. Therefore, by Corollary 1 we have the following assertion.

Corollary 2. *Let M^n $(n > 1)$ be a Legendrian normally flat submanifold of an $\mathcal{S}$-space form $\widetilde{M}^{2n+s}(c)$, with $c < -3s$. If M^n is semi-parallel then it is totally geodesic.*

Theorem 2 ([4])**.** *Let M^m $(m \leq n)$ be a minimal anti-invariant submanifold of an $\mathcal{S}$-space form $\widetilde{M}^{2n+s}(c)$ normal to the structure vector fields. Then the following assertions are equivalent.*

1. *M^m is totally geodesic.*
2. *M^m is of constant curvature $k = \frac{c+3s}{4}$.*
3. *The Ricci tensor $S = \frac{1}{4}(m-1)(c+3s)g$.*
4. *The scalar curvature $\rho = \frac{1}{4}m(m-1)(c+3s)$.*

By the hypothesis of flat normal connection, M^n is of constant curvature $k = \frac{c+3s}{4}$, in view of Corollary 1 we get

Corollary 3. *Let M^n be a Legendrian normally flat submanifold of an $\mathcal{S}$-space form $\widetilde{M}^{2n+s}(c)$ with $c \neq -3s$. If M^n is semi-parallel, then the following statements are equivalent.*

1. *M^n is totally geodesic.*
2. *The Ricci tensor $S = \frac{1}{4}(n-1)(c+3s)g$.*
3. *The scalar curvature $\rho = \frac{1}{4}n(n-1)(c+3s)$.*

It is well known that the equation of Ricci shows that the triviality of the normal connection of M into space form $\widetilde{M}^{n+d}(c)$ (and more generally, for submanifolds in a locally conformally flat space) is equivalent to the fact that all second fundamental tensors are mutually commute, or that all second fundamental tensors are mutually diagonalizable (see [6]).

So, for any $p \in M$ there exists a local orthogonal frame $\{e_i\}$ of M^n such that all the second fundamental form tensors are mutually diagonalizable, then

$$A_N(e_i) = \lambda_i^N e_i$$

for any unit normal vector field N and λ_i^N are the principle curvatures of M with respect to N.

Next, we assume that M^n is a Legendrian normally flat submanifold of an $\mathcal{S}$-space form $\widetilde{M}^{2n+s}(c)$, with $c \neq -3s$. In this case, from (10) and (15) we have

$$R(X,Y)Z = \widetilde{R}(X,Y)Z = \frac{c+3s}{4}\Big\{g(Y,Z)X - g(X,Z)Y\Big\} \tag{21}$$

for any vector $X, Y, Z \in TM$. For an orthonormal frame $\{e_1, \dots, e_n\}$ of M, the Ricci tensor S of M is defined by $S(X,Y) = \sum_{i=1}^n g(R(e_i, X)Y, e_i)$. So, from (21) we have

$$S(X,Y) = \frac{c+3s}{4}(n-1)g(X,Y). \tag{22}$$

Putting $B = S$, $T = \sigma$ in (2) and (3), we get

$$\begin{aligned} Q(S,\sigma)(Z,W;X,Y) = &- S(Y,Z)\sigma(X,W) + S(X,Z)\sigma(Y,W) \\ &- S(Y,W)\sigma(Z,X) + S(X,W)\sigma(Z,Y). \end{aligned} \tag{23}$$

From (14), (5), (11) and (23), the condition (4) turns into

$$\begin{aligned} -A_{\varphi W}R(X,Y)Z - A_{\varphi Z}R(X,Y)W = L\{ &- S(Y,Z)A_{\varphi X}W + S(X,Z)A_{\varphi Y}W \\ &- S(Y,W)A_{\varphi X}Z + S(X,W)A_{\varphi Y}Z\}. \end{aligned} \tag{24}$$

So, a Legendrian normally flat submanifold M^n of an $\mathcal{S}$-space form $\widetilde{M}^{2n+s}(c)$ is Ricci generalized pseudo-parallel if and only if the equation (24) holds.

Theorem 3. *Let $\widetilde{M}^{2n+s}(c)$, $c \neq -3s$, be an $\mathcal{S}$-space form of constant φ-sectional curvature c and M^n be a Legendrian normally flat submanifold of $\widetilde{M}^{2n+s}(c)$. If M^n is Ricci generalized pseudo-parallel, then either M^n is minimal or $L = \frac{1}{n-1}$.*

Proof. Let M be a Ricci generalized pseudo-parallel, since M is a Legendrian normally flat submanifold, we choose an orthonormal basis of $T_p^{\perp}M$ of the form $\{e_{n+1} = \varphi e_1, \dots, e_{2n} = \varphi e_n, e_{2n+1} = \xi_1, \dots, e_{2n+s} = \xi_s\}$ and for any $i, j \in \{1, \dots, n\}$, $\alpha \in \{1, \dots, s\}$ denote λ_i^{n+j} by the principle curvatures with respect to the normal vector field φe_j, i.e.

$$A_{\varphi e_j}(e_i) = \lambda_i^{n+j} e_i. \tag{25}$$

In this case the mean curvature vector can be written as

$$H^{n+j} = \frac{1}{n}\sum_{i=1}^{n} \lambda_i^{n+j}.$$

In view of (24), setting $X = e_i, Y = e_j, Z = e_k, W = e_l$ we obtain

$$\begin{aligned} -A_{\varphi e_l}R(e_i,e_j)e_k - A_{\varphi e_k}R(e_i,e_j)e_l = L\big\{ &- S(e_j,e_k)A_{\varphi e_i}e_l + S(e_i,e_k)A_{\varphi e_{l_j}}e_l \\ &- S(e_j,e_l)A_{\varphi e_i}e_k + S(e_i,e_l)A_{\varphi e_j}e_k\big\}. \end{aligned} \tag{26}$$

Substituting (22), (25) into (26) and for any $e_m \in TM$, we get

$$\begin{aligned} -\lambda_m^{n+l}R_{ijkm} - \lambda_m^{n+k}R_{ijlm} = \frac{c+3s}{4}(n-1)L\big\{ &- \lambda_l^{n+i}\delta_{jk}\delta_{lm} + \lambda_l^{n+j}\delta_{ik}\delta_{lm} \\ &- \lambda_k^{n+i}\delta_{jl}\delta_{km} + \lambda_k^{n+j}\delta_{il}\delta_{km}\big\}, \end{aligned} \tag{27}$$

where $g(e_i,e_j) = \delta_{ij}$ and $1 \leq i,j,k,l,m \leq n$. Since,

$$R_{ijkm} = \frac{c+3s}{4}\Big\{\delta_{jk}\delta_{im} - \delta_{ik}\delta_{jm}\Big\}, \qquad R_{ijlm} = \frac{c+3s}{4}\Big\{\delta_{jl}\delta_{im} - \delta_{il}\delta_{jm}\Big\}, \tag{28}$$

by the use of (28), equation (27) turns into

$$\begin{aligned} &-\lambda_m^{n+l}(\delta_{jk}\delta_{im} - \delta_{ik}\delta_{jm}) - \lambda_m^{n+k}(\delta_{jl}\delta_{im} - \delta_{il}\delta_{jm}) \\ &\qquad = (n-1)L\big\{ - \lambda_l^{n+i}\delta_{jk}\delta_{lm} + \lambda_l^{n+j}\delta_{ik}\delta_{lm} - \lambda_k^{n+i}\delta_{jl}\delta_{km} + \lambda_k^{n+j}\delta_{il}\delta_{km}\big\}. \end{aligned}$$

Hence, if we put $k = i$, $m = j$, we get

$$\begin{aligned} &-\lambda_j^{n+l}(\delta_{ij} - \delta_{ii}\delta_{jj}) - \lambda_j^{n+j}\delta_{ij}(\delta_{jl}\delta_{ij} - \delta_{il}\delta_{jj}) \\ &\qquad = (n-1)L\big\{ - \lambda_l^{n+l}\delta_{il}\delta_{ij}\delta_{jl} + \lambda_l^{n+l}\delta_{jl}\delta_{ii} - \lambda_i^{n+i}\delta_{jl}\delta_{ij} + \lambda_i^{n+i}\delta_{ij}\delta_{il}\big\}, \end{aligned} \tag{29}$$

because it follows from (11) that

$$\lambda_i^{n+j} = g(A_{\varphi e_j}e_i,e_i) = g(A_{\varphi e_i}e_i,e_j) = \lambda_i^{n+i}\delta_{ij}\,.$$

Summing over $i = 1,\ldots,n$ and $j = 1,\ldots,n$ in (29) respectively, we have

$$H^{n+l} = \frac{n-1}{n}L\lambda_l^{n+l}. \tag{30}$$

On the other hand, by substituting (21) and (22) in (24), we obtain

$$[(n-1)L-1]\{-g(Y,Z)A_{\varphi X}W + g(X,Z)A_{\varphi Y}W - g(Y,W)A_{\varphi X}Z + g(X,W)A_{\varphi Y}Z\} = 0. \tag{31}$$

By setting $X = e_i, Y = e_j, Z = e_k, W = e_l$ and substituting (25) into (31), for any $e_m \in TM$ we get

$$[(n-1)L-1]\{\lambda_l^{n+j}\delta_{ik}\delta_{lm} - \lambda_l^{n+i}\delta_{jk}\delta_{lm} + \lambda_k^{n+j}\delta_{il}\delta_{km} - \lambda_k^{n+i}\delta_{jl}\delta_{km}\} = 0.$$

In the same way, we put $k = i$, $m = j$ in the above equation

$$[(n-1)L-1]\{\lambda_l^{n+l}\delta_{ii}\delta_{jl} - \lambda_l^{n+l}\delta_{il}\delta_{ij}\delta_{jl} + \lambda_i^{n+i}\delta_{ij}\delta_{il} - \lambda_i^{n+i}\delta_{jl}\delta_{ij}\} = 0. \tag{32}$$

Furthermore, by summing over $i = 1,\ldots,n$ and $j = 1,\ldots,n$ in (32), we obtain

$$[(n-1)L-1](n-1)\lambda_l^{n+l} = 0.$$

As $n > 1$ we have

$$[(n-1)L-1]\lambda_l^{n+l} = 0. \tag{33}$$

Comparing (30) and (33), we deduce that if $L = 0$ then $H^{n+l} = 0$ for any $1 \leq l \leq n$, i.e. M is minimal. If $L \neq 0$, then $\frac{[(n-1)L-1]n}{(n-1)L}H^{n+l} = 0$, which implies $H^{n+l} = 0$ or $L = \frac{1}{n-1}$. $\square$

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Махі Ф., Белхельфа М. *Лежандрові нормально плоскі підмноговиди $\mathcal{S}$-просторових форм* // Карпатські матем. публ. — 2020. — Т.12, №1. — С. 69–78.

У поданому дослідженні ми розглядаємо лежандровий нормально плоский підмноговид M $(2n+s)$-вимірної $\mathcal{S}$-просторової форми $\widetilde{M}^{2n+s}(c)$ сталої φ-секційної кривизни c. Ми показали, що якщо M є псевдопаралельним, то M є напівпаралельним або тотально геодезичним.

Ми також довели, що якщо M є узагальнено псевдопаралельним підмноговидом Річчі, то або M є мінімальним, або $L = \frac{1}{n-1}$ при $c \neq -3s$.

Ключові слова і фрази: $\mathcal{S}$-просторова форма, лежандровий підмноговид, нормально плоский підмноговид, псевдопаралельний підмноговид, узагальнено псевдопаралельний підмноговид Річчі.

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MALYTSKA G.P., BURTNYAK I.V.

CONSTRUCTION OF THE FUNDAMENTAL SOLUTION OF A CLASS OF DEGENERATE PARABOLIC EQUATIONS OF HIGH ORDER

In the article, using the modified Levy method, a Green's function for a class of ultraparabolic equations of high order with an arbitrary number of parabolic degeneration groups is constructed. The modified Levy method is developed for high-order Kolmogorov equations with coefficients depending on all variables, while the frozen point, which is a parametrix, is chosen so that an exponential estimate of the fundamental solution and its derivatives is conveniently used.

Key words and phrases: degenerated parabolic equations, modified Levy's method, Kolmogorov's equation, fundamental solution, parametrix.

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INTRODUCTION

A fundamental solution of the inverse Cauchy problem for degenerate parabolic equations of second-order variables with smooth coefficients was constructed first by M. Weber [10]. Under the same conditions on the coefficients, a fundamental solution of the Cauchy problem was constructed in [5], in the case of Holder coefficients for second-order equations with two degenerate groups. The Levy method was modified in [7], and in Banach spaces in [8], for the second order Kolmogorov systems with one degeneracy group [4]. The Kolmogorov equation of high order has features that make it easy to use the Levy method for constructing a fundamental solution. The parametric method was applied to a degenerate parabolic equation of high order with one group of parabolic degeneracy variables in [2, 3, 9] and with two degenerate groups in [1] and with four degenerate groups for Kolmogorov type systems of the second order in [6]. We modified the Levy method with respect to the properties of a fundamental solution of high-order Kolmogorov-type equations with coefficients dependent only on t, in particular a selected point which is a parameter so that an exponential estimate of the fundamental solution and its derivatives is conveniently used.

1 DESIGNATION, TASK STATEMENT AND MAIN RESULTS

Let us denote by $n_j \in N$, $j = \overline{1,p}$, $n_1 \geq n_2 \geq \cdots \geq n_p$, $n_0 = \sum\limits_{j=1}^{p} n_j$, $x = (x_1, \ldots, x_p)$, $x_j = (x_{j1}, \ldots, x_{jn_j})$, $x_j \in R^{n_j}$, $x \in R^{n_0}$, $\xi = (\xi_1, \ldots, \xi_p)$, $\xi_j = (\xi_{j1}, \ldots, \xi_{jn_j})$, $\xi_j \in R^{n_j}$, $\xi \in R^{n_0}$,

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$x^{(j)} = \left(x_1, \ldots, x_j\right) \in R^{\sum\limits_{k=1}^{j} n_k}$, $\xi^{(j)} = \left(\xi_1, \ldots, \xi_j\right) \in R^{\sum\limits_{k=1}^{j} n_k}$, $j = \overline{2,p}$. $\Gamma(\alpha)$ is Euler's gamma function and $B(a,b)$ is Euler's beta function.

$$x_j - \xi_j + \sum_{k=1}^{j-1} x_k \frac{(t-\tau)^{j-k}}{(j-k)!} = \left(x_{j1} - \xi_{j1} + \sum_{k=1}^{j-1} x_{k1} \frac{(t-\tau)^{j-k}}{(j-k)!}, .., x_{jn_j} - \xi_{jn_j} + \sum_{k=1}^{j-1} x_{kn_j} \frac{(t-\tau)^{j-k}}{(j-k)!}\right),$$

$$\rho_1\left(t, x_1, \tau, \xi_1\right) = \left(\left|x_1 - \xi_1\right| (t-\tau)^{-\frac{1}{2b}}\right)^q, \; q = \frac{2b}{2b-1}, \; b \in N,$$

$$\rho_j\left(t, x^{(j)}, \tau, \xi^{(j)}\right) = \left(\left|x_j - \xi_j + \sum_{k=1}^{j-1} x_k \frac{(t-\tau)^{j-k}}{(j-k)!}\right| (t-\tau)^{-\left(j-1+\frac{1}{2b}\right)}\right)^q, \; j = \overline{2,p},$$

$$\xi(t,\tau) = \left(\xi_1, \xi_2 - \xi_1 (t-\tau), \ldots, \xi_p + \sum_{k=1}^{p-1} (-1)^{p-k} \xi_k \frac{(t-\tau)^{p-k}}{(p-k)!}\right).$$

We investigate the Cauchy problem for the equation

$$\partial_t u(t,x) - \sum_{j=1}^{p-1} \sum_{\mu=1}^{n_{j+1}} x_{j\mu} \partial_{x_{j+1}\mu} u(t,x) = \sum_{|k| \le 2b} a_k(t,x) D_{x_1}^k u(t,x), \tag{1}$$

with the initial condition

$$u(t,x)\Big|_{t=\tau} = u_0(x), \qquad 0 \le \tau \le t \le T, \tag{2}$$

where τ is a fixed number, and operator

$$\partial_t - \sum_{|k| \le 2b} a_k(t,x) D_{x_1}^k, \quad D_{x_1}^k = \frac{(-1)^k \partial^{k_1 + \cdots + k_{n_1}}}{\partial x_1^{k_1} \ldots \partial x_{n_1}^{k_{n_1}}}, \quad |k| = k_1 + \cdots + k_{n_1}, \tag{3}$$

is uniformly parabolic in the sense of Petrovsky in the strip $\Pi_{[0,T]} = (t,x), x \in R^{n_0}, 0 \le t \le T$.

Let us suppose that

1) $a_k(t,x), \partial_{x_j} a_k(t,x), j = \overline{2,p}$, are continuous and bounded in $\Pi_{[0,T]}$,

2) there are constants $\alpha \in (0,1], r \in (0,1]$, such that for any $x \in R^{n_0}, \xi \in R^{n_0}$ and $t \in [0,T]$

$$\left|a_k(t,x) - a_k(t,\xi)\right| \le c_1 \left(\left|x_1 - \xi_1\right|^\alpha + \sum_{j=2}^{p} \left|x_j - \xi_j\right|\right),$$

$$\left|\partial_{x_j} a_k(t,x) - \partial_{x_j} a_k(t,\xi)\right| \le c_1 |x - \xi|^r, \qquad j = \overline{2,p}.$$

Theorem 1. *If conditions 1)–2) are satisfied, then equation* (1) *has a fundamental solution of the Cauchy problem* (1)–(2) $Z(t,x;\tau,\xi)$ *at* $t > \tau$ *and the following estimations hold:*

$$\left|\partial_{x_j} Z(t,x;\tau,\xi)\right| \le A(t-\tau)^{-\sum\limits_{s=1}^{p} \frac{2b(s-1)+1}{2b}\left(n_s + |m_s|\right)} \Phi(t,x;\tau,\xi),$$

$m_s = 0$, *at* $s \ne j$, $m_j = 1$, $j = \overline{2,p}$;

$$\left|\partial_{x_1}^{m_1} Z(t,x;\tau,\xi)\right| \le A_{m_1}(t-\tau)^{-\frac{n_1 + |m_1|}{2b} - \sum\limits_{s=2}^{p} \frac{2b(s-1)+1}{2b} n_s} \Phi(t,x;\tau,\xi),$$

$|m_1| \le 2b,\ x \in R^{n_0},\ \ \xi \in R^{n_0},\ \ 0 \le \tau < t \le T$, *where*

$$\Phi(t,x;\tau,\xi) = \sum_{i=1}^{\infty} A^s \Gamma\left(1+\frac{s\alpha^*}{2b}\right)\Gamma\left(\frac{\alpha^*}{2b}\right)\Gamma^{-1}\left(1+\frac{\alpha^*(1+s)}{2b}\right) \times \exp\left\{-c_0\rho_1(t,x_1,\tau,\xi_1) - 2^{-2sp}c_0\sum_{j=2}^{p}\rho_j\left(t,x^{(j)},\tau,\xi^{(j)}\right)\right\},$$

and positive constants A, A_{m_1}, c_0 depend on n_0, $2b$, c_1, α, r, and the constant of parabolicity of the operator (3) *is* $\sup\limits_{(t,x)\in\Pi_{[0,T]}} |a_k(t,x)|$ *and* $\alpha^* = \min(\alpha, r)$.

Proof. To prove the theorem, we write equation (1) in the form

$$\partial_t u(t,x) - \sum_{j=1}^{p-1}\sum_{\mu=1}^{n_{j+1}} x_{j\mu}\partial_{x_{j+1}\mu}u(t,x) = \sum_{|k|=2b} a_k(t,\xi(t,\tau))D_{x_1}^k u(t,x) + \sum_{|k|=2b}[a_k(t,x) - a_k(t,\xi(t,\tau))]D_{x_1}^k u(t,x) + \sum_{|k|<2b} a_k(t,x)D_{x_1}^k u(t,x). \tag{4}$$

Let us denote by $Z_0(t,x;\tau,\xi;\xi(t,\tau))$ the fundamental solution of equation

$$\partial_t u(t,x) - \sum_{j=1}^{p-1}\sum_{\mu=1}^{n_{j+1}} x_{j\mu}\partial_{x_{j+1}\mu}u(t,x) = \sum_{|k|=2b} a_k(t,\xi(t,\tau))D_{x_1}^k u(t,x). \tag{5}$$

Fundamental solution $Z_0(t,x,;\tau,\xi;\xi(t,\tau))$ of equation (5) is constructed in [5], where ξ is fixed. For derivatives of $Z_0(t,x;\tau,\xi;\xi(t,\tau))$ the following inequalities are performed

$$|\partial_x^m Z_0(t,x;\tau,\xi;\xi(t,\tau))| \le C_m(t-\tau)^{-\sum\limits_{s=1}^{p}\frac{2b(s-1)+1}{2b}(n_s+|m_s|)} \times \exp\left\{-c_0\left(\sum_{j=2}^{p}\rho_j\left(t,x^{(j)},\tau,\xi^{(j)}\right) + \rho_1(t,x_1,\tau,\xi_1)\right)\right\}, \tag{6}$$

where $|m| = \sum\limits_{j=1}^{p}|m_j|$, $|m_j| = \sum\limits_{k=1}^{n_j} m_{jk}$, $t > \tau, C_m > 0$.

Fundamental solution $Z(t,x;\tau,\xi)$ of equation (1) will be sought in the form

$$Z(t,x;\tau,\xi) = Z_0(t,x;\tau,\xi;\xi(t,\tau)) + \int_\tau^t d\beta\int_{R^{n_0}} Z_0(t,x;\beta,\gamma;\gamma(t,\beta))\,\varphi(\beta,\gamma;\tau,\xi)\,d\gamma, \tag{7}$$

where $\varphi(t,x;\tau,\xi)$ is an unknown absolutely integrable on R^{n_0} function at $t > \tau$.

We substitute (6) into equation (1) with respect to the function $\varphi(t,x;\tau,\xi)$, then

$$\varphi(t,x;\tau,\xi) = K(t,x;\tau,\xi) + \int_\tau^t K(t,x;\beta,\gamma)\varphi(\beta,\gamma;\tau,\xi)\,d\gamma, \tag{8}$$

where

$$K(t,x;\tau,\xi) = \sum_{|k|=2}(a_k(t,x) - a_k(t,\xi(t,\tau)))D_{x_1}^k Z_0(t,x;\tau,\xi;\xi(t,\tau)) + \sum_{|k|<2b} a_k(t,x)D_{x_1}^k Z_0(t,x;\tau,\xi;\xi(t,\tau)).$$

The solution of equation (8) can be represented by a Neumann series

$$\varphi\left(t,x;\tau,\xi\right)=\sum_{n=1}^{\infty}K_n\left(t,x;\tau,\xi\right), \tag{9}$$

where

$$K\left(t,x;\tau,\xi\right)=K_1\left(t,x;\tau,\xi\right);\quad K_n\left(t,x;\tau,\xi\right)=\int_{\tau}^{t}d\beta\int\limits_{R^{n_0}}K(t,x;\beta,\gamma)K_{n-1}\left(\beta,\gamma;\tau,\xi\right)d\gamma. \tag{10}$$

Let us show the convergence of series (9) and the required estimation of the function for the Levy method $\varphi\left(t,x;\tau,\xi\right)$ and its increments.

Using the following lemma, which generalizes Lemma 2 and Lemma 1 in [3] for equation (1), we can obtain estimates for $K_n\left(t,x;\tau,\xi\right)$ and $K\left(t,x;\tau,\xi\right)$.

Lemma 1. *For any points* (t,x), (β,ξ), (τ,y), $0\le\tau<\beta<t$, $x\in R^{n_0}$, $\xi\in R^{n_0}$, $y\in R^{n_0}$, $b\in N$, $2b>2$ *the following inequality holds*

$$\begin{aligned}\rho_1\left(t,x_1,\beta,\xi_1\right)+\sum_{j=2}^{p}\rho_j\left(t,x^{(j)},\beta,\xi^{(j)}\right)+\rho_1\left(\beta,\xi_1,\tau,y_1\right)+\sum_{j=2}^{p}\rho_j\left(\beta,\xi^{(j)},\tau,y^{(j)}\right)\\ \ge 2^{-2p}\left(\sum_{j=2}^{p}\rho_j\left(t,x^{(j)},\tau,y^{(j)}\right)+\rho_1\left(t,x_1,\tau,y_1\right)\right).\end{aligned} \tag{11}$$

The proof of Lemma 1 is based on the inequalities

$$\begin{aligned}&\rho_p\left(t,x^{(p)},\beta,\xi^{(p)}\right)+\rho_p\left(\beta,\xi^{(p)},\tau,y^{(p)}\right)\\ &\quad\ge 2^{-2}\left(\left|x_p-y_p+\sum_{j=1}^{p-1}\left[x_k(t-\beta)^{p-k}+\xi_k(\beta-\tau)^{p-k}\right]\frac{1}{(p-k)!}\right|(t-\tau)^{-p+1-\frac{1}{2b}}\right)^q.\end{aligned} \tag{12}$$

From (12) we can get

$$\begin{aligned}&\left(\left|x_p-y_p+\sum_{k=1}^{p-1}\left[x_k(t-\beta)^{p-k}+\xi_k(\beta-\tau)^{p-k}\right]((p-k)!)^{-1}\right|(t-\tau)^{-p+1-\frac{1}{2b}}\right)^q\\ &\ge 2^{-2}\left(\left|x_p-y_p+\sum_{k=1}^{p-1}\left[x_k\left(t-\beta\right)^{p-k}+\xi_k\left(\beta-\tau\right)^{p-k}\right]((p-k)!)^{-1}\right.\right.\\ &\qquad\left.\left.\times\ \frac{x_1((\beta-\tau)^{p-1}+(t-\beta)^{p-1})}{(p-1)!}\right|(t-\tau)^{-p+1-\frac{1}{2b}}\right)^q\\ &-\sum_{\mu=1}^{n_p}\left(\left[|x_{1\mu}-\xi_{1\mu}|(\beta-\tau)^{p-1}\right]((p-k)!)^{-1}(t-\tau)^{-p+1-\frac{1}{2b}}\right)^q.\end{aligned} \tag{13}$$

Applying (12) to the first part of (13) $(p-2)$ times, we have

$$\left(\left|x_p-y_p+\sum_{k=1}^{p-1}(x_k(t-\beta)^{p-k}+\xi_k\left(\beta-\tau\right)^{p-k})((p-k)!)^{-1}\right|(t-\tau)^{-p+1-\frac{1}{2b}}\right)^q$$

$$\geq 2^{-2(p-1)}\rho_p\left(t,x^{(p)},\tau,y^{(p)}\right)-\sum_{\mu=1}^{n_p}\left(\left|x_{j\mu}-\xi_{j\mu}\right|(\beta-\tau)^{p-1})\left((p-1)!\right)^{-1}(t-\tau)^{-p+1-\frac{1}{2b}}\right)^q$$
$$-\sum_{j=2}^{p-1}\sum_{\mu=2}^{n_j}2^{-2(j-1)}\left(\left|x_{1\mu}-\xi_{1\mu}+\sum_{k=2}^{j-1}x_{k\mu}(t-\beta)^{j-k}((j-k)!)^{-1}\right|\frac{(\beta-\tau)^{p-j}}{(p-j)!}(t-\tau)^{-p+1-\frac{1}{2b}}\right)^q. \tag{14}$$

Taking into account the inequalities (11)–(14), we get

$$\begin{gathered}\rho_p\left(t,x^{(p)},\beta,\xi^{(p)}\right)+\rho_p\left(\beta,\xi^{(p)},\tau,y^{(p)}\right)\geq 2^{-2p}\rho_p\left(t,x^{(p)},\tau,y^{(p)}\right)\\ -\sum_{j=2}^{p-1}\sum_{\mu=1}^{n_p}2^{-2(j-1)}\left(\left|x_{j\mu}-\xi_{j\mu}+\sum_{k=2}^{j-1}x_{k\mu}(t-\beta)^{j-k}\ ((j-k)!)^{-1}\right|(\beta-\tau)^{p-j}((p-j)!)^{-1}\right.\\ \left.\times(t-\tau)^{-p+j-\frac{1}{2b}}\right)^q-2^{-2}\sum_{\mu=1}^{n_p}\left(\left|x_{1\mu}-\xi_{1\mu}\right|(\beta-\tau)^{p-1}((p-1)!)^{-1}(t-\tau)^{-p+1-\frac{1}{2b}}\right)^q.\end{gathered} \tag{15}$$

We will collect all of the terms that contain $x_{p-1}-\xi_{p-1}$

$$\begin{gathered}\rho_{p-1}\left(\beta,x^{(p-1)},\beta,\xi^{(p-1)}\right)-2^{-2(p-1)}\sum_{\mu=1}^{n_p}\left(\left|x_{p-1\mu}-\xi_{p-1\mu}+\sum_{k=1}^{p-2}x_k(t-\beta)^{p-1-k}\right.\right.\\ \left.\left.\times\frac{1}{((p-1-k)!)}\right|(\beta-\tau)\ (t-\beta)^{-p+1-\frac{1}{2b}}\right)^q\geq\sum_{\mu=n_{p+1}}^{n_{p-1}}\left(\left|x_{p-1\mu}-\xi_{p-1\mu}\right.\right.\\ \left.\left.+\sum_{k=1}^{p-2}x_{k\mu}\frac{(t-\beta)}{(p-1-k)!}^{p-1-k}\right|(t-\beta)^{-p+3-\frac{1}{2b}}\right)^q+\sum_{\mu=1}^{n_p}\left(1-2^{-2(p-1)}\right)\\ \times\left(\left|x_{p-1\mu}-\xi_{p-1\mu}+\sum_{k=1}^{p-2}x_{k\mu}(t-\beta)^{p-1-k}((p-1-k)!)^{-1}\right|(t-\tau)^{-p+2-\frac{1}{2b}}\right)^q.\end{gathered} \tag{16}$$

Repeating all inequalities (12), (16) for the terms $\rho_j\left(t,x^{(j)},\beta,\xi^{(j)}\right)+\rho_j\left(\beta,\xi^{(j)},\tau,y^{(j)}\right)$, $j=\overline{1,p-1}$, and adding their together we have

$$\begin{gathered}\rho_1(t,x_1,\beta,\xi_1)+\rho_1(\beta,\xi_1,\tau,y_1)+\sum_{j=2}^{p}\left(\rho_j(t,x^(j),\beta,\xi^(j))+\rho_j\left(\beta,\xi^{(j)},\tau,y^{(j)}\right)\right)\\ \geq 2^{-2p}\left(\sum_{j=2}^{p}\rho_j\left(t,x^{(j)},\tau,y^{(j)}\right)+\rho_1(t,x_1,\tau,y_1)\right).\end{gathered}$$

Lemma 2. *The following estimations are performed for reproducing kernels:*

$$\begin{gathered}|K_m(t,x;\tau,\xi)|\leq A_m^m(t-\tau)^{-\sum\limits_{j=1}^{p}\frac{(1+2b(j-1))n_j}{2b}-1+\frac{m\alpha}{2b}}\\ \times\exp\left\{\rho_1(t,x_1,\beta,\xi_1)-2^{-2pm}c\sum_{j=2}^{p}\rho_j\left(t,x^{(j)},\tau,\xi^{(j)}\right)\right\},\end{gathered} \tag{17}$$

at $m\leq m^*=\left[\sum\limits_{j=1}^{p}\frac{\left((1+2b(j-1))n_j+2b\right)}{\alpha}\right]+1;$

$$
\begin{aligned}
|K_{m+l}(t,x;\tau,\xi)| \leq A_m^{m+l} \prod_{k=0}^{l-1} B\left(\frac{\alpha}{2b}, 1+\frac{\alpha k}{2b}\right)(t-\tau)^{\frac{\alpha l}{2b}} \\
\times \exp\left\{-c\rho_1(t,x_1,\beta,\xi_1) - 2^{-2p(m+l)}\sum_{j=2}^{p}\rho_j\left(t,x^{(j)},\tau,\xi^{(j)}\right)\right\},
\end{aligned}
\tag{18}
$$

at $m+l > m^*$.

From (17), (18) it follows the convergence of a series (9) following for $\varphi(t,x;\tau,\xi)$

$$
|\varphi(t,x;\tau,\xi)| \leq A(t-\tau)^{-\sum\limits_{j=1}^{p}\frac{(1+2b(j-1))n_j+2b-\alpha}{2b}}\Phi(t,x;\tau,\xi). \tag{19}
$$

Let us prove the existence of derivatives $\partial_{x_j}\varphi(t,x;\tau,\xi)$, $j=\overline{2,p}$, at $t>\tau$.

Under the assumption 1), there are continuous derivatives $\partial_{x_j}K(t,x;\tau,\xi)$, $j=\overline{2,p}$ satisfying the estimations

$$
\begin{aligned}
\left|\partial_{x_j}K(t,x;\tau,\xi)\right| \leq A\exp\left\{-c\left(\sum_{j=2}^{p}\rho_j\left(t,x^{(j)},\tau,\xi^{(j)}\right)+\rho_1(t,x_1,\tau,\xi_1)\right)\right\} \\
\times(t-\tau)^{-\sum\limits_{s=1}^{p}\frac{(2b(s-1)+1)n_s}{2b}-\frac{j-(1-\alpha^*)}{2b}}, \quad t>\tau.
\end{aligned}
\tag{20}
$$

To prove the existence of derivatives $\partial_{x_j}K(t,x;\tau,\xi)$, $j=\overline{2,p}$, we use the following property of the fundamental solution of equation (5) with $\xi(t,\tau)=y$, where y is a parameter

$$
\partial_t u(t,x) - \sum_{j=1}^{p-1}\sum_{\mu=1}^{n_{j+1}} x_{j\mu}\partial_{x_{j+1}\mu}u(t,x) = \sum_{|k|\leq 2b} a_k(t,\xi(t,\tau))D_{x_1}^k u(t,x).
$$

Property 1. If $a_k(t,y)$ have continuous bounded derivatives by the parameter y up to the order r, then there are continuous derivatives by y, $\partial_y^s\partial_{x_1}^m Z_0(t,x;\tau,\xi;y)$, $s\in\overline{0,r}$, and

$$
\begin{aligned}
\left|\partial_{x_1}^m\partial_y^s Z_0(t,x;\tau,\xi;y)\right| \leq C_m \exp\left\{-c\left(\sum_{j=2}^{p}\rho_j\left(t,x^{(j)},\tau,\xi^{(j)}\right)+\rho_1(t,x_1,\tau,\xi_1)\right)\right\} \\
\times(t-\tau)^{-\sum\limits_{j=1}^{p}\frac{(1+2b(j-1))n_j}{2b}-\frac{|m|}{2b}}.
\end{aligned}
\tag{21}
$$

Let us consider $\partial_{x_{p\mu}}K(t,x;\beta,\gamma)$, $\mu=\overline{1,n_p}$. Then

$$
\begin{aligned}
\partial_{x_{p\mu}}K(t,x;\beta,\gamma) &= \sum_{|k|=2b}\left(\partial_{x_{p\mu}}a_k(t,x) - \partial_{\gamma_{p\mu}}a_k(t,\gamma(t,\beta))\right) \\
&\quad\times D_{x_1}^k Z_0(t,x;\beta,\gamma;\gamma(t,\beta)) + \sum_{|k|=2b}\left(\partial_{\gamma_{p\mu}}a_k(t,\gamma(t,\beta))\right)D_{x_1}^k Z_0(t,x;\beta,\gamma;\gamma(t,\beta)) \\
&+ \sum_{|k|=2b}\left(a_k(t,x)-a_k(t,\gamma(t,\beta))\right)\partial_{x_{p\mu}}D_{x_1}^k Z_0(t,x;\beta,\gamma;\gamma(t,\beta)) \\
&+ \sum_{|k|<2b}\partial_{x_{p\mu}}a_k(t,x)\,D_{x_1}^k Z_0(t,x;\beta,\gamma;\gamma(t,\beta)).
\end{aligned}
\tag{22}
$$

Let us rewrite (22) by a convenient form for applications

$$\begin{aligned}
&\partial_{x_{p\mu}} K(t,x;\beta,\gamma) = \sum_{|k|=2b} \left(\partial_{x_{p\mu}} a_k(t,x) - \partial_{\gamma_{p\mu}} a_k(t,\gamma(t,\beta))\right) \\
&\quad \times D^k_{x_1} Z_0(t,x;\beta,\gamma;\gamma(t,\beta)) - \partial_{\gamma_{p\mu}} \left(\sum_{|k|=2b} \left(a_k(t,x) - a_k(t,\gamma(t,\beta))\right) \right. \\
&\quad \times D^k_{x_1} Z_0(t,x;\beta,\gamma;\gamma(t,\beta)) + \sum_{|k|=2b} \left(a_k(t,x) - a_k(t,\gamma(t,\beta))\right) \\
&\quad \times D^k_{x_1} \partial_{\overline{\gamma}_{p\mu}} D^k_{x_1} Z_0(t,x;\beta,\gamma;\overline{\gamma}(t,\beta)) \,|_{\overline{\gamma}=\gamma} + \sum_{|k|<2b} \left(a_k(t,x)\right)'_{x_{p\mu}} D^k_{x_1} Z_0(t,x;\beta,\gamma;\gamma(t,\beta)) \\
&+ \sum_{|k|<2b} a_k \partial_{\overline{\gamma}_{p\mu}} D^k_{x_1} Z_0(t,x;\beta,\gamma;\overline{\gamma}(t,\beta)) \,|_{\overline{\gamma}=\gamma} - \sum_{|k|<2b} a_k(t,x) D^k_{x_1} \partial_{\gamma_{p\mu}} Z_0(t,x;\beta,\gamma;\gamma(t,\beta)),
\end{aligned} \tag{23}$$

where $\mu = \overline{1, n_p}$, $\overline{\gamma} = \left(\gamma_1, \dots, \gamma_{p-1}, \overline{\gamma_p}\right)$. Using the images (23), estimates (6) and (21) and integrating by parts of the terms with $\partial_{\gamma_{p\mu}}$, we can get $\partial_{x_{p\mu}} K_2(t,x;\tau,\xi) = \lim_{h\to 0} \int_0^{t-h} d\beta \int_{R^{n_0}} \partial_{x_{p\mu}} K(t,x;\ \beta,\gamma)\, K(\beta,\gamma;\tau,\xi)\, d\gamma$.

From the estimations of reproducing kernel (18), estimations of derivatives of the kernel (20) and Lemma 1, we obtain $\left|\partial_{x_{p\mu}} K_2(t,x;\tau,\xi)\right| \le A_2 \exp\Big\{ -c_2(1-\varepsilon)\Big(\rho_1(t,x_1,\tau,\xi_1) + 2^{-2p}\sum_{j=2}^{p} \rho_j\left(t,x^{(j)},\tau,\xi^{(j)}\right) + \rho_1(t,x_1,\tau,\xi_1)\Big)\Big\}(t-\tau)^{-\sum_{j=1}^{p}\frac{(1+2b(j-1))n_j}{2b} - p - (1-\alpha^*)/2b}$ at $t > \tau$. By the method of mathematical induction we can prove the existence $\partial_{x_{p\mu}} K_m(t,x;\tau,\xi)$ for any m and evaluation

$$\begin{aligned}
\left|\partial_{x_{p\mu}} K_m(t,x;\tau,\xi)\right| &\le A_m(\varepsilon) \exp\{-c_2(1-\varepsilon m)(\rho_1(t,x_1,\tau,\xi_1) + 2^{-mp} \sum_{j=2}^{p} \rho_j\left(t,x^{(j)},\tau,\xi^{(j)}\right) \\
&+ \rho_1(t,x_1,\tau,\xi_1)\}\,(t-\tau)^{-\sum_{j=1}^{p}\frac{(1+2b(j-1))n_j}{2b} - p - (1-\alpha m)/2b}, \qquad \mu = \overline{1, n_p}.
\end{aligned} \tag{24}$$

Taking into account the estimation (24), we can estimate the series $\sum_{m=1}^{\infty} \partial_{x_{p\mu}} K_m(t,x;\ \tau,\xi)$ by a converging series:

$$\begin{aligned}
&\left|\sum_{m=1}^{\infty} \partial_{x_{p\mu}} K_m(t,x;\tau,\xi)\right| \le \sum_{m=1}^{l} A_m (t-\tau)^{-\sum_{j=1}^{p}\frac{(1+2b(j-1))n_j}{2b} - p - (1-\alpha^* m)/2b} \\
&\times \exp\left\{-c_2(1-m\varepsilon)\left(c_m \rho_1(t,x_1,\tau,\xi_1) + 2^{-2mp}\left(\rho_1(t,x_1,\tau,\xi_1) + \sum_{j=2}^{p} \rho_j\left(t,x^{(j)},\tau,\xi^{(j)}\right)\right)\right)\right\} \\
&\quad + \sum_{k=1}^{\infty} A_0 \left(\Gamma\left(\frac{\alpha^*}{2b}\right) F A_0\right)^k (t-\tau)^{\frac{\alpha^* k}{2b}} \Gamma^{-1}\left(1 + \frac{k\alpha^*}{2b}\right) \\
&\times \exp\left\{-c_5\left(c_{4l+k+1} \rho_1(t,x_1,\tau,\xi_1) + 2^{-2p(l+k-1)}\left(\rho_1(t,x_1,\tau,\xi_1) + \sum_{j=2}^{p} \rho_j\left(t,x^{(j)},\tau,\xi^{(j)}\right)\right)\right)\right\},
\end{aligned} \tag{25}$$

where $l = \left[\sum_{j=1}^{p} \frac{(1+2b(j-1))n_j + 2bp + 1}{\alpha^*}\right] + 1$, and A_0, F are positive constants,

$$F = \left(2\int_0^{\infty} \exp\left\{-\frac{\alpha^2}{2}\right\} d\alpha\right)^{n_0}.$$

The series $\sum\limits_{m=1}^{\infty} \partial_{x_p} K_m (t, x; \tau, \xi)$ at $0 < \delta \le t - \tau \le T$ is convergent uniformly and absolutely. Then $\partial_{x_p} \varphi (t, x; \tau, \xi) = \sum\limits_{m=1}^{\infty} \partial_{x_p} K (t, x; \tau, \xi)$ and $\partial_{x_p} K_m (t, x; \tau, \xi)$ are continuous, then in the domain of convergence and $\partial_{x_p} \varphi (t, x; \tau, \xi)$ continuous function. Inequality (25) will be written in the form

$$\left|\partial_{x_p} \varphi (t, x; \tau, \xi)\right| \le A(t-\tau)^{-\sum\limits_{j=1}^{p} \frac{(1+2b(j-1))n_j}{2b} - p - (1-\alpha^*)/2b} \Phi (t, x; \tau, \xi) .$$

Let us consider $\partial_{x_{j\mu}} K (t, x; \beta, \gamma)$, $j = \overline{2, p-1}$, $\mu = \overline{1, n_j}$. For $\mu = \overline{n_{j-1}+1, n_j}$ formula (23) is true with the corresponding replacing p by j. For $\mu = \overline{1, n_{j-1}}$, $\partial_{x_{j\mu}} K (t, x; \beta, \gamma)$ can be written in the form

$$\begin{aligned}
\partial_{x_{j\mu}} K (t, x; \beta, \gamma) &= \sum_{|k|=2b} \left[\partial_{x_{j\mu}} a_k (t, x) - \partial_{y_{j\mu}} a_k (t, y) \Big|_{y=\gamma(t,\beta)}\right] D^k_{x_1} Z_0 (t, x; \beta, \gamma; \gamma (t, \beta)) \\
&- \partial_{\gamma_{j\mu}} \left(\sum_{|k|=2b} [a_k(t, x) - a_k (t, \gamma (t, \beta))] D^k_{x_1} Z_0 (t, x; \beta, \gamma; \gamma (t, \beta)) \right) \\
&+ \sum_{|k|=2b} [a_k (t, x) - a_k (t, \gamma (t, \beta))]\ \partial_{\overline{\gamma}_{j\mu}} D^k_{x_1} Z_0 (t, x; \beta, \gamma; \overline{\gamma} (t, \beta)) \big|_{\overline{\gamma}=\gamma} \\
&+ \sum_{l=1}^{p-j} \sum_{|k|=2b} [a_k (t, x) - a_k (t, \gamma (t, \beta))]\, \partial_{\gamma_{j+l,\mu}} D^k_{x_1} Z_0 (t, x; \beta, \gamma; \overline{\gamma} (t, \beta)) \big|_{\overline{\gamma}=\gamma} \\
&\times (-1)^l \frac{(t-\tau)^{p-l-l}}{(p-j-l)!} \sum_{|k|=2b} \partial_{y_{j+l\ \mu}} a_k (t, y) \Big|_{y=\gamma(t,\beta)} (-1)^l \frac{(t-\beta)^{p-j-l}}{(p-j-l)!} \\
&\times D^k_{x_1} \partial_{\gamma_{j\mu}} Z_0 (t, x; \beta, \gamma; \gamma (t, \beta)) + \sum_{|k|<2b} (a_k (t, x))'_{x_{j\mu}} D^k_{x_1} Z_0 (t, x; \beta, \gamma; \gamma (t, \beta)) \\
&+ \sum_{|k|<2b} \partial_{\gamma_{j\mu}} a_k (t, x) D^k_{x_1} Z_0 (t, x; \beta, \gamma; \gamma (t, \beta)) \\
&+ \sum_{|k|<2b} a_k (t, x)\, D^k_{x_1} \partial_{\overline{\gamma}_{j\mu}} Z_0 (t, x; \beta, \gamma; \overline{\gamma} (t, \beta)) \big|_{\overline{\gamma}=\gamma} \\
&+ (-1)^l \frac{(t-\beta)^{p-j-l}}{(p-j-l)!} \sum_{l=1}^{p-j} \sum_{|k|=2b} a_k (t, x) D^k_{x_1} \partial_{\gamma_{j+l}} Z_0 (t, x; \beta, \gamma; \gamma (t, \beta)) .
\end{aligned} \tag{26}$$

Kernels have the highest singularity at the variable x_p. Also, using (26) we have the existence of $\partial_{x_j} \varphi (t, x; \tau, \xi)$, $j = \overline{2, p-1}$ and the following estimations

$$\left|\partial_{x_j} \varphi (t, x; \tau, \xi)\right| \le A(t-\tau)^{-\sum\limits_{s=1}^{p} \frac{(1+2b(j-1))n_s - \alpha^* + 1}{2b} - j} \Phi (t, x; \tau, \xi) , \qquad j = \overline{2, p-1}.$$

Using arguments like in [1] we can get

$$\Delta_{hx_1} \varphi (t, x; \tau, \xi) = \Delta_{hx_1} K (t, x; \tau, \xi) + \int_\tau^t d\beta \int_{R^{n_0}} \Delta_{hx_1} K (t, x; \beta, \gamma) K (\beta, \gamma; \tau, \xi)\, d\gamma.$$

Applying the technique developed for parabolic systems in [6], and the evaluation of reproducing kernels, we obtain

$$\left|\Delta_{hx_1}\varphi\left(t,x;\tau,\xi\right)\right| \leq |h_{x_1}|^{\alpha_1}(t-\tau)^{-\sum\limits_{s=1}^{p}\frac{(1+2b(s-1))n_s-(1-\alpha_2)}{2b}-j}\Phi\left(t,x;\tau,\xi\right),$$

$$\alpha_1>0,\ \alpha_2>0,\ \alpha_1+\alpha_1=\alpha.$$

The existence and evaluation of $\partial^k_{x_1}Z\left(t,x;\tau,\xi\right)$, $|k|\leq 2b$, at $t>\tau$, are established for both of the cases of parabolic equations and systems in [6]. The theorem is proved. $\square$

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Малицька Г.П., Буртняк І.В. *Побудова фундаментального розв'язку одного класу вироджених параболічних рівнянь високого порядку* // Карпатські матем. публ. — 2020. — Т.12, №1. — С. 79–87.

У статті модифікованим методом Леві побудовано функцію Ґріна для одного класу ультрапараболічних рівнянь високого порядку з довільною кількістю груп виродження параболічності. Модифікований метод Леві розроблено для рівнянь Колмогорова високого порядку з коефіцієнтами залежними від усіх змінних, при цьому заморожена точка, яка є параметриксом, підібрана так, щоб зручно використовувалася експоненціальна оцінка фундаментального розв'язку та його похідних.

Ключові слова і фрази: вироджені параболічні рівняння, модифікований метод Леві, рівняння Колмогорова, фундаментальний розв'язок, параметрикс.

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ISOMORPHIC SPECTRUM AND ISOMORPHIC LENGTH OF A BANACH SPACE

We prove that, given any ordinal $\delta < \omega_2$, there exists a transfinite δ-sequence of separable Banach spaces $(X_\alpha)_{\alpha<\delta}$ such that X_α embeds isomorphically into X_β and contains no subspace isomorphic to X_β for all $\alpha < \beta < \delta$. All these spaces are subspaces of the Banach space $E_p = \left(\bigoplus_{n=1}^\infty \ell_p\right)_2$, where $1 \le p < 2$. Moreover, assuming Martin's axiom, we prove the same for all ordinals δ of continuum cardinality.

Key words and phrases: Banach space, isomorphic embedding, Martin axiom.

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INTRODUCTION

We use the standard terminology of Banach spaces theory, see [1]. Let X and Y be Banach spaces. We write $X \hookrightarrow Y$ if X embeds isomorphically into Y, and $X \simeq Y$ if X and Y are isomorphic.

Isomorphic spectrum

By the *isomorphic spectrum* of an infinite dimensional Banach space X we mean the set $\mathrm{sp}\,(X)$ of all isomorphic types of infinite dimensional subspaces of X.

Consider the following equivalence relation on the set $\mathcal{B}$ of separable infinite dimensional Banach spaces. We say that Banach spaces $X, Y \in \mathcal{B}$ are *equispectral* and write $X \overset{\mathrm{sp}}{\sim} Y$ provided that $X \hookrightarrow Y$ and $X \hookrightarrow Y$ (notice that Banach [2, p. 193] used a different terminology for equispectral Banach spaces X and Y, he said that X and Y have *equal linear dimension* and used the notation $\dim_l X = \dim_l Y$). It is immediate that $X \overset{\mathrm{sp}}{\sim} Y$ if and only if $\mathrm{sp}\,(X) = \mathrm{sp}\,(Y)$. It is a well known fact that $X \overset{\mathrm{sp}}{\sim} Y$ does not imply that $X \simeq Y$, however $X \simeq Y$ easily implies that $X \overset{\mathrm{sp}}{\sim} Y$. For instance, $L_1 \oplus \ell_2 \overset{\mathrm{sp}}{\sim} L_1$, however $L_1 \oplus \ell_2 \not\simeq L_1$.

Observe that if $X \in \{c_0, \ell_p : 1 \le p < \infty\}$ and Y is any infinite dimensional subspace of X then $X \overset{\mathrm{sp}}{\sim} Y$.

Denote by $\widetilde{\mathcal{B}}$ the set of all equivalence classes in $\mathcal{B}$ modulo the relation $\overset{\mathrm{sp}}{\sim}$, and for every $X \in \mathcal{B}$ by $\widetilde{X}$ we denote the equivalence class containing X.

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Given Banach spaces X and Y, we write $X \prec Y$ to express that $X \hookrightarrow Y$, while $Y \not\hookrightarrow X$. It is easy to see that, for every $X_i, Y_i \in \mathcal{B}$, $i = 1, 2$ with $X_1 \overset{\text{sp}}{\sim} X_2$ and $Y_1 \overset{\text{sp}}{\sim} Y_2$ the relation $X_1 \prec Y_1$ is equivalent to $X_2 \prec Y_2$. So, the same relation $\prec$ is well defined on $\widetilde{\mathcal{B}}$ by setting $\mathcal{X} \prec \mathcal{Y}$ provided $X \prec Y$ for some (or, equivalently, any) representatives $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$.

Observe that $\prec$ is a strict partial relation on $\widetilde{\mathcal{B}}$, and that $X \prec Y$ is equivalent to the strict inclusion $\operatorname{sp}(X) \subset \operatorname{sp}(Y)$.

By the solution of the homogeneous Banach space problem obtained by a combination of results of Gowers [5,6] and Komorowski–Tomczak-Jaegermann [9,10], ℓ_2 is the unique element X of $\mathcal{B}$ with $\operatorname{sp}(X) = \{X\}$. Although the spaces c_0 and ℓ_p with $1 \le p < \infty$, $p \ne 2$ have more than one-element isomorphic spectrum, all of them are equispectral, as mentioned above. So, $\widetilde{c_0}$ and $\widetilde{\ell_p}$ with $1 \le p < \infty$ are minimal elements of $\widetilde{\mathcal{B}}$. On the other hand, it is easy to see that $\widetilde{C[0,1]}$ is the unique maximal element of $\widetilde{\mathcal{B}}$, which is, moreover, the greatest element of $\widetilde{\mathcal{B}}$.

Set-theoretical preliminaries

We use the standard set-theoretical terminology and notation of [7], where the reader can also find necessary background. By $\mathfrak{c}$ we denote the cardinality of continuum. We say that A *meets* B provided that $A \cap B \ne \varnothing$.

Let $(M, <)$ be a partially ordered set. Following [11], the *length* of M is defined to be the supremum of ordinals α which are isomorphic to a subset of M, and is denoted by $L(M)$. For instance, $L(\alpha) = \alpha$ for every ordinal α and $L(\mathbb{R}) = \omega_1$.

Let ω_α be any infinite cardinal. We endow the power-set $\mathcal{P}(\omega_\alpha)$ with the partial order $A < B$ if and only if $|A \setminus B| < \aleph_\alpha = |B \setminus A|$.

Let us recall the statement of Martin's axiom (MA). A subset D of a partially ordered set P is said to be *dense* if for every $p \in P$ there is $d \in D$ such that $d \le p$. A subset $Q \subseteq P$ is said to be *consistent* provided for every finite subset $F \subseteq Q$ there exists $p \in P$ such that $p \le f$ for every $f \in F$. Elements p, q of P are said to be *consistent* if the two-element subset $\{p, q\}$ is consistent. A subset $Q \subseteq P$ consisting of more than two elements is said to be *pairwise inconsistent* if every two distinct elements of Q are not consistent. P is said to have the *countable chain condition* (CCC in short) if every pairwise inconsistent subset of P is at most countable.

Martin's axiom. *Let P be a partially ordered set possessing the CCC. Let $\mathfrak{M}$ be a collection of dense subsets of P of cardinality $< \mathfrak{c}$. Then there exists a consistent subset $Q \subseteq P$ which meets every element of $\mathfrak{M}$.*

We remark that MA is independent of the usual axioms ZFC. It follows from the Continuum Hypothesis (CH) and sometimes allows to extend results, previously established under the assumption of CH.

We need the following combinatorial lemma proved in [11].

Lemma 1. *(i) For every regular cardinal ω_δ one has $L(\mathcal{P}(\omega_\delta)) \ge \omega_{\delta+2}$.*

(ii) Let ω_c be the cardinal of cardinality $\mathfrak{c}$. Then (MA) $L(\mathcal{P}(\omega_0)) = \omega_{c+1}$.

Here (MA) in item *(ii)* means that the proof of *(ii)* uses Martin's axiom.

Isomorphic length of a Banach space

Let X be a separable infinite dimensional Banach space. By the *isomorphic length* of X we mean the length of the subset $\widetilde{\mathcal{B}}_X$ of the partially ordered set $\widetilde{\mathcal{B}}$ consisting of all equivalence classes containing all infinite dimensional subspaces of X: $IL(X) = L(\widetilde{\mathcal{B}}_X)$. Since by the above $\widetilde{\mathcal{B}}_{\ell_p}$ and $\widetilde{\mathcal{B}}_{c_0}$ are singletons, we have that $IL(\ell_p) = IL(c_0) = 1$ for every $p \in [1, +\infty)$. In the next section, we show that for $E_p = \left(\bigoplus_{n=1}^{\infty} \ell_p\right)_2$ with $1 \leq p < 2$ one has $IL(E_p) \geq \omega_2$, and Martin's axiom implies that $IL(E_p) = \omega_{c+1}$. Of course, the same could be said about the universal Banach space $C[0,1]$, which has the maximal possible length.

1 TRANSFINITE $\prec$-INCREASING SEQUENCES OF SPACES

Theorem 1. *Let $1 \leq p < 2$ and $E_p = \left(\bigoplus_{n=1}^{\infty} \ell_p\right)_2$. Then*

1) *for every ordinal γ of cardinality $\aleph_1$ there is a transfinite sequence $(X_\alpha)_{\alpha<\gamma}$ of subspaces of E_p such that $X_\alpha \prec X_\beta$ for all $\alpha < \beta < \gamma$,*

2) (MA) *for every ordinal γ of cardinality $\mathfrak{c}$ there is a transfinite sequence $(X_\alpha)_{\alpha<\gamma}$ of subspaces of E_p such that $X_\alpha \prec X_\beta$ for all $\alpha < \beta < \gamma$.*

Proof. Let $(p_n)_{n=1}^{\infty}$ be any sequence on numbers with $p < p_1 < p_2 < \ldots$ and $\lim_{n\to\infty} p_n = 2$.

Lemma 2. *For every finite dimensional Banach space X and every $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that for every into isomorphism $T : \ell_{p_n}^m \to X \oplus_2 \left(\bigoplus_{j>n} \ell_{p_j}\right)_2$ one has $\|T\|\|T^{-1}\| \geq n$.*

Proof of Lemma 2. Recall the standard definition (see, for example, [12, p. 54]): a Banach space Z is said to have *Rademacher type p*, $1 \leq p \leq 2$ (or just *type p*) if there exists a constant $T_p(Z) < \infty$ such that for every $k \in \mathbb{N}$ and for every $x_1, \ldots, x_k \in Z$,

$$\left(\int_0^1 \left\|\sum_{i=1}^{k} r_i(t) x_i\right\|_Z^p dt\right)^{1/p} \leq T_p(Z) \left(\sum_{i=1}^{k} ||x_i||_Z^p\right)^{1/p}, \tag{1}$$

where $\{r_i\}$ are Rademacher functions.

The Khinchin-Kahane inequality (see e.g. [12, p. 57]) implies that we can replace the value $\left(\int_0^1 \left\|\sum_{i=1}^{k} r_i(t) x_i\right\|_Z^p dt\right)^{1/p}$ with $\left(\int_0^1 \left\|\sum_{i=1}^{k} r_i(t) x_i\right\|_Z^2 dt\right)^{1/2}$ in the left-hand side of inequality (1), it will not change the class of spaces of type p, but may change the constant $T_p(Z)$, let us denote this new constant $T_{p,2}(Z)$.

Now we shall check (recall that $p \leq 2$) that the fact that spaces $\{Z_n\}_{n=1}^{\infty}$ have type p with uniformly bounded constants $\{T_{p,2}(Z_n)\}_{n=1}^{\infty}$, then $Z := \left(\bigoplus_{n=1}^{\infty} Z_n\right)_2$ also has type p with constant $T_{p,2}(Z)$ bounded from above by $\mathbf{T} := \sup_n T_{p,2}(Z_n)$.

So let $z_i = \{z_{i,n}\}_{n=1}^{\infty} \in \left(\bigoplus_{n=1}^{\infty} Z_n\right)_2$, so $z_{i,n} \in Z_n$. We have

$$\begin{aligned}\left(\int_0^1 \left\|\sum_{i=1}^k r_i(t) z_i\right\|_Z^2 dt\right)^{1/2} &= \left(\int_0^1 \sum_{n=1}^{\infty} \left\|\sum_{i=1}^k r_i(t) z_{i,n}\right\|_{Z_n}^2 dt\right)^{1/2}\\ &\le \mathbf{T}\left(\sum_{n=1}^{\infty}\left(\sum_{i=1}^k \|z_{i,n}\|_{Z_n}^p\right)^{2/p}\right)^{1/2}\\ &\le \mathbf{T}\left(\sum_{i=1}^{k}\left(\sum_{n=1}^{\infty} \|z_{i,n}\|_{Z_n}^2\right)^{p/2}\right)^{1/p} = \mathbf{T}\left(\sum_{i=1}^k \|z_i\|_Z^p\right)^{1/p},\end{aligned}$$

where in the first line we use the definition of Z as a direct sum; in the second line we use the fact that Z_n have type p with constant $\mathbf{T}$; in the third line we use the triangle inequality for the space $\ell_{2/p}$ (recall that $2/p \ge 1$), and in the last line we use the definition of Z again.

Now we return to the proof of Lemma 2. Since X is finite-dimensional, it has type p_{n+1} with sufficiently large constant. We need the well-known fact that ℓ_p has type p if $p \in [1, 2]$ (see e.g. [12, p. 63]) and an easy-to-see fact (consider the unit vectors) that ℓ_p does not have a larger type.

We conclude that $X \oplus_2 \left(\bigoplus_{j>n} \ell_{p_j}\right)_2$ has type p_{n+1} with some constant C, but ℓ_{p_n} does not have type p_{n+1}. Therefore the type constant of $\ell_{p_n}^m$ for type p_{n+1} and sufficiently large m is $> Cn$. It is easy to see that this implies that for every into isomorphism $T : \ell_{p_n}^m \to X \oplus_2 \left(\bigoplus_{j>n} \ell_{p_j}\right)_2$ one has $\|T\|\|T^{-1}\| \ge n$. □

We continue the proof of Theorem 1. Using Lemma 2, construct recurrently a sequence $(m_n)_{n\in\mathbb{N}}$ of positive integers so that

for every $n \in \mathbb{N}$ and every into isomorphism

$$U : \ell_{p_n}^{m_n} \to \left(\bigoplus_{i=1}^{n-1} \ell_{p_i}^{m_i}\right)_2 \oplus_2 \left(\bigoplus_{j>n} \ell_{p_j}\right)_2 \tag{2}$$

one has $\|U\|\|U^{-1}\| \ge n$.

It is known that for every $\varepsilon > 0$, every $m \in \mathbb{N}$ and every $q \in (p, 2]$ there exists a subspace F of ℓ_p which is $(1+\varepsilon)$-isomorphic to ℓ_q^m (see [8] for tight estimates of the parameters involved, the result itself follows from [4]). Using this fact for $\varepsilon = 1$, $m = m_n$ and $q = p_n$, for every $n \in \mathbb{N}$ we choose a subspace F_n of n-th summand of E_p (which is isometric to ℓ_p) which is 2-isomorphic to $\ell_{p_n}^{m_n}$, say, by means of an isomorphism $J_n : F_n \to \ell_{p_n}^{m_n}$ with $\|J_n\|\|J_n^{-1}\| \le 2$.

Fix any ordinal γ of cardinality $\aleph_1$ (or $\mathfrak{c}$, respectively). Using items (i) and (ii) of Lemma 1, respectively, choose a transfinite sequence $(N_\alpha)_{\alpha<\gamma}$ of subsets of $\mathbb{N}$ so that $|N_\alpha \setminus N_\beta| < \aleph_0 = |N_\beta \setminus N_\alpha|$ for all $\alpha < \beta < \gamma$. For each $\alpha < \gamma$ set

$$X_\alpha = \left(\bigoplus_{n\in N_\alpha} F_n\right)_2.$$

We consider each X_α as a subspace of E_p. Let us show that $(X_\alpha)_{\alpha<\gamma}$ has the desired properties. Fix any $\alpha < \beta < \gamma$. Set $N' = N_\alpha \setminus N_\beta$, $N'' = N_\alpha \cap N_\beta$, $N''' = N_\beta \setminus N_\alpha$. Then $N_\alpha = N' \sqcup N''$, $N_\beta = N'' \sqcup N'''$, $|N'| < \aleph_0 = |N'''|$. Hence,

$$X_\alpha = \Big(\bigoplus_{n\in N'} F_n\Big)_2 \oplus_2 \Big(\bigoplus_{n\in N''} F_n\Big)_2, \ X_\beta = \Big(\bigoplus_{n\in N''} F_n\Big)_2 \oplus_2 \Big(\bigoplus_{n\in N'''} F_n\Big)_2.$$

Since $|N'| < \aleph_0 = |N'''|$, we have that

$$\dim\Big(\bigoplus_{n\in N'} F_n\Big)_2 < \infty = \dim\Big(\bigoplus_{n\in N'''} F_n\Big)_2$$

and hence, X_α embeds isomorphically into X_β.

Prove that X_β does not embed isomorphically into X_α. Assume, on the contrary, that there is an into isomorphism $T : X_\beta \to X_\alpha$. Take any $n_0 \in N'''$ and consider the restriction $T_{n_0} = T|_{F_{n_0}}$ of T to F_{n_0}.

Observe that

$$X_\alpha \subseteq \Big(\bigoplus_{i=1}^{n_0-1} F_i\Big)_2 \oplus_2 \Big(\bigoplus_{j>n_0} \ell_{p_j}\Big)_2.$$

Let

$$S : \Big(\bigoplus_{i=1}^{n-1} F_i\Big)_2 \oplus_2 \Big(\bigoplus_{j>n} \ell_{p_j}\Big)_2 \to \Big(\bigoplus_{i=1}^{n-1} \ell_{p_i}^{m_i}\Big)_2 \oplus_2 \Big(\bigoplus_{j>n} \ell_{p_j}\Big)_2$$

be an operator which sends $\big((f_i)_{i=1}^{n_0-1}, g\big)$ to $\big((J_i f_i)_{i=1}^{n_0-1}, g\big)$. Since J_i are isomorphisms with $\|J_i\|\|J_i^{-1}\| \le 2$, so is S with $\|S\|\|S^{-1}\| \le 2$. Hence,

$$\|T\|\|T^{-1}\| \ge \|T_0\|\|T_0^{-1}\| \ge \frac{1}{2}\|S\circ T_0\|\|(S\circ T_0)^{-1}\| \overset{\text{by}(2)}{\ge} \frac{1}{2} n_0.$$

This is impossible for large enough $n_0 \in N'''$. $\square$

The next corollary follows from Theorem 1 and the observation that a separable infinite dimensional Banach space X has only continuum many closed subspaces, and hence, $IL(X) \le \omega_{c+1}$.

Corollary 1. (MA) $IL(E_p) = IL\big(C[0,1]\big) = \omega_{c+1}$.

2 REMARKS AND AN OPEN PROBLEM

It would be interesting to find the isomorphic length of the classical spaces $L_p = L_p[0,1]$.

Problem 1. *Evaluate $IL(L_p)$ for $1 \le p < \infty$, $p \ne 2$.*

The embeddability of L_r into L_p for $1 \le p < r \le 2$ [4] together with impossibility of the embedding L_p into L_r for the same values of p, r [2, p. 206] imply the inequality $IL(L_p) \ge \omega_1$ for $1 \le p < 2$, because every countable ordinal $\alpha < \omega_1$ is isomorphic to a subset of any interval (a, b) in the reverse order. The same inequality $IL(L_p) \ge \omega_1$ for all values $1 \le p < \infty$, $p \ne 2$ is a corollary of the following result.

Theorem 2 (Bourgain, Rosenthal, Schechtman, [3]). *Let $1 < p < \infty$, $p \ne 2$. There exists a family $(X_\alpha^p)_{\alpha<\omega_1}$ of complemented subspaces of L_p so that for all $\alpha < \beta < \omega_1$ one has $X_\alpha^p \prec X_\beta^p$. Moreover, if B is a separable Banach space such that $X_\alpha^p \hookrightarrow B$ for all $\alpha < \omega_1$ then $L_p \hookrightarrow B$.*

Observe that Theorem 2 gives a strictly $\prec$-increasing ω_1-sequence of subspaces of L_p for $1 < p < 2$ directly. The same holds also for $p = 1$ due to the fact ([4]) that L_r $(1 < r < 2)$ embeds isometrically into L_1. On the other hand, the argument based on embeddability/non-embeddability of L_r into L_p does not provide an uncountable sequence. However, both arguments provide the same estimate for $IL(L_p)$ if $1 < p < 2$.

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Доведено, що для кожного ординалу $\delta < \omega_2$ існує трансфінітна δ-послідовність сепарабельних банахових просторів $(X_\alpha)_{\alpha<\delta}$ така, що X_α вкладається ізоморфно в X_β і не містить підпросторів, ізоморфних до X_β для всіх $\alpha < \beta < \delta$. Всі ці простори є підпросторами банахового простору $E_p = \left(\bigoplus_{n=1}^\infty \ell_p\right)_2$, де $1 \le p < 2$. Більш того, у припущенні аксіоми Мартіна доведено дане твердження для всіх ординалів δ потужності континуум.

Ключові слова і фрази: банахів простір, ізоморфне вкладення, аксіома Мартіна.

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RELATED FIXED POINT RESULTS VIA C_*-CLASS FUNCTIONS ON C^*-ALGEBRA-VALUED G_b-METRIC SPACES

We initiate the concept of C^*-algebra-valued G_b-metric spaces. We study some basic properties of such spaces and then prove some fixed point theorems for Banach and Kannan types via C_*-*class* functions. Also, some nontrivial examples are presented to ensure the effectiveness and applicability of the obtained results.

Key words and phrases: fixed point, C_*-*class* function, C^*-algebra-valued G_b-metric space.

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1 INTRODUCTION

One of the main directions to obtain possible generalizations of fixed point results is the introduction of new types of spaces. For instance, Ma Z., Jiang, L. and Sun, H. in [21] initiated the notion of C^*-algebra-valued metric spaces, where the set of nonnegative reals was replaced by the set of positive elements of a unital C^*-algebra. Going in the same direction, many papers appeared. See, for example, [16,17,20,22,23,34,36].

In [19], the concept of a C^*-algebra-valued modular space has been introduced. It generalized the concept of a modular space. Now, let $T : X_\rho \to X_\rho$ be a self-mapping on a complete C^*-algebra-valued modular space such that there are $c \in \mathbb{A}$ and $\lambda, \sigma \in \mathbb{R}^+$ with $\|c\| < 1$ and $\lambda > \sigma$ so that

$$\rho(\lambda(T\mu - T\nu)) \preceq c^*\rho(\sigma(\mu - \nu))c, \qquad \forall \mu, \nu \in \mathcal{X}_\rho.$$

Then T admits a unique fixed point in X_ρ ([19]).

Bakhtin [10] considered the class of b-metric spaces. Later, many works such as [5–7, 11, 12, 30] have been provided. In [9], the concept of complex valued metric spaces was initiated. Rao et al. [32] initiated the concept of complex valued b-metric spaces. Mustafa and Sims [24] considered the class of G-metric spaces, where the considered metric depends on three variables. For other related papers, see [1,2,8,18,27,29,31,35].

The notion of G_b-metric spaces was presented by Aghajani et al. [3] (see also [25]). Later, Ege [14] introduced the notion of complex valued G_b-metric spaces and proved the related Banach and Kannan type fixed point theorems. In [15], Ege proved a common fixed point theorem via α-series. For other results on G_b-metric spaces, see [26,28,33].

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Very recently, Ansari et al. [4] defined the concept of complex valued *C-class* functions. Also, Moeini et al. [22] presented the notion of C_*-class functions.

In this presented work, we introduce the C^*-algebra-valued G_b-metric spaces which generalize the complex valued G_b-metric spaces. By using C_**-class* functions, we establish Banach and Kannan type fixed point theorems in C^*-algebra-valued G_b-metric spaces. To support our results, some nontrivial examples are also given.

Definition 1 ([3]). *Let E be a nonempty set and $s \geq 1$. If the function $G : E \times E \times E \to \mathbb{R}_+$ verifies:*

(G_b1) $G(\mu, \eta, \xi) = 0$ if $\mu = \eta = \xi$;

(G_b2) $0 < G(\mu, \mu, \eta)$ for all $\mu, \eta \in E$ with $\mu \neq \eta$;

(G_b3) $G(\mu, \mu, \eta) \leq G(\mu, \eta, \xi)$ for all $\mu, \eta, \xi \in E$ with $\eta \neq \xi$;

(G_b4) $G(\mu, \eta, \xi) = G(p\{\mu, \eta, \xi\})$, where p is a permutation of μ, η, ξ;

(G_b5) $G(\mu, \eta, \xi) \leq s(G(\mu, a, a) + G(a, \eta, \xi))$ for all $\mu, \eta, \xi, a \in E$,

then G is said to be a G_b-metric and (E, G) is called a G_b-metric space.

Mention that any G-metric space is a G_b-metric space with $s = 1$.

Proposition 1 ([3]). *Let (E, G) be a G_b-metric space. For any $\mu, \eta, \xi, a \in E$, we have*

(i) if $G(\mu, \eta, \xi) = 0$, then $\mu = \eta = \xi$;

(ii) $G(\mu, \eta, \xi) \leq s(G(\mu, \mu, \eta) + G(\mu, \mu, \xi))$;

(iii) $G(\mu, \eta, \eta) \leq 2sG(\eta, \mu, \mu)$;

(iv) $G(\mu, \eta, \xi) \leq s(G(\mu, a, \xi) + G(a, \eta, \xi))$.

Definition 2 ([3]). *Let (E, G) be a G_b-metric space and $\{\mu_n\}$ be a sequence in E.*

(i) $\{\mu_n\}$ is G_b-convergent to μ if for each $\varepsilon > 0$, there is $p_0 \in \mathbb{N}$ so that $G(\mu, \mu_p, \mu_q) < \varepsilon$, $p, q \geq p_0$.

(ii) $\{\mu_n\}$ is said to be G_b-Cauchy if for every $\varepsilon > 0$, there is $p_0 \in \mathbb{N}$ so that $G(\mu_p, \mu_q, \mu_i) < \varepsilon$, $p, q, i \geq p_0$.

(iii) If each G_b-Cauchy sequence G_b-converges in (E, G), then (E, G) is called G_b-complete.

Proposition 2 ([3]). *Let E be a G_b-metric space. We have the following equivalences:*

(1) $\{\mu_n\}$ G_b-converges to μ;

(2) $G(\mu_p, \mu_p, \mu) \to 0$ as $p \to \infty$;

(3) $G(\mu_p, \mu, \mu) \to 0$ as $p \to \infty$.

A Banach algebra $\mathbb{A}$ (over the field of complex numbers $\mathbb{C}$) is called a C^*-algebra if there exists an involution $*$ in $\mathbb{A}$ (i.e., an operator $*: \mathbb{A} \to \mathbb{A}$ verifying $a^{**} = a$ for every $a \in \mathbb{A}$) so that, for all $c, d \in \mathbb{A}$ and $\eta, \nu \in \mathbb{C}$, we have:

(i) $(\eta c + \nu d)^* = \bar{\eta} c^* + \bar{\nu} d^*$;

(ii) $(cd)^* = d^* c^*$;

(iii) $\|c^* c\| = \|c\|^2$.

By (iii), we have $\|c\| = \|c^*\|$ for each $c \in \mathbb{A}$. also, $(\mathbb{A}, *)$ is said to be a unital $*$-algebra if the identity element $1_{\mathbb{A}}$ is contained in $\mathbb{A}$. An element $c \in \mathbb{A}$ is called positive if $c^* = c$ and its spectrum $\sigma(c) = \{\lambda \in \mathbb{R} : \lambda 1_{\mathbb{A}} - c \text{ is noninvertible}\} \subset \mathbb{R}_+$. Denote by $\mathbb{A}_+$ the family of positive elements in $\mathbb{A}$. Define the partial order $'\succeq'$ on $\mathbb{A}$ as

$$d \succeq c \ \text{ iff } \ d - c \in \mathbb{A}_+.$$

If $c \in \mathbb{A}$ is positive, we write $c \succeq 0_{\mathbb{A}}$, where $0_{\mathbb{A}}$ is the zero element of $\mathbb{A}$. Each positive element a of a C^*-algebra $\mathbb{A}$ has a unique positive square root. Denote by $\mathbb{A}$ a unital C^*-algebra with identity element $1_{\mathbb{A}}$. Moreover, $\mathbb{A}_+ = \{c \in \mathbb{A} : c \succeq 0_{\mathbb{A}}\}$ and $(c^* c)^{\frac{1}{2}} = |c|$.

Lemma 1 ([13])**.** *Let $\mathbb{A}$ be a unital C^*-algebra ($1_{\mathbb{A}}$ is its unit).*

(1) For each $z \in \mathbb{A}_+$, $z \preceq 1_{\mathbb{A}}$ iff $\|z\| \leq 1$.

(2) If $c \in \mathbb{A}_+$ with $\|c\| < \frac{1}{2}$, then $1_{\mathbb{A}} - c$ is invertible and $\|c(1_{\mathbb{A}} - c)^{-1}\| < 1$.

(3) Let $c, d \in \mathbb{A}$ so that $c, d \succeq 0_{\mathbb{A}}$ and $cd = dc$. We have $cd \succeq 0_{\mathbb{A}}$.

(4) Put $\mathbb{A}' = \{c \in \mathbb{A} : cd = dc, \forall d \in \mathbb{A}\}$. Let $c \in \mathbb{A}'$, $d, e \in \mathbb{A}$ with $d \succeq e \succeq 0_{\mathbb{A}}$ and $1_{\mathbb{A}} - c \in \mathbb{A}'$ is an invertible operator. We have

$$(1_{\mathbb{A}} - c)^{-1} d \succeq (1_{\mathbb{A}} - c)^{-1} e.$$

Note that if $0_{\mathbb{A}} \preceq c, d$, we have not $0_{\mathbb{A}} \preceq cd$ in a C^*-algebra. Indeed, take the C^*-algebra $\mathbb{M}_2(\mathbb{C})$ with $c = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$, $d = \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}$, then $cd = \begin{pmatrix} -1 & 2 \\ -4 & 8 \end{pmatrix}$. Clearly c, d are in $\mathbb{M}_2(\mathbb{C})_+$, while cd is not.

The notion of complex C-class functions has been initiated by Ansari et al. [4].

Definition 3. *Define $S = \{z \in \mathbb{C} : z \succeq 0\}$. Let $F : S^2 \to \mathbb{C}$ be a continuous function. Such F is said to be a complex C-class function if for all $p, q \in S$*

(1) $F(p, q) \preceq p$;

(2) $F(p, q) = p$ implies that either $p = 0$ or $q = 0$.

For examples of these functions, see [4].

2 MAIN RESULTS

First, we initiate the concept of C^*-algebra-valued G_b-metric spaces.

Definition 4. *Let $\mathbb{A}$ be a unital C^*-algebra and E be a nonempty set. Let $s \in \mathbb{A}$ be such that $\|s\| \geq 1$. A mapping $G : E \times E \times E \to \mathbb{A}_+$ is said to be a C^*-algebra-valued G_b-metric on E if*

(CG_b1) $G(\mu, \eta, \xi) = 0_{\mathbb{A}}$ if $\mu = \eta = \xi$;

(CG_b2) $0_{\mathbb{A}} \prec G(\mu, \mu, \eta)$ for all $\mu, \eta \in E$ with $\mu \neq \eta$;

(CG_b3) $G(\mu, \mu, \eta) \preceq G(\mu, \eta, \xi)$ for all $\mu, \eta, \xi \in E$ with $\eta \neq \xi$;

(CG_b4) $G(\mu, \eta, \xi) = G(p\{\mu, \eta, \xi\})$, where p is a permutation of μ, η, ξ;

(CG_b5) $G(\mu, \eta, \xi) \preceq s(G(\mu, a, a) + G(a, \eta, \xi))$ for all $\mu, \eta, \xi, a \in E$.

The triplet $(E, \mathbb{A}, G)$ is called a C^*-algebra-valued G_b-metric space.

Remark 1. *By taking $\mathbb{A} = \mathbb{R}$, a C^*-algebra-valued G_b-metric space is a (real) G_b-metric space.*

As in Proposition 1, we have the following.

Proposition 3. *Let $(E, \mathbb{A}, G)$ be a C^*-algebra-valued G_b-metric space. For all $\mu, \eta, \xi \in E$, we have*

(i) $G(\mu, \eta, \xi) \preceq s(G(\mu, \mu, \eta) + G(\mu, \mu, \xi))$;

(ii) $G(\mu, \eta, \eta) \preceq 2sG(\eta, \mu, \mu)$.

Definition 5. *Let $(E, \mathbb{A}, G)$ be a C^*-algebra-valued G_b-metric space and $\{\mu_n\}$ be a sequence in E.*

(i) $\{\mu_n\}$ is G_b-convergent to $x \in E$ with respect to the algebra $\mathbb{A}$ iff for each $a \in \mathbb{A}$ with $0_{\mathbb{A}} \prec a$, there is $k \in \mathbb{N}$ so that $G(x, \mu_p, \mu_q) \prec a$ for all $p, q \geq k$.

(ii) $\{\mu_n\}$ is called G_b-Cauchy with respect to $\mathbb{A}$ if for each $a \in \mathbb{A}$ with $0_{\mathbb{A}} \prec a$, there is $k \in \mathbb{N}$ so that $G(\mu_p, \mu_q, \mu_i) \prec a$, $p, q, i \geq k$.

(iii) If each G_b-Cauchy sequence with respect to $\mathbb{A}$ G_b-converges with respect to $\mathbb{A}$, then $(E, \mathbb{A}, G)$ is said to be complete.

Proposition 4. *Let $(E, \mathbb{A}, G)$ be a C^*-algebra-valued G_b-metric space and $\{\mu_n\}$ be a sequence in E. Then $\{\mu_n\}$ is G_b-convergent to μ with respect to $\mathbb{A}$ iff $\|G(\mu, \mu_n, \mu_m)\| \to 0$ as $n, m \to \infty$.*

Proof. ($\Rightarrow$) Let $\{\mu_n\}$ be G_b-convergent to μ with respect to $\mathbb{A}$ and let $a = \varepsilon.1_{\mathbb{A}}$ (where $\varepsilon > 0$). Then $0_{\mathbb{A}} \prec a \in \mathbb{A}$ and there is an integer k so that $G(\mu, \mu_n, \mu_m) \prec a$ for all $n, m \geq k$. Thus, $\|G(\mu, \mu_n, \mu_m)\| < \|a\| = \varepsilon$ and so $\|G(\mu, \mu_n, \mu_m)\| \to 0$ as $n, m \to \infty$.

($\Leftarrow$) Suppose that $\|G(\mu, \mu_n, \mu_m)\| \to 0$ as $n, m \to \infty$. For $a \in \mathbb{A}$ with $0_{\mathbb{A}} \prec a$, there is $\delta > 0$ so that for $z \in \mathbb{A}$,

$$\|z\| < \delta \quad \Rightarrow \quad z \prec a.$$

For such a $\delta > 0$, there is an integer k so that $\|G(x, \mu_n, \mu_m)\| < \delta$, i.e., $G(\mu, \mu_n, \mu_m) \prec a$ for all $n, m \geq k$, i.e., $\{\mu_n\}$ is G_b-convergent to μ with respect to $\mathbb{A}$. $\square$

From Proposition 3 and Proposition 4, we state the following.

Theorem 1. *Let $(E, \mathbb{A}, G)$ be a C^*-algebra-valued G_b-metric space. Let $\{\mu_n\}$ be a sequence in E and $\mu \in E$. We have equivalence of the following:*

(1) $\{\mu_n\}$ is G_b-convergent to μ with respect to $\mathbb{A}$;

(2) $\|G(\mu_p, \mu_p, \mu)\| \to 0$ when $p \to \infty$;

(3) $\|G(\mu_p, \mu, \mu)\| \to 0$ when $p \to \infty$;

(4) $\|G(\mu_q, \mu_p, \mu)\| \to 0$ when $p, q \to \infty$.

Proof. $(1) \Rightarrow (2)$. It follows from Proposition 4.
$(2) \Rightarrow (3)$. From Proposition 3, one writes

$$G(\mu_n, \mu, \mu) \preceq s(G(\mu_n, \mu_n, \mu) + G(\mu_n, \mu_n, \mu)).$$

Using (2), we get

$$\|G(\mu_n, \mu, \mu)\| \to 0 \text{ as } n \to \infty.$$

$(3) \Rightarrow (4)$. Using (CG_b4) and Proposition 3,

$$G(\mu_m, \mu_n, \mu) = G(\mu_n, \mu, \mu_m) \preceq s(G(\mu_m, \mu, \mu) + G(\mu, \mu_n, \mu)) = s(G(\mu_n, \mu, \mu) + G(\mu, \mu, \mu_m)).$$

Then $\|G(\mu_m, \mu_n, \mu)\| \to 0$ as $m, n \to \infty$.
$(4) \Rightarrow (1)$. By (CG_b3) and (CG_b4), we have

$$\begin{aligned} G(\mu, \mu_n, \mu_n) = G(\mu_n, \mu, \mu_n) &\preceq s(G(\mu_n, \mu, \mu_m) + G(\mu_m, \mu_m, \mu_n)) \\ &\preceq sG(\mu_n, \mu, \mu_m) + 2s^2 G(\mu_m, \mu_n, \mu)). \end{aligned}$$

Using the equivalence in Proposition 4, $\|G(\mu_m, \mu_n, \mu)\| \to 0$ as $m, n \to \infty$. Therefore, $\|G(\mu, \mu_n, \mu_n)\| \to 0$ as $n \to \infty$. □

Theorem 2. *Let $(E, \mathbb{A}, G)$ be a C^*-algebra-valued G_b-metric space and $\{\mu_n\}$ be a sequence in E. Then $\{\mu_n\}$ is G_b-Cauchy with respect to $\mathbb{A}$ if and only if $\|G(\mu_n, \mu_m, \mu_p)\| \to 0$ as $n, m, p \to \infty$.*

Proof. $(\Rightarrow)$ Let $b = \varepsilon \cdot 1_{\mathbb{A}}$ and $\varepsilon > 0$ be a real number. Then $0_{\mathbb{A}} \prec b \in \mathbb{A}$ and so there is an integer k such that $G(\mu_n, \mu_m, \mu_l) \prec b$ for all $n, m, l \geq k$. Thus, $\|G(\mu_n, \mu_m, \mu_l)\| < \|b\| = \varepsilon$ for all $n, m, l \geq k$.

$(\Leftarrow)$ Assume that $\|G(\mu_n, \mu_m, \mu_l)\| \to 0$ as $n, m, l \to \infty$. For $b \in \mathbb{A}$ with $0_{\mathbb{A}} \prec a$, there is $\gamma > 0$ so that for $z \in \mathbb{A}$

$$\|z\| < \gamma \text{ implies } z \prec b.$$

For such a γ, there is an integer k so that $\|G(\mu_n, \mu_m, \mu_l)\| < \gamma$ for all $n, m, l \geq k$. That is, $G(\mu_n, \mu_m, \mu_l) \prec b$ for all $n, m \geq k$. Then $\{\mu_n\}$ is G_b-Cauchy with respect to $\mathbb{A}$. □

Example 1. *Let $E = \mathbb{R}$ and $\mathbb{A} = M_2(\mathbb{R})$ the set of all 2×2 matrices. Consider the usual operations: scalar multiplication, addition and matrix multiplication. For $A \in \mathbb{A}$, consider*

$\|A\| = \left(\sum_{i,j=1}^{2} |a_{ij}|^2\right)^{\frac{1}{2}}$. *The operator* $* : \mathbb{A} \to \mathbb{A}$ *given as* $A^* = A$, *is a convolution on* $\mathbb{A}$. *Thus* $\mathbb{A}$ *becomes a unital* C^**-algebra. For*

$$a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad b = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \in \mathbb{A} = M_2(\mathbb{R}),$$

consider $a \preceq b$ *iff* $(a_{ij} - b_{ij}) \leq 0$, *for all* $i, j = 1, 2$.

Define $d(x,y) = diag(|x-y|, |x-y|)$ *with "diag" is a diagonal matrix and* $x, y \in \mathbb{R}$. *Suppose* $D_m(d)(x,y,z) = \max\{d(x,y), d(y,z), d(x,z)\}$ *for all* $x, y, z \in E$. *Define* $G : E \times E \times E \to \mathbb{A}_+$ *by*

$$G(x,y,z) = \Big(D_m(d)(x,y,z)\Big)^p,$$

where $p > 1$ *is an integer. It can be proved that* $(E, \mathbb{A}, G)$ *is a* C^**-algebra-valued* G_b*-metric space with* $s = 2^{p-1}.1_{\mathbb{A}}$.

To define the set of C_*-class functions (which contains complex valued C-class functions of [4]), it suffices to use the family of elements of a unital C^*-algebra instead of the set of complex numbers.

Definition 6 ([22]). *Let* $\mathbb{A}$ *be a unital* C^**-algebra and* $F : \mathbb{A}_+ \times \mathbb{A}_+ \to \mathbb{A}$ *be a continuous function. Such* F *is said to be a* C_**-class function if for all* $c, d \in \mathbb{A}_+$*:*

(1) $F(c,d) \preceq c$;

(2) $F(c,d) = c$ *implies that either* $c = 0_{\mathbb{A}}$ *or* $d = 0_{\mathbb{A}}$.

Let $\mathcal{C}_*$ be the set of C_*-class functions.

Remark 2. *If we replace* $\mathbb{A}$ *by* $\mathbb{C}$ *in Definition 6, the class* $\mathcal{C}_*$ *corresponds to the set of complex* C*-class functions.*

Example 2. *Consider* $\mathbb{A} = M_2(\mathbb{R})$ *as defined in Example 1.*

(1) Given $F_* : \mathbb{A}_+ \times \mathbb{A}_+ \to \mathbb{A}$ *as* $F_*(c,d) = c - d$, *that is,*

$$F_*\left(c = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}, d = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}\right) = \begin{pmatrix} c_{11} - d_{11} & c_{12} - d_{12} \\ c_{21} - d_{21} & c_{22} - d_{22} \end{pmatrix}$$

for all $c_{p,q}, d_{p,q} \in \mathbb{R}_+$, $(p, q \in \{1,2\})$. *Then* $F_* \in \mathcal{C}_*$.

(2) Given $F_* : \mathbb{A}_+ \times \mathbb{A}_+ \to \mathbb{A}$ *as*

$$F_*\left(\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}, \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}\right) = \lambda \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$$

for all $c_{p,q}, d_{p,q} \in \mathbb{R}_+$ *with* $(p, q \in \{1,2\})$, *where* $\lambda \in (0,1)$. *Then* $F_* \in \mathcal{C}_*$.

Example 3. *Let* $E = L^\infty(M)$ *and* $U = L^2(M)$, *where* M *is a Lebesgue measurable set. Let* $\mathcal{B}(U)$ *be the family of bounded linear operators on a Hilbert space* H. *Note that* $\mathcal{B}(U)$ *is a* C^**-algebra (with the usual operator norm). Given* $F_* : \mathcal{B}(U)_+ \times \mathcal{B}(U)_+ \to \mathcal{B}(U)$ *as*

$$F_*(P,Q) = P - \psi(P),$$

where $\psi : \mathcal{B}(U)_+ \to \mathcal{B}(U)_+$ *is continuous so that* $\psi(P) = 0_{\mathcal{B}(U)}$ *iff* $P = 0_{\mathcal{B}(U)}$. *Then* $F_* \in \mathcal{C}_*$.

Let Σ be the set of the functions $\sigma : \mathbb{A}_+ \to \mathbb{A}_+$ so that:

(a) σ is continuous;

(b) $\sigma(t) \succ 0_{\mathbb{A}}$ iff $t \succ 0_{\mathbb{A}}$ and $\sigma(0_{\mathbb{A}}) = 0_{\mathbb{A}}$.

Our first result is as follows.

Theorem 3. *Let $(E, \mathbb{A}, G)$ be a complete C^*-algebra-valued G_b-metric space with $s = (b.1_{\mathbb{A}}) \succ 1_{\mathbb{A}}$ and $T : E \to E$ be so that*

$$\sigma((b^\varepsilon.1_{\mathbb{A}})G(T\mu, T\eta, T\xi)) \preceq F_*\Big(\sigma(G(\mu,\eta,\xi)), \vartheta(G(\mu,\eta,\xi))\Big), \tag{1}$$

for all $\mu, \eta, \xi \in E$, where $F_* \in \mathcal{C}_*, \sigma, \vartheta \in \Sigma$ and $\varepsilon \in (1, \infty)$. Then T possesses a unique fixed point.

Proof. Let T verify (1). Consider $\mu_0 \in E$ and define $\mu_n = T^n\mu_0$. By (1), one writes

$$\sigma((b^\varepsilon.1_{\mathbb{A}})G(\mu_n, \mu_{n+1}, \mu_{n+1})) \preceq F_*\Big(\sigma(G(\mu_{n-1}, \mu_n, \mu_n)), \vartheta(G(\mu_{n-1}, \mu_n, \mu_n))\Big).$$

We have

$$G(\mu_n, \mu_{n+1}, \mu_{n+1}) \preceq (b^\varepsilon.1_{\mathbb{A}})^{-1}G(\mu_{n-1}, \mu_n, \mu_n), \quad \text{for all } n \geq 1. \tag{2}$$

The inequality (2) implies that

$$G(\mu_n, \mu_{n+1}, \mu_{n+1}) \preceq (b^\varepsilon.1_{\mathbb{A}})^{-2}G(\mu_{n-2}, \mu_{n-1}, \mu_{n-1}), \quad \text{for all } n \geq 2.$$

If the same process is continued, we get

$$G(\mu_n, \mu_{n+1}, \mu_{n+1}) \preceq (b^\varepsilon.1_{\mathbb{A}})^{-n}G(\mu_0, \mu_1, \mu_1), \quad \text{for all } n \geq 0. \tag{3}$$

Using (CG_b5) together with (3) ($n, m \in \mathbb{N}$ with $n < m$),

$$\begin{aligned}
&G(\mu_n, \mu_m, \mu_m) \preceq (b.1_{\mathbb{A}})[G(\mu_n, \mu_{n+1}, \mu_{n+1}) + G(\mu_{n+1}, \mu_m, \mu_m)] \\
&\preceq (b.1_{\mathbb{A}})[G(\mu_n, \mu_{n+1}, \mu_{n+1})] + (b.1_{\mathbb{A}})^2[G(\mu_{n+1}, \mu_{n+2}, \mu_{n+2}) + G(\mu_{n+2}, \mu_m, \mu_m)] \\
&\preceq (b.1_{\mathbb{A}})[G(\mu_n, \mu_{n+1}, \mu_{n+1})] + (b.1_{\mathbb{A}})^2[G(\mu_{n+1}, \mu_{n+2}, \mu_{n+2})] + \dots \\
&+ (b.1_{\mathbb{A}})^{m-n}[G(\mu_{m-1}, \mu_m, \mu_m)] \\
&\preceq (b.1_{\mathbb{A}})((b.1_{\mathbb{A}})^\varepsilon)^{-n}G(\mu_0, \mu_1, \mu_1) + (b.1_{\mathbb{A}})^2((b.1_{\mathbb{A}})^\varepsilon)^{-n-1}G(\mu_0, \mu_1, \mu_1) + \dots \\
&+ (b.1_{\mathbb{A}})^{m-n}((b.1_{\mathbb{A}})^\varepsilon)^{-m+1}G(\mu_0, \mu_1, \mu_1) \\
&\preceq [(b.1_{\mathbb{A}})((b.1_{\mathbb{A}})^\varepsilon)^{-n} + (b.1_{\mathbb{A}})^2((b.1_{\mathbb{A}})^\varepsilon)^{-n-1} + \dots + (b.1_{\mathbb{A}})^{m-n}((b.1_{\mathbb{A}})^\varepsilon)^{-m+1}]G(\mu_0, \mu_1, \mu_1) \\
&\preceq (b.1_{\mathbb{A}})((b.1_{\mathbb{A}})^\varepsilon)^{-n}[1_{\mathbb{A}} + (((b.1_{\mathbb{A}})^{\varepsilon-1})^{-1})^1 + \dots + (((b.1_{\mathbb{A}})^{\varepsilon-1})^{-1})^{m-n-1}]G(\mu_0, \mu_1, \mu_1) \\
&= (b.1_{\mathbb{A}})((b.1_{\mathbb{A}})^\varepsilon)^{-n}G(\mu_0, \mu_1, \mu_1)\sum_{k=1}^{m-n}((s^{\varepsilon-1})^{-1})^{k-1}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\|G(\mu_n, \mu_m, \mu_m)\| &\leq \|b\|\|(b^\varepsilon)^{-n}\|\|G(\mu_0, \mu_1, \mu_1)\|\sum_{k=1}^{m-n}\|((b^{\varepsilon-1})^{-1})\|^{k-1} \\
&\leq \|b\|\left(\frac{1}{\|b^\varepsilon\|}\right)^n\|G(\mu_0, \mu_1, \mu_1)\|\frac{\|b^{\varepsilon-1}\|}{\|b^{\varepsilon-1}\| - 1}.
\end{aligned}$$

If we take $n \to \infty$, then

$$\|b\| \left(\frac{1}{\|b^\varepsilon\|}\right)^n \|G(\mu_0, \mu_1, \mu_1)\| \frac{\|b^{\varepsilon-1}\|}{\|b^{\varepsilon-1}\| - 1} \to 0,$$

because $\varepsilon \in (1, +\infty)$ and $\|b\| > 1$. We deduce that

$$\lim_{n,m\to\infty} \|G(\mu_n, \mu_m, \mu_m)\| = 0. \tag{4}$$

From Proposition 3, we have

$$G(\mu_n, \mu_m, \mu_l) \preceq G(\mu_n, \mu_m, \mu_m) + G(\mu_m, \mu_m, \mu_l),$$

for $n, m, l \in \mathbb{N}$. Consequently,

$$\|G(\mu_n, \mu_m, \mu_l)\| \leq \|G(\mu_n, \mu_m, \mu_m)\| + \|G(\mu_m, \mu_m, \mu_l)\|.$$

By (4), we conclude that $\|G(\mu_n, \mu_m, \mu_l)\| \to 0$ as $n, m, l \to \infty$. Thus, $\{\mu_n\}$ is G_b-Cauchy with respect to $\mathbb{A}$. The completeness of $(E, \mathbb{A}, G)$ implies that there is some $u \in E$ so that $\{\mu_n\}$ is G_b-convergent to u with respect to $\mathbb{A}$.

We claim that $Tu = u$. Assume on the contrary $u \neq Tu$. By (1), we have

$$\sigma((b^\varepsilon.1_{\mathbb{A}})G(\mu_{n+1}, Tu, Tu)) \preceq F_*\Big(\sigma(G(\mu_n, u, u)), \vartheta(G(\mu_n, u, u))\Big), \quad \text{for all } n \geq 0.$$

Therefore,

$$G(\mu_{n+1}, Tu, Tu) \preceq (b^\varepsilon.1_{\mathbb{A}})^{-1} G(\mu_n, u, u), \quad \text{for all } n \geq 2,$$

and so

$$\|G(\mu_{n+1}, Tu, Tu)\| \leq \frac{1}{\|b^\varepsilon\|} \|G(\mu_n, u, u)\|.$$

Taking $n \to \infty$, we obtain $\lim_{n\to\infty} \|G(\mu_{n+1}, Tu, Tu)\| = 0$. Thus, $\{\mu_n\}$ G_b-converges to Tu. By uniqueness of limit $u = Tu$. Let $\zeta \neq u$ be another fixed point of T. From (1),

$$\sigma((b^\varepsilon.1_{\mathbb{A}})G(u, \zeta, \zeta)) = \sigma((b^\varepsilon.1_{\mathbb{A}})G(Tu, T\zeta, T\zeta)) \preceq F_*\Big(\sigma(G(u, \zeta, \zeta)), \vartheta(G(u, \zeta, \zeta))\Big),$$

so

$$G(u, \zeta, \zeta) \preceq (b^\varepsilon.1_{\mathbb{A}})^{-1} G(u, \zeta, \zeta).$$

Thus,

$$\|G(u, \zeta, \zeta)\| \leq \frac{1}{\|b^\varepsilon\|} \|G(u, \zeta, \zeta)\|.$$

We conclude that $\|G(u, \zeta, \zeta)\| \leq 0$ because $\frac{1}{\|b^\varepsilon\|} \in [0, \frac{1}{\|b\|}) \subset [0, 1)$. Therefore, u is the unique fixed point of T. $\square$

Taking $F_*(s, t) = k^* s k$ (where $k \in \mathbb{A}$ with $\|k\| < 1$ and $s \in \mathbb{A}_+$) in Theorem 3, we have the following.

Corollary 1. *Let $(E, \mathbb{A}, G)$ be a complete C^*-algebra-valued G_b-metric space with $s = (b.1_{\mathbb{A}}) \succ 1_{\mathbb{A}}$. Let $T : E \to E$ be so that*

$$\sigma((b^\varepsilon.1_{\mathbb{A}})G(T\mu, T\eta, T\xi)) \preceq k^* \sigma(G(\mu, \eta, \xi))k, \tag{5}$$

for all $\mu, \eta, \xi \in E$, where $k \in \mathbb{A}$ with $\|k\| < 1$, $\sigma \in \Sigma, \varepsilon \in (1, +\infty)$. Then T admits a unique fixed point.

Example 4. *Let $E = \mathbb{R}$. Consider $\mathbb{A} = M_2(\mathbb{R})$ as defined in Example 1. Let $G : E \times E \times E \to M_2(\mathbb{R})$ be defined as*

$$G(\mu,\eta,\xi) = diag\Big(\frac{1}{9}(|\mu-\eta|+|\eta-\xi|+|\xi-\mu|)^2, \frac{1}{9}(|\mu-\eta|+|\eta-\xi|+|\xi-\mu|)^2\Big)$$

$$= \begin{pmatrix} \frac{1}{9}(|\mu-\eta|+|\eta-\xi|+|\xi-\mu|)^2 & 0 \\ 0 & \frac{1}{9}(|\mu-\eta|+|\eta-\xi|+|\xi-\mu|)^2 \end{pmatrix}$$

for all $\mu,\eta,\xi \in E$. Then $(E, M_2(\mathbb{R}), G)$ is a complete C^*-algebra-valued G_b-metric space with coefficient

$$s = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = (2.1_{\mathbb{A}}).$$

Given $T : E \to E$ as $T\mu = \frac{\mu}{3^\varepsilon}$ for all $\mu \in E$, where $\varepsilon \in (1,+\infty)$. Take $\sigma : \mathbb{A}_+ \to \mathbb{A}_+$ as $\sigma(t) = t$. For all $\mu,\eta,\xi \in E$,

$$\sigma((2^\varepsilon.1_{\mathbb{A}})G(T\mu,T\eta,T\xi)) = \sigma\left((2^\varepsilon.1_{\mathbb{A}})G\left(\frac{\mu}{3^\varepsilon},\frac{\eta}{3^\varepsilon},\frac{\xi}{3^\varepsilon}\right)\right) = \sigma\left(\left(\frac{2^\varepsilon}{9^\varepsilon}.1_{\mathbb{A}}\right)G(\mu,\eta,\xi)\right)$$

$$= \sigma\left(\begin{pmatrix} \frac{2^\varepsilon}{9^\varepsilon} & 0 \\ 0 & \frac{2^\varepsilon}{9^\varepsilon} \end{pmatrix}\begin{pmatrix} \frac{1}{9}(|\mu-\eta|+|\eta-\xi|+|\xi-\mu|)^2 & 0 \\ 0 & \frac{1}{9}(|\mu-\eta|+|\eta-\xi|+|\xi-\mu|)^2 \end{pmatrix}\right)$$

$$= \begin{pmatrix} \frac{2^\varepsilon}{9^\varepsilon} & 0 \\ 0 & \frac{2^\varepsilon}{9^\varepsilon} \end{pmatrix}\begin{pmatrix} \frac{1}{9}(|\mu-\eta|+|\eta-\xi|+|\xi-\mu|)^2 & 0 \\ 0 & \frac{1}{9}(|\mu-\eta|+|\eta-\xi|+|\xi-\mu|)^2 \end{pmatrix}$$

$$\preceq k^*\sigma(G(\mu,\eta,\xi))k,$$

where $k = \begin{pmatrix} \sqrt{\frac{2^\varepsilon}{9^\varepsilon}} & 0 \\ 0 & \sqrt{\frac{2^\varepsilon}{9^\varepsilon}} \end{pmatrix}$, $\|k\| < 1$, $\varepsilon \in (1,+\infty)$. The inequality (5) *holds. From Corollary 1, $\mu = 0$ is the unique fixed point of T in E.*

A related Kannan type fixed point theorem is stated as follows.

Theorem 4. *Let $(E,\mathbb{A},G)$ be a complete C^*-algebra-valued G_b-metric space. Let $T : E \to E$ verifies for all $\mu,\eta \in E$,*

$$\sigma(G(T\mu,T\eta,T\eta)) \preceq F_*\Big(\sigma(m(\mu,\eta)),\vartheta(m(\mu,\eta))\Big), \tag{6}$$

where $F_* \in \mathcal{C}_*, \sigma,\vartheta \in \Sigma$, and

$$m(\mu,\eta) = b(G(\mu,T\mu,T\mu) + G(\eta,T\eta,T\eta)),$$

where $b \in \mathbb{A}'_+$ and $\|b\| < \frac{1}{2}$. Then T possesses a unique fixed point.

Proof. Assume that $b \neq 0_{\mathbb{A}}$. Then $b \in \mathbb{A}'_+$, and so $b(G(\mu, T\mu, T\mu) + G(\eta, T\eta, T\eta))$ is also a positive element. Let μ_0 be in E. Take $\mu_{n+1} = T\mu_n = T^{n+1}\mu_0$ for all $n \geq 0$. We claim that $\{\mu_n\}$ is a G_b-Cauchy sequence with respect to $\mathbb{A}$. In case of $\mu_n = \mu_{n+1}$ for some n, μ_n is a fixed point of T. Therefore, assume that $\mu_n \neq \mu_{n+1}$ for all $n \geq 0$. Choose $G(\mu_n, \mu_{n+1}, \mu_{n+1}) = G_n$. It follows from (6) that

$$\begin{aligned}
\sigma(G(\mu_n, \mu_{n+1}, \mu_{n+1})) &= \sigma(G(T\mu_{n-1}, T\mu_n, T\mu_n)) \\
&\preceq F_*\Big(\sigma(b(G(\mu_{n-1}, T\mu_{n-1}, T\mu_{n-1}) + G(\mu_n, T\mu_n, T\mu_n))), \\
&\vartheta(b(G(\mu_{n-1}, T\mu_{n-1}, T\mu_{n-1}) + G(\mu_n, T\mu_n, T\mu_n)))\Big) \\
&\preceq F_*\Big(\sigma(b(G(\mu_{n-1}, \mu_n, \mu_n) + G(\mu_n, \mu_{n+1}, \mu_{n+1}))), \\
&\vartheta(b(G(\mu_{n-1}, \mu_n, \mu_n) + G(\mu_n, \mu_{n+1}, \mu_{n+1})))\Big) \\
&\preceq \sigma(b(G(\mu_{n-1}, \mu_n, \mu_n) + G(\mu_n, \mu_{n+1}, \mu_{n+1}))).
\end{aligned}$$

Hence,

$$G(\mu_n, \mu_{n+1}, \mu_{n+1}) \preceq b(G(\mu_{n-1}, \mu_n, \mu_n) + G(\mu_n, \mu_{n+1}, \mu_{n+1})),$$

and thus

$$G(\mu_n, \mu_{n+1}, \mu_{n+1}) \preceq (1_{\mathbb{A}} - b)^{-1} bG(\mu_{n-1}, \mu_n, \mu_n) = tG(\mu_{n-1}, \mu_n, \mu_n),$$

where $t = (1_{\mathbb{A}} - b)^{-1}b$. Inductively, we conclude that

$$\begin{aligned}
G(\mu_n, \mu_{n+1}, \mu_{n+1}) &\preceq tG(\mu_{n-1}, \mu_n, \mu_n) \preceq t^2 G(\mu_{n-2}, \mu_{n-1}, \mu_{n-1}) \preceq \ldots \preceq t^n G(\mu_0, \mu_1, \mu_1) \\
&= t^n G_0.
\end{aligned}$$

Since $|b| < \frac{1}{2}$, we have $\|t\| < 1$. Hence

$$\lim_{n\to\infty} \|G(\mu_n, \mu_{n+1}, \mu_{n+1})\| = 0.$$

and so

$$\lim_{n\to\infty} G(\mu_n, \mu_{n+1}, \mu_{n+1}) = 0_{\mathbb{A}}.$$

Now,

$$\begin{aligned}
\sigma(G(\mu_n, \mu_m, \mu_m)) &= \sigma(G(T\mu_{n-1}, T\mu_{m-1}, T\mu_{m-1})) \\
&\preceq F_*\left(\sigma\left(\frac{G(\mu_{n-1}, T\mu_{n-1}, T\mu_{n-1}) + G(\mu_{m-1}, T\mu_{m-1}, T\mu_{m-1})}{2}\right),\right. \\
&\quad \left.\vartheta\left(\frac{G(\mu_{n-1}, T\mu_{n-1}, T\mu_{n-1}) + G(\mu_{m-1}, T\mu_{m-1}, T\mu_{m-1})}{2}\right)\right) \\
&\preceq F_*\left(\sigma\left(\frac{G(\mu_{n-1}, \mu_n, \mu_n) + G(\mu_{m-1}, \mu_m, \mu_m)}{2}\right),\right. \\
&\quad \left.\vartheta\left(\frac{G(\mu_{n-1}, \mu_n, \mu_n) + G(\mu_{m-1}, \mu_m, \mu_m)}{2}\right)\right) \\
&\preceq \sigma\left(\frac{G(\mu_{n-1}, \mu_n, \mu_n) + G(\mu_{m-1}, \mu_m, \mu_m)}{2}\right) \to \sigma(0_{\mathbb{A}}).
\end{aligned}$$

This shows that $\{\mu_n\}$ is G_b-Cauchy with respect to $\mathbb{A}$. Since E is complete, there is $u \in E$ so that $\mu_n \to u$. We have

$$\begin{aligned}\sigma(G(Tu,\mu_{n+1},\mu_{n+1})) &= \sigma(G(T\mu_{n-1},T\mu_n,T\mu_n)) \\ &\preceq F_*\left(\sigma\left(\frac{G(u,Tu,Tu)+G(\mu_n,T\mu_n,T\mu_n)}{2}\right),\right. \\ &\quad \left.\vartheta\left(\frac{G(u,Tu,Tu)+G(\mu_n,T\mu_n,T\mu_n)}{2}\right)\right) \\ &\preceq F_*\left(\sigma\left(\frac{G(u,Tu,Tu)+G(\mu_n,\mu_{n+1},\mu_{n+1})}{2}\right),\right. \\ &\quad \left.\vartheta\left(\frac{G(u,Tu,Tu)+G(\mu_n,\mu_{n+1},\mu_{n+1})}{2}\right)\right).\end{aligned}$$

Letting $n \to \infty$, we get

$$\sigma(G(Tu,u,u)) \preceq F_*\Big(\sigma(G(Tu,u,u)),\vartheta(G(Tu,u,u))\Big).$$

That is, $\sigma(G(Tu,u,u))$=$F_*\Big(\sigma(G(Tu,u,u)),\vartheta(G(Tu,u,u))\Big)$. So,

$$\sigma(G(Tu,u,u)) = 0_{\mathbb{A}} \quad \text{or} \quad \vartheta(G(Tu,u,u)) = 0_{\mathbb{A}}.$$

That is, $G(Tu,u,u) = 0_{\mathbb{A}}$, i.e., $u = Tu$.

Let v be in E so that $v = Tv$. We have

$$\begin{aligned}\sigma(G(v,u,u)) &= \sigma(G(Tv,Tu,Tu)) \\ &\preceq F_*\left(\sigma\left(\frac{G(v,Tv,Tv)+G(u,Tu,Tu)}{2}\right),\vartheta\left(\frac{G(v,Tv,Tv)+G(u,Tu,Tu)}{2}\right)\right) \\ &= F_*\left(\sigma\left(\frac{G(u,u,u)+G(v,v,v)}{2}\right),\vartheta\left(\frac{G(u,u,u)+G(v,v,v)}{2}\right)\right) \\ &= F_*\Big(\sigma(0_{\mathbb{A}}),\vartheta(0_{\mathbb{A}})\Big) \preceq \sigma(0_{\mathbb{A}}) = 0_{\mathbb{A}},\end{aligned}$$

which implies that $u = v$. □

If we consider $F_*(s,t) = s - t$ (for $s,t \in \mathbb{A}_+$) in Theorem 4, we get the following.

Corollary 2. *Let (E,G) be a complete C^*-algebra-valued G_b-metric space. Let $T : E \to E$ be so that*

$$\sigma(G(T\mu,T\eta,T\eta)) \preceq \sigma(m(\mu,\eta)) - \vartheta(m(\mu,\eta)),$$

for all $\mu,\eta \in E$, where $\sigma,\vartheta \in \Sigma$ and

$$m(\mu,\eta) = \frac{G(\mu,T\mu,T\mu)+G(\eta,T\eta,T\eta)}{2}.$$

Then T admits a unique fixed point.

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Моейні Б., Ішік Г., Айді Г. *Стосовно результатів про нерухому точку для функцій класу C_* на C^*-алгеброзначних G_b-метричних просторах* // Карпатські матем. публ. — 2020. — Т.12, №1. — С. 94–106.

Запропоновано концепцію C^*-алгеброзначних G_b-метричних просторів. Досліджено деякі основні властивості таких просторів і доведено деякі теореми про нерухому точку типу Банаха і Каннана для функцій класу C_*. Також наведено деякі нетривіальні приклади, щоб показати ефективність і застосовність отриманих результатів.

Ключові слова і фрази: нерухома точка, функція класу C_*, C^*-алгеброзначний G_b-метричний простір.

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MYTROFANOV M.A., RAVSKY A.V.

A NOTE ON APPROXIMATION OF CONTINUOUS FUNCTIONS ON NORMED SPACES

Let X be a real separable normed space X admitting a separating polynomial. We prove that each continuous function from a subset A of X to a real Banach space can be uniformly approximated by restrictions to A of functions, which are analytic on open subsets of X. Also we prove that each continuous function to a complex Banach space from a complex separable normed space, admitting a separating $*$-polynomial, can be uniformly approximated by $*$-analytic functions.

Key words and phrases: normed space, continuous function, analytic function, $*$-analytic function, uniform approximation, separating polynomial.

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The first known result on uniform approximation of continuous functions was obtained by Weierstrass in 1885. Namely, he showed that any continuous real-valued function on a compact subset K of a finitely dimensional real Euclidean space X can be uniformly approximated by restrictions on K of polynomials on X. For a compact subset K of a finitely dimensional complex Euclidean space X holds a counterpart of Stone-Weierstrass' theorem, according to which any continuous complex-valued function on K can be approximated by elements of any algebra, containing restrictions on K of polynomials on X and their conjugated functions. A general direction of investigations is to try to extend these results to topological linear spaces. Most of the obtained results concern separable Banach spaces, although in the paper [4] the authors obtained partial positive results for separable Fréchet spaces. A negative result belongs to Nemirovskii and Semenov, who in [7] built a continuous real-valued function on the unit ball K of the real space ℓ_2, which cannot be uniformly approximated by restrictions onto K of polynomials on ℓ_2. This result showed that in order to uniformly approximate continuous functions on Banach spaces we need a bigger class of functions than polynomials. The following fundamental result was obtained by Kurzweil [3].

Theorem 1. *Let X be any separable real Banach space that admits a separating polynomial, G be any open subset of X, and F be any continuous map from G to any real Banach space Y. Then for any $\varepsilon > 0$ there exists an analytic map H from G to Y such that $\|F(x) - H(x)\| < \varepsilon$ for all $x \in G$.*

Separating polynomials were introduced in [3] and are considered in reviews [2] and [6]. In order to define them and to obtain a counterpart of Kurzweil's Theorem for a complex Banach space X, in paper [5] were introduced notions, which we adapt below for complex normed spaces X and Y.

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A map B_{km} from X^{k+m} to Y is a map of type (k,m) if $B_{km}(x_1,\dots,x_k,x_{k+1},\dots,x_{k+m})$ is a nonzero map, which is k-linear with respect to x_i, $1 \le i \le k$, and m-antilinear with respect to x_{k+j}, $1 \le j \le m$.

Definition 1. *A map $B_n : X^n \to Y$ is $*$-n-linear if*

$$B_n(x_1,\dots,x_k,x_{k+1},\dots,x_{k+m}) = \sum_{k+m=n} c_{km} B_{km}(x_1,\dots,x_k,x_{k+1},\dots,x_{k+m}),$$

where for each k and m such that $k+m=n$, B_{km} is a map of type (k,m) and c_{km} is either 0 or 1, and at least one of c_{km} is non-zero.

Definition 2. *A map $F_n : X \to Y$ is called an n-homogeneous $*$-polynomial if there exists a $*$-n-linear map $B_n : X^n \to Y$ such that $F_n(x) = B_n(x,\dots,x)$ for all $x \in X$. Remark that F_0 is a constant map.*

Definition 3. *A map $F : X \to Y$ is a $*$-polynomial of degree j, if*

$$F = \sum_{n=0}^{j} F_n,$$

where F_n is an n-homogeneous continuous $*$-polynomial for each n and $F_j \neq 0$.

Definition 4. *A map $H : X \to Y$ is $*$-analytic if every point $x \in X$ has a neighborhood V such that*

$$H(x) = \sum_{n=0}^{\infty} F_n(x),$$

where for each n we have that F_n is an n-homogeneous continuous $*$-polynomial and the series $\sum\limits_{n=0}^{\infty} F_n(x)$ converges in V uniformly with respect to the norm of the space Y.

Definition 5. *Let X be a complex (resp. real) normed space. A $*$-polynomial (resp. polynomial) $P : X \to \mathbb{C}$ (resp. to $\mathbb{R}$) is called a separating $*$-polynomial (resp. polynomial) if $P(0) = 0$ and $\inf\limits_{\|x\|=1} P(x) > 0$.*

Denote by $\widetilde{\mathcal{H}}(X,Y)$ the normed space of $*$-analytic functions from X to Y.

Theorem 2 ([5])**.** *Let X be any separable complex Banach space that admits a separating $*$-polynomial, Y be any complex Banach space, and $F : X \to Y$ be any continuous map. Then for any $\varepsilon > 0$ there exists a map $H \in \widetilde{\mathcal{H}}(X,Y)$ such that $\|F(x) - H(x)\| < \varepsilon$ for all $x \in X$.*

The aim of the present paper is to generalize Theorems 1 and 2 to normed spaces. To this end we need the following technical result.

Lemma 1. *If a real normed space X admits a separating polynomial q then its completion $\widehat{X}$ admits a separating polynomial too.*

Proof. Let $q = \sum\limits_{i \in I} q_i$ be a sum of homogeneous polynomials q_i on the space X. For each $i \in I$ there exists a polylinear form $h_i : X^{n_i} \to \mathbb{R}$ such that $q_i(x) = h_i(x,\dots,x)$ for each $x \in X$. Since h_i is a Lipschitz function on X^{n_i}, by [1, Theorem 4.3.17], it admits a continuous extension

$\widehat{h}_i$ on the space $\widehat{X}^{n_i}$, which is polylinear by the polylinearity of h_i. The map $\widehat{q}_i : \widehat{X} \to \mathbb{R}$ defined as $\widehat{q}_i(x) = \widehat{h}_i(x, \dots, x)$ for each $x \in \widehat{X}$ is an extension of the map q_i. Then the map $\widehat{q} = \sum_{i \in I} \widehat{q}_i$ is a continuous polynomial extension of the map q onto the space X. It is easy to show that the unit sphere S of the space X is dense in the unit sphere $\widehat{S}$ of the space $\widehat{X}$. Therefore $\inf_{x \in \widehat{S}} \widehat{q}(x) = \inf_{x \in S} q(x) > 0$, so $\widehat{q}$ is a separating polynomial for the space $\widehat{X}$. □

Theorem 3. *Let X be a separable real normed space that admits a separating polynomial, Y be a real Banach space, $A \subset X$, $f : A \to Y$ be a continuous function, and $\varepsilon > 0$. Then there are an open set $A_\varepsilon \supset A$ of X and an analytic function $f_\varepsilon : A_\varepsilon \to Y$ such that $\|f(x) - f_\varepsilon(x)\| < \varepsilon$ for all $x \in A$.*

Proof. Let $\widehat{X}$ be a completion of X. We build a cover of the set A by open in $\widehat{X}$ sets as follows. For each point $x \in A$ pick its neighborhood $O(x)$ open in $\widehat{X}$ such that $\|f(x') - f(x)\| < \varepsilon/3$ for all $x' \in O(x) \cap A$.

Put $\widehat{A}_\varepsilon = \bigcup_{x \in A} O(x)$. The topological space $\widehat{A}_\varepsilon$ is metrizable, and therefore paracompact, [1, 5.1.3]. Therefore, by [1, 5.1.9] there is a locally finite partition $\{\varphi_s : s \in S\}$ of the unity, subordinated to the cover $\{O(x) : x \in A\}$.

Now we construct an auxiliary function $f'_\varepsilon : \widehat{A}_\varepsilon \to Y$. First, for each index $s \in S$ we define a real number a_s as follows. If $\operatorname{supp} \varphi_s \cap A \neq \varnothing$, then we pick an arbitrary point $x_s \in \operatorname{supp} \varphi_s \cap A$, and we put $a_s = f(x_s)$. Otherwise, we put $a_s = 0$. Finally, put $f'_\varepsilon = \sum_{s \in S} a_s \varphi_s$.

Let $x \in A$. Put $S_x = \{s \in S : x \in \operatorname{supp} \varphi_s\}$. Then $\sum_{s \in S_x} \varphi_s(x) = 1$. Let $s \in S_x$ be any index. Thus there is an element $x_0 \in A$ such that $x \in \operatorname{supp} \varphi_s \subset O(x_0)$. Hence $x_s \in O(x_0)$ and

$$\|f(x) - a_s\| = \|f(x) - f(x_s)\| \leqslant \|f(x) - f(x_0)\| + \|f(x_0) - f(x_s)\| < 2\varepsilon/3.$$

Then

$$\begin{aligned}
\|f(x) - f'_\varepsilon(x)\| &= \left\| f(x) - \sum_{s \in S} a_s \varphi_s(x) \right\| = \left\| \sum_{s \in S} f(x)\varphi_s(x) - \sum_{s \in S} a_s \varphi_s(x) \right\| \\
&= \left\| \sum_{s \in S_x} f(x)\varphi_s(x) - \sum_{s \in S_x} a_s \varphi_s(x) \right\| \leqslant \sum_{s \in S_x} \|f(x)\varphi_s(x) - a_s \varphi_s(x)\| \\
&= \sum_{s \in S_x} \|f(x) - a_s\| \, \varphi_s(x) < \sum_{s \in S_x} (2\varepsilon/3)\varphi_s(x) = 2\varepsilon/3.
\end{aligned}$$

The function f'_ε is continuous on $\widehat{A}_\varepsilon$ as a sum of a family of continuous functions with a locally finite family of supports.

By Lemma 1, the space $\widehat{X}$ admits a separating polynomial. Therefore the space X satisfies the conditions of Theorem 1, so there exists a function $\widehat{f}_\varepsilon$ analytic on $\widehat{A}_\varepsilon$ such that $\|\widehat{f}_\varepsilon(x) - f'_\varepsilon(x)\| < \varepsilon/3$ for all $x \in \widehat{A}_\varepsilon$. Then for all $x \in A$ we have

$$\|f(x) - \widehat{f}_\varepsilon(x)\| \leqslant \|f(x) - f'_\varepsilon(x)\| + \|f'_\varepsilon(x) - \widehat{f}_\varepsilon(x)\| < \varepsilon.$$

It remains to put $A_\varepsilon = \widehat{A}_\varepsilon \cap X$ and let f_ε be the restriction of the map $\widehat{f}_\varepsilon$ to the set A_ε. □

For a complex normed space X we denote by $\widetilde{X}$ itself, considered as a real normed space, and by $\mathcal{H}(\widetilde{X}, Y)$ the real normed space of analytic functions from $\widetilde{X}$ to a Banach space Y.

Theorem 4. *Let X be any separable complex normed space that admits a separating $*$-polynomial, Y be any complex Banach space, and $F : X \to Y$ be any continuous map. Then for any $\varepsilon > 0$ there exists a map $H \in \widetilde{\mathcal{H}}(X, Y)$ such that $\|F(x) - H(x)\| < \varepsilon$ for each $x \in X$.*

Proof. The proof is almost identical to the proof of Theorem 4 from [5] with the following modifications. Instead of the application of Kurzweil's Theorem we apply Theorem 3. Instead of [5, Lemma 2] we use the fact (proof of which is similar to that of [5, Lemma 2]) that the identity map from a complex normed space $\widetilde{\mathcal{H}}(X, Y)$ to the real normed space $\mathcal{H}(\widetilde{X}, Y)$ is an isomorphism of real normed spaces. □

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Нехай X є дійсним сепарабельним нормованим простором, що допускає відокремлювальний поліном. Показано, що непервні функції з підмножини A в X в дійсний банахів простір можуть бути рівномірно наближені аналітичними на відкритих підмножинах X. Також показано, що неперервні функції у комплексний банахів простір з комплексного сепарабельного нормованого простору, що допускає відокремлювальний $*$-поліном, можуть бути рівномірно наближені $*$-аналітичними функціями.

Ключові слова і фрази: нормований простір, неперервна функція, аналітична функція, $*$-аналітична функція, рівномірна апроксимація, відокремлювальний поліном.

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LAL S., SHARMA V.K.

ON THE ESTIMATION OF FUNCTIONS BELONGING TO LIPSCHITZ CLASS BY BLOCK PULSE FUNCTIONS AND HYBRID LEGENDRE POLYNOMIALS

In this paper, block pulse functions and hybrid Legendre polynomials are introduced. The estimators of a function f having first and second derivative belonging to $Lip_\alpha[a, b]$ class, $0 < \alpha \leq 1$, and a, b are finite real numbers, by block pulse functions and hybrid Legendre polynomials have been calculated. These calculated estimators are new, sharp and best possible in wavelet analysis. An example has been given to explain the validity of approximation of functions by using the hybrid Legendre polynomials approximation method. A real-world problem of radioactive decay is solved using this hybrid Legendre polynomials approximation method. Moreover, the Hermite differential equation of order zero is solved by using hybrid Legendre polynomials approximation method to explain the importance and the application of the technique of this method.

Key words and phrases: block pulse function, Legendre polynomial, hybrid Legendre polynomial.

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INTRODUCTION

In recent years, researchers like Marzban and Razzaghi [8, 10], Hsiao [4] defined and then used hybrid functions (HFs) for the numerical solutions of differential equations and integral equations. Working in the same direction, Marzban et al. [11] derived an operational matrix for a detailed analysis of HFs. In the continuation of their efforts, Merzban [9] studied the optimal control of linear delay systems applying HFs.

Objectives of this research paper are:

(i) to introduce block pulse functions and hybrid Legendre polynomials;

(ii) to estimate the error bounds of the functions of a certain class by hybrid functions;

(iii) to estimate the approximations of a function $f \in Lip_\alpha[a, b]$ by the partial sums of the block function series and hybrid Legendre series.

This research paper is organized as follows. In Section 1, block pulse functions and their some properties, block pulse functions expansion, hybrid Legendre polynomials, hybrid Legendre polynomials expansion, and $Lip_\alpha[a, b]$ class have been explained. In Section 2, the approximation of a function $f \in Lip_\alpha[0, 1]$ by block pulse functions expansion, Legendre polynomials expansion and hybrid Legendre polynomials expansion have been estimated and appropriate detailed proofs are provided. In Section 3, hybrid Legendre approximation is explained with the help of an example. Section 4 is introduced to explain the application of this expansion in solving the Hermite differential equation of order zero and in solving some real-world problems. Eventually, some conclusions are mentioned in Section 5.

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1 DEFINITIONS AND PRELIMINARIES

1.1 Block pulse functions and their expansion

Let n be an arbitrary fixed positive integer. Define functions $\beta_i, i = 1, 2, \ldots, n$, on the interval $[0,1]$ by (see [7])

$$\beta_i(t) = \begin{cases} 1, & \frac{i-1}{n} \leq t < \frac{i}{n}; \\ 0, & \text{otherwise.} \end{cases}$$

These functions are referred as block pulse functions (or BPFs).

Let $\langle \cdot, \cdot \rangle$ denotes the inner product over the field F ($\mathbb{R}$ or $\mathbb{C}$). Block pulse functions expansion of an $f \in L^2[0,1)$ is given by (see [3])

$$f(t) = \sum_{i=1}^{\infty} f_{\beta_i} \beta_i(t), \quad f_{\beta_i} := n\langle f, \beta_i \rangle, \tag{1}$$

where n is an arbitrary fixed positive integer associated with block pulse function β_i. Let S_n denotes the n^{th} partial sum of the series in (1) and it is given by

$$S_n(t) = \sum_{i=1}^{n} f_{\beta_i} \beta_i(t).$$

1.2 Properties of block pulse functions

An n-set of BPFs defined above satisfies the following properties.

1. *Disjointness*, i.e., $\beta_i(t)\beta_j(t) = \delta_{ij}\beta_i(t)$, where $1 \leq i, j \leq n$, δ_{ij} is the Kronecker delta.

2. *Orthogonality*, i.e.,

$$\langle \beta_i, \beta_j \rangle = \begin{cases} 0, & i \neq j; \\ \frac{1}{n}, & i = j, \end{cases} \qquad 1 \leq i, j \leq n.$$

3. *Completeness*, i.e., for every $f \in L^2[0,1)$ Parseval's identity

$$\int_0^1 f^2(t)dt = \sum_{i=1}^{\infty} |f_{\beta_i}|^2 ||\beta_i||^2$$

satisfied, where f_{β_i} is defined in (1).

1.3 Legendre and hybrid Legendre polynomials

Legendre differential equation is given by (see [1])

$$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0,$$

where n is a positive integer. Legendre polynomial $L_n(x)$ is the solution of above differential equation and it is written in the form (see [2])

$$L_n(x) = \sum_{r=0}^{[\frac{n}{2}]} (-1)^r \frac{(2n-2r)!}{2^n r!(n-r)!(n-2r)!} x^{n-2r},$$

where

$$\left[\frac{n}{2}\right] = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even;} \\ \frac{n-1}{2}, & \text{if } n \text{ is odd.} \end{cases}$$

Rodrigue' s formula for $L_n(x)$ is given by

$$L_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n, \qquad n = 0, 1, 2, \ldots.$$

Let n and m be the arbitrary fixed positive integers. Hybrid Legendre polynomials, denoted by h_{ij}, $i = 1, 2, \ldots, n, j = 0, 1, \ldots, m-1$, on the interval $[0, 1)$ are defined by

$$h_{ij}(t) = \begin{cases} L_j(2nt - 2i + 1), & \frac{i-1}{n} \le t < \frac{i}{n}; \\ 0, & \text{otherwise,} \end{cases}$$

where i and j are the orders of BPFs and Legendre polynomials respectively.

1.4 Hybrid Legendre polynomials expansion

If $f \in L^2[0, 1)$, then associated hybrid Legendre polynomial infinite series is (see [6])

$$f(t) = \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} c_{ij}h_{ij}(t), \qquad c_{ij} = \frac{\langle f, h_{ij}\rangle}{\langle h_{ij}, h_{ij}\rangle}. \tag{2}$$

The $(n, m)^{th}$ partial sums of the series (2) is given by

$$s_{n,m}(t) = \sum_{i=1}^{n}\sum_{j=0}^{m-1} c_{ij}h_{ij}(t).$$

1.5 $Lip_\alpha[a, b]$ class

A function f belongs to $Lip_\alpha[a, b]$ class for $0 < \alpha \le 1$ if

$$|f(x+t) - f(x)| = O(|t|^\alpha), \ \ 0 < \alpha \le 1.$$

If $0 < \alpha < \beta \le 1$, then $Lip_\beta[0, 1] \subsetneq Lip_\alpha[0, 1]$.

Example. *Let* $\alpha = \frac{1}{3}$, $\beta = \frac{1}{2}$ *and* $f(x) = x^{\frac{1}{3}}$, $g(x) = x^{\frac{1}{2}}$, $\forall x \in [0, 1]$, *then* $g \in Lip_\beta[0, 1] \Rightarrow g \in Lip_\alpha[0, 1]$. *Here,*

$$|g(x+t) - g(x)| = \left|(x+t)^{\frac{1}{2}} - x^{\frac{1}{2}}\right| \le |(x+t) - x|^{\frac{1}{2}} = t^{\frac{1}{2}}.$$

Hence, $|g(x+t) - g(x)| = O(t^{\frac{1}{2}})$ *and* $g \in Lip_{\frac{1}{2}}[0, 1]$. *Also,*

$$|g(x+t) - g(x)| = \left|(x+t)^{\frac{1}{2}} - x^{\frac{1}{2}}\right| \le |(x+t) - x|^{\frac{1}{2}} = t^{\frac{1}{2}}\frac{t^{\frac{1}{3}}}{t^{\frac{1}{3}}} = t^{\frac{1}{3}}t^{\frac{1}{6}} \le t^{\frac{1}{3}}, \quad \forall\, t \in [0, 1].$$

Hence, $|g(x+t) - g(x)| = O(t^{\frac{1}{3}})$ *and* $g \in Lip_{\frac{1}{3}}[0, 1]$. *Now,*

$$|f(x+t) - f(x)| = \left|(x+t)^{\frac{1}{3}} - x^{\frac{1}{3}}\right| \le |(x+t) - x|^{\frac{1}{3}} = t^{\frac{1}{3}}, \quad \forall\, t \in [0, 1].$$

Thus, $|f(x+t) - f(x)| = O(t^{\frac{1}{3}})$ *and* $f \in Lip_{\frac{1}{3}}[0, 1]$. *But*

$$|f(x+t) - f(x)| = \left|(x+t)^{\frac{1}{3}} - x^{\frac{1}{3}}\right| \le |(x+t) - x|^{\frac{1}{3}} = t^{\frac{1}{3}}\frac{t^{\frac{1}{2}}}{t^{\frac{1}{2}}} = t^{\frac{1}{2}}\ t^{-\frac{1}{6}}.$$

Hence,

$$\lim_{t\to 0+}\left|\frac{f(x+t) - f(x)}{t^{\frac{1}{2}}}\right| \to +\infty.$$

This shows that $f \notin Lip_{\frac{1}{2}}[0, 1]$. *Therefore* $Lip_{\frac{1}{2}}[0, 1] \subsetneq Lip_{\frac{1}{3}}[0, 1]$.

2 MAIN RESULTS

Theorem 1. *Let f be a differentiable function on the interval $[0,1]$ such that its first derivative $f' \in Lip_\alpha[0,1]$ and the block pulse functions expansion of f be $f(t) = \sum_{i=1}^{\infty} f_{\beta_i}\beta_i(t)$, where $f_{\beta_i} = \frac{\langle f, \beta_i\rangle}{\langle \beta_i, \beta_i\rangle}$, and β_i is a block pulse function. Then the error of approximation of f by $(S_m f)(t) = \sum_{i=1}^{m} f_{\beta_i}\beta_i(t)$ is*

$$E^{(BP)}(f) = \min \|f - S_m f\|_2 = O\left[\frac{1}{m}\left(1 + \frac{1}{m^\alpha}\right)\right],$$

where $0 < \alpha \leq 1$ and m be an arbitrary fixed positive integer.

Proof. Since

$$e_i = f_{\beta_i}\beta_i(t) - f(t)\chi_{[\frac{i-1}{m},\frac{i}{m})}, \qquad \forall \left[\frac{i-1}{m}, \frac{i}{m}\right), \quad i = 1, 2, \ldots, m,$$

where m is an arbitrary fixed positive integer associated with the BPFs and $\chi_{[\frac{i-1}{m},\frac{i}{m})}$ is a characterstic function defined on the interval $\left[\frac{i-1}{m}, \frac{i}{m}\right)$. Then

$$e_i^2 = f_{\beta_i}^2\beta_i^2(t) + f^2(t)\chi_{[\frac{i-1}{m},\frac{i}{m})} - 2f_{\beta_i}f(t)\chi_{[\frac{i-1}{m},\frac{i}{m})}.$$

Now, by Taylor theorem (see [5])

$$\|e_i\|^2 = f_{\beta_i}^2 \int_{\frac{i-1}{m}}^{\frac{i}{m}} \beta_i^2(t)dt + \int_{\frac{i-1}{m}}^{\frac{i}{m}} f^2(t)dt - 2f_{\beta_i}\int_{\frac{i-1}{m}}^{\frac{i}{m}} \beta_i(t)f(t)dt = \frac{f_{\beta_i}^2}{m} + \int_{\frac{i-1}{m}}^{\frac{i}{m}} f^2(t)dt - 2f_{\beta_i}\int_{\frac{i-1}{m}}^{\frac{i}{m}} f(t)dt$$

$$= \frac{f_{\beta_i}^2}{m} + \int_0^{\frac{1}{m}} f^2\left(\frac{i-1}{m} + u\right)du - 2f_{\beta_i}\int_0^{\frac{1}{m}} f\left(\frac{i-1}{m} + u\right)du$$

$$= \frac{f_{\beta_i}^2}{m} + \int_0^{\frac{1}{m}} \left\{ f^2\left(\frac{i-1}{m}\right) + u^2\left(f'\left(\frac{i-1}{m} + \theta u\right)\right)^2 \right\} du$$

$$+ 2\int_0^{\frac{1}{m}} f\left(\frac{i-1}{m}\right) u f'\left(\frac{i-1}{m} + \theta u\right)du - 2f_{\beta_i}\int_0^{\frac{1}{m}} \left(f\left(\frac{i-1}{m}\right) + uf'\left(\frac{i-1}{m} + \theta u\right)\right)du$$

$$= \frac{f_{\beta_i}^2}{m} + \frac{f^2\left(\frac{i-1}{m}\right)}{m} + \int_0^{\frac{1}{m}} u^2\left(f'\left(\frac{i-1}{m} + \theta u\right)\right)^2 du$$

$$+ 2f\left(\frac{i-1}{m}\right)\int_0^{\frac{1}{m}} f'\left(\frac{i-1}{m} + \theta u\right)udu - 2f_{\beta_i}\frac{f\left(\frac{i-1}{m}\right)}{m} - 2f_{\beta_i}\int_0^{\frac{1}{m}} uf'\left(\frac{i-1}{m} + \theta u\right)du,$$

where $0 < \theta < 1$. Also,

$$f_{\beta_i} = m\langle f, \beta_i\rangle = m\int\limits_{\frac{i-1}{m}}^{\frac{i}{m}} f(t)dt = m\int\limits_{0}^{\frac{1}{m}} f\left(\frac{i-1}{m} + u\right) du$$

$$= m\int\limits_{0}^{\frac{1}{m}} \left(f\left(\frac{i-1}{m}\right) + uf'\left(\frac{i-1}{m} + \theta u\right)\right) du = f\left(\frac{i-1}{m}\right) + m\int\limits_{0}^{\frac{1}{m}} uf'\left(\frac{i-1}{m} + \theta u\right) du.$$

From the above formulas we get

$$\begin{aligned}
\|e_i\|_2^2 &= \int_0^{\frac{1}{m}} u^2 \left(f'\left(\frac{i-1}{m} + \theta u\right)\right)^2 du - m\Big(\int_0^{\frac{1}{m}} uf'\left(\frac{i-1}{m} + \theta u\right)\Big)^2 du \\
&= \int_0^{\frac{1}{m}} u^2\Big\{\left(f'\left(\frac{i-1}{m} + \theta u\right) - f'\left(\frac{i-1}{m}\right) + f'\left(\frac{i-1}{m}\right)\right)\Big\}^2 du \\
&\quad - m\Big(\int_0^{\frac{1}{m}} u\left\{\left(f'\left(\frac{i-1}{m} + \theta u\right) - f'\left(\frac{i-1}{m}\right) + f'\left(\frac{i-1}{m}\right)\right)\right\}\Big)^2 du \\
&\le \int_0^{\frac{1}{m}} u^2\left(A_1 u^\alpha + f'\left(\frac{i-1}{m}\right)\right)^2 du + m\Big(\int_0^{\frac{1}{m}} u\left\{A_1 u^\alpha + f'\left(\frac{i-1}{m}\right)\right\} du\Big)^2 \\
&= \frac{A_1^2}{m^{2\alpha+3}}\left(\frac{1}{2\alpha+3} + \frac{1}{(\alpha+2)^2}\right) + \frac{\left(f'\left(\frac{i-1}{m}\right)\right)^2}{12m^3} + \frac{2A_1 f'\left(\frac{i-1}{m}\right)}{m^{\alpha+3}}\left(\frac{1}{\alpha+3} + \frac{1}{2\alpha+4}\right),
\end{aligned}$$

where A_1 is a positive constant.

Hence,

$$\begin{aligned}
\|e\|_2^2 = \sum_{i=1}^{m}\|e_i\|_2^2 &= \frac{A_1^2}{m^{2\alpha+2}}\Big(\frac{1}{2\alpha+3} + \frac{1}{(\alpha+2)^2}\Big) + \frac{(f'(\frac{i-1}{m}))^2}{12m^2} + \frac{2A_1 f'(\frac{i-1}{m})}{m^{\alpha+2}}\left(\frac{1}{\alpha+3} + \frac{1}{2\alpha+4}\right) \\
&\le \frac{2A_1^2}{m^{2\alpha+2}} + \frac{(f'(\frac{i-1}{m}))^2}{12m^2} + \frac{4A_1 f'(\frac{i-1}{m})}{m^{\alpha+2}} \\
&= 2\Big(\frac{A_1^2}{m^{2\alpha+2}} + \frac{(f'(\frac{i-1}{m}))^2}{24m^2} + \frac{2A_1 f'(\frac{i-1}{m})}{m^{\alpha+2}}\Big) \le 2\Big(\frac{A_1}{m^{\alpha+1}} + \frac{f'(\frac{i-1}{m})}{m}\Big)^2.
\end{aligned}$$

Therefore,

$$\|e\|_2 = O\left[\frac{1}{m}\left(1 + \frac{1}{m^\alpha}\right)\right].$$

So, the proof of Theorem 1 is completely established. □

Theorem 2. *Let f be a differentiable function defined on the interval $[-1, 1]$ such that its second derivative $f'' \in Lip_\alpha[-1, 1]$ and Legendre expansion of function f be*

$$f(t) = \sum_{j=0}^{\infty} c_j P_j(t), \tag{3}$$

where P_j is a Legendre polynomial and $c_j = \dfrac{\langle f, P_j\rangle}{\langle P_j, P_j\rangle}$. Then the error of the approximation of f by $(U_m f)(t) = \sum\limits_{j=0}^{m-1} c_j P_j(t)$, $m = 1, 2, \ldots$, is

$$E^{(LP)}(f) = \min\|f - U_m f\|_2 \le M\left(1 + \frac{1}{\sqrt{2\alpha+1}}\right)\frac{1}{(2m-3)^{\frac{3}{2}}},$$

where $m \geq 2$ *is an integer,* M *is a positive constant and* $0 < \alpha \leq 1$.

Proof. Legendre expansion of a function $f(t)$ is given by (3). Let

$$U_m(t) = \sum_{j=0}^{m-1} c_j P_j(t)$$

denotes the m^{th} partial sum of (3). Then for arbitrary $m \geq 2$, we have

$$\begin{aligned} \|f - U_m\|_2^2 = \int_{-1}^{1} (f - U_m)^2 dt &= \int_{-1}^{1} \left(\sum_{j=0}^{m-1} c_j P_j(t) \right)^2 dt \\ &= \sum_{j=m}^{\infty} c_j^2 \int_{-1}^{1} P_j^2 dt = \sum_{j=m}^{\infty} c_j^2 \left(\frac{2}{2j+1} \right), \end{aligned}$$

and for arbitrary $j \geq m$, we obtain

$$\begin{aligned} c_j &= \frac{\int_{-1}^{1} f(t) P_j(t) dt}{\int_{-1}^{1} P_j^2 dt} = \frac{2j+1}{2} \int_{-1}^{1} f(t) P_j(t) dt = \frac{2j+1}{2} \int_{-1}^{1} f(t) \frac{P'_{j+1} - P'_{j-1}}{2j+1} dt \\ &= \frac{1}{2} \int_{-1}^{1} f(t) \left(P'_{j+1} - P'_{j-1} \right) dt = \frac{1}{2} \int_{-1}^{1} f'(t) \left(P_{j-1} - P_{j+1} \right) dt \\ &= \frac{1}{2} \int_{-1}^{1} f'(t) P_{j-1} dt - \int_{-1}^{1} f'(t) P_{j+1} dt = \frac{1}{2(2j-1)} \int_{-1}^{1} f'(t) \left(P'_j - P'_{j-2} \right) dt \\ &\qquad - \frac{1}{2(2j+3)} \int_{-1}^{1} f'(t) \left(P'_{j+2} - P'_j \right) dt \\ &= \frac{1}{2(2j-1)} \int_{-1}^{1} f''(t) \left(P_{j-2} - P_j \right) dt + \frac{1}{2(2j+3)} \int_{-1}^{1} f''(t) \left(P_{j+2} - P_j \right) dt = \frac{1}{2} \int_{-1}^{1} f''(t) B_j(t) dt, \end{aligned}$$

where $B_j(t) = \dfrac{P_{j-2} - P_j}{2j-1} + \dfrac{P_{j+2} - P_j}{2j+3}$. Hence,

$$\begin{aligned} |c_j| &\leq \frac{1}{2} \int_{-1}^{1} \left| f''(t) - f''(0) \right| \left| B_j(t) \right| dt + \frac{1}{2} \int_{-1}^{1} \left| f''(0) \right| \left| B_j(t) \right| dt \\ &\leq \frac{M_1}{2} \int_{-1}^{1} |t|^{\alpha} \left| B_j(t) \right| dt + \frac{|f''(0)|}{2} \int_{-1}^{1} \left| B_j(t) \right| dt, \end{aligned} \tag{4}$$

where M_1 is a positive constant. Next, applying the Cauchy-Schwarz inequality, we get

$$\begin{aligned}\left(\int_{-1}^{1}|t|^{\alpha}|B_j|dt\right)^2 &\leq \int_{-1}^{1}|t|^{2\alpha}dt\int_{-1}^{1}|B_j|^2dt = 2\int_{0}^{1}|t|^{2\alpha}dt\int_{-1}^{1}|B_j|^2dt\\ &= \frac{2}{2\alpha+1}\int_{-1}^{1}\left(\frac{P_{j+2}^2+P_j^2}{(2j+3)^2}+\frac{P_{j-2}^2+P_j^2}{(2j-1)^2}\right)dt\\ &= \frac{2}{(2\alpha+1)(2j+3)^2}\left(\frac{2}{2j+5}+\frac{2}{2j+1}\right)+\frac{2}{(2\alpha+1)(2j-1)^2}\left(\frac{2}{2j-3}+\frac{2}{2j+1}\right)\\ &\leq \frac{16}{(2\alpha+1)(2j-3)^3}.\end{aligned}$$

Hence,

$$\int_{-1}^{1}|t|^{\alpha}|B_j|dt \leq \frac{4}{\sqrt{2\alpha+1}(2j-3)^{\frac{3}{2}}}. \tag{5}$$

Also,

$$\left(\int_{-1}^{1}|B_j|dt\right)^2 \leq \left(\int_{-1}^{1}1^2dt\right)^2\left(\int_{-1}^{1}|B_j|^2dt\right) \leq \frac{16}{(2j-3)^3}.$$

Therefore,

$$\int_{-1}^{1}|B_j|dt \leq \frac{4}{(2j-3)^{\frac{3}{2}}}. \tag{6}$$

By (4)–(6) we have

$$|c_j| \leq \frac{1}{2}\frac{4M_1}{\sqrt{2\alpha+1}}\frac{1}{(2j-3)^{\frac{3}{2}}}+\frac{|f''(0)|}{2}\frac{4}{(2j-3)^{\frac{3}{2}}} \leq M\left(\frac{1}{\sqrt{2\alpha+1}}+1\right)\frac{1}{(2j-3)^{\frac{3}{2}}},$$

where $M = \max\{2M_1, 2|f''(0)|\}$ and

$$\begin{aligned}\|f-U_m\|_2^2 &\leq \sum_{j=m}^{\infty}M^2\left(1+\frac{1}{\sqrt{2\alpha+1}}\right)^2\frac{1}{(2j-3)^3}\frac{2}{(2j+1)}\\ &\leq 2M^2\left(1+\frac{1}{\sqrt{2\alpha+1}}\right)^2\sum_{j=m}^{\infty}\frac{1}{(2j-3)^4}\\ &\leq 2M^2\left(1+\frac{1}{\sqrt{2\alpha+1}}\right)^2\frac{1}{(2m-3)^3}, \qquad m \geq 2.\end{aligned}$$

Hence,

$$\|f-U_m\|_2 \leq \sqrt{2}M\left(1+\frac{1}{\sqrt{2\alpha+1}}\right)\frac{1}{(2m-3)^{\frac{3}{2}}}.$$

Thus Theorem 2 is completely proved. □

Theorem 3. *Let f be a differentiable function on $[0,1]$ such that $f'' \in Lip_\alpha[0,1]$ and hybrid Legendre polynomials expansion of f be*

$$f(t) = \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} c_{ij} h_{ij}(t), \tag{7}$$

where $c_{ij} = \dfrac{\langle f, h_{ij} \rangle}{\langle h_{ij}, h_{ij} \rangle}$ and h_{ij} is the hybrid Legendre polynomials, and

$$(S_{n,m}f)(t) = \sum_{i=1}^{n} \sum_{j=0}^{m-1} c_{ij} h_{ij}(t)$$

be the $(n,m)^{th}$ partial sum of the series (7). Then the error of approximation f by $S_{n,m}f$ is

$$E_{n,m}^{(HFs)} f = \min \|f - S_{n,m}f\|_2 = O\left[\left(\frac{1}{n^{\alpha+2}} + \frac{1}{2n^2}\right) \frac{1}{(2m-3)^{\frac{3}{2}}}\right],$$

where $m \geq 2$, n is a positive integer and $0 < \alpha \leq 1$.

Proof. We see that hybrid Legendre polynomials expansion of f is given by (7). Now, suppose n and m are the arbitrary fixed positive integers. Then for $i = 1, 2, \ldots, n$ and $j = 0, 1, \ldots, m-1$, we have

$$\begin{aligned}
c_{ij} &= \frac{\langle f, h_{ij} \rangle}{\langle h_{ij}, h_{ij} \rangle} = \frac{\int\limits_{\frac{i-1}{n}}^{\frac{i}{n}} f(t) h_{ij}(t) dt}{\int\limits_{\frac{i-1}{n}}^{\frac{i}{n}} h_{ij}^2(t) dt} = \frac{\int\limits_{\frac{i-1}{n}}^{\frac{i}{n}} f(t) P_j(2nt - 2i + 1) dt}{\int\limits_{\frac{i-1}{n}}^{\frac{i}{n}} P_j^2(2nt - 2i + 1) dt} \\
&= \frac{\int\limits_{-1}^{1} f\left(\frac{u+2i-1}{2n}\right) P_j(u) \frac{du}{2n}}{\int\limits_{-1}^{1} P_j^2(u) \frac{du}{2n}} = \frac{2j+1}{2} \int\limits_{-1}^{1} f\left(\frac{u+2i-1}{2n}\right) P_j(u) du \\
&= \frac{2j+1}{2} \int\limits_{-1}^{1} f\left(\frac{t+2i-1}{2n}\right) P_j(t) dt = \frac{2j+1}{2} \int\limits_{-1}^{1} f\left(\frac{t+2i-1}{2n}\right) \left(\frac{P'_{j+1} - P'_{j-1}}{2j+1}\right) dt \\
&= \frac{1}{2} \int\limits_{-1}^{1} f\left(\frac{t+2i-1}{2n}\right) \left(P'_{j+1} - P'_{j-1}\right) dt = \frac{-1}{4n} \int\limits_{-1}^{1} f'\left(\frac{t+2i-1}{2n}\right) \left(P_{j+1} - P_{j-1}\right) dt \\
&= \frac{-1}{4n} \int\limits_{-1}^{1} f'\left(\frac{t+2i-1}{2n}\right) P_{j+1} dt + \frac{1}{4n} \int\limits_{-1}^{1} f'\left(\frac{t+2i-1}{2n}\right) P_{j-1} dt = I_1 + I_2.
\end{aligned} \tag{8}$$

Now,

$$I_1 = \frac{-1}{4n}\int_{-1}^{1} f'\left(\frac{t+2i-1}{2n}\right)P_{j+1}dt = \frac{-1}{4n}\int_{-1}^{1} f'\left(\frac{t+2i-1}{2n}\right)\left(\frac{P'_{j+2}-P'_j}{2j+3}\right)dt$$
$$= \frac{1}{8n^2(2j+3)}\int_{-1}^{1} f''\left(\frac{t+2i-1}{2n}\right)\left(P_{j+2}-P_j\right)dt \tag{9}$$
$$= \frac{1}{8n^2}\int_{-1}^{1} f''\left(\frac{t+2i-1}{2n}\right)\left(\frac{P_{j+2}-P_j}{2j+3}\right)dt.$$

Next,

$$I_2 = \frac{1}{4n}\int_{-1}^{1} f'\left(\frac{t+2i-1}{2n}\right)P_{j-1}dt = \frac{1}{8n^2}\int_{-1}^{1} f''\left(\frac{t+2i-1}{2n}\right)\left(\frac{P_{j-2}-P_j}{2j-1}\right)dt. \tag{10}$$

By (8)–(10) we have

$$\begin{aligned} c_{ij} &= \frac{1}{8n^2}\int_{-1}^{1} f''\left(\frac{t+2i-1}{2n}\right)\left(\frac{P_{j+2}-P_j}{2j+3}\right)dt + \frac{1}{8n^2}\int_{-1}^{1} f''\left(\frac{t+2i-1}{2n}\right)\left(\frac{P_{j-2}-P_j}{2j-1}\right)dt \\ &= \frac{1}{8n^2}\int_{-1}^{1} f''\left(\frac{t+2i-1}{2n}\right)\left(\frac{P_{j+2}-P_j}{2j+3}+\frac{P_{j-2}-P_j}{2j-1}\right)dt = \frac{1}{8n^2}\int_{-1}^{1} f''\left(\frac{t+2i-1}{2n}\right)B_j dt, \end{aligned}$$

where $B_j = \dfrac{P_{j+2}-P_j}{2j+3} + \dfrac{P_{j-2}-P_j}{2j-1}$. Hence,

$$|c_{ij}| \le \frac{1}{8n^2}\int_{-1}^{1} \left|f''\left(\frac{t+2i-1}{2n}\right)\right| |B_j| dt$$
$$\le \frac{1}{8n^2}\int_{-1}^{1} \left|f''\left(\frac{t+2i-1}{2n}\right) - f''\left(\frac{2i-1}{2n}\right)\right| |B_j| dt + \frac{1}{8n^2}\int_{-1}^{1} \left|f''\left(\frac{2i-1}{2n}\right)\right| |B_j|\, dt \tag{11}$$
$$\le \frac{M_1}{2^{\alpha+3}n^{\alpha+2}}\int_{-1}^{1} |t|^{\alpha}|B_j|dt + \frac{1}{8n^2}\left|f''\left(\frac{2i-1}{2n}\right)\right|\int_{-1}^{1} |B_j|\, dt,$$

where M_1 is a positive constant.

Now, by (5), (6) and (11) we get

$$|c_{ij}| \le \frac{M_1}{2^{\alpha+3}n^{\alpha+2}}\frac{4}{\sqrt{2\alpha+1}(2j-3)^{\frac{3}{2}}} + \frac{1}{8n^2}\left|f''\left(\frac{2i-1}{2n}\right)\right| \frac{4}{(2j-3)^{\frac{3}{2}}}$$
$$\le \frac{M_1}{2^{\alpha+3}n^{\alpha+2}}\frac{4}{\sqrt{2\alpha+1}(2j-3)^{\frac{3}{2}}} + \frac{1}{8n^2}M_2\frac{4}{(2j-3)^{\frac{3}{2}}} \tag{12}$$
$$\le B\left(\frac{1}{\sqrt{2\alpha+1}\,2^{\alpha+1}\,n^{\alpha+2}} + \frac{1}{2n^2}\right)\frac{1}{(2j-3)^{\frac{3}{2}}} \le B\left(\frac{1}{n^{\alpha+2}} + \frac{1}{2n^2}\right)\frac{1}{(2j-3)^{\frac{3}{2}}},$$

where $B = \max\{M_1, M_2\}$. Since $f'' \in Lip_\alpha[0,1]$, $0 < \alpha \leq 1$, it is continuous on $[0,1]$. Therefore it is bounded on $[0,1]$. Thus, there exists a constant M_2 independent of t such that $|f''(t)| \leq M_2$ $\forall t \in [0,1]$. Also $n \in \mathbb{N}$ and $i = 1, 2, \ldots, n$, so $0 \leq \dfrac{i-1}{n} < 1$, i.e. $\dfrac{i-1}{n} \in [0,1]$ for each n and $i = 1, 2, \ldots, n$. Hence $\left| f''\left(\dfrac{i-1}{n}\right)\right| \leq M_2$.

Let $S_{n,m}f$ denotes the $(n,m)^{th}$ partial sum of the series (7) as given in theorem 3. Now,

$$f - S_{n,m}f = \left(\sum_{i=1}^{n} + \sum_{i=n+1}^{\infty}\right)\left(\sum_{j=0}^{m-1} + \sum_{j=m}^{\infty}\right) c_{ij}h_{ij} - \sum_{i=1}^{n}\sum_{j=0}^{m-1} c_{ij}h_{ij} = \sum_{i=1}^{n}\sum_{j=m}^{\infty} c_{ij}h_{ij}.$$

Then

$$\begin{aligned}\|f - S_{n,m}f\|_2^2 &= \int_0^1 \left(\sum_{i=1}^{n}\sum_{j=m}^{\infty} c_{ij}h_{ij}\right)^2 dt = \sum_{i=1}^{n}\sum_{j=m}^{\infty} c_{ij}^2 \int_0^1 h_{ij}^2 dt \\ &= \sum_{i=1}^{n}\sum_{j=m}^{\infty} c_{ij}^2 \int_{\frac{i-1}{m}}^{\frac{i}{m}} P_j^2(2nt - 2i + 1)dt = \sum_{i=1}^{n}\sum_{j=m}^{\infty} c_{ij}^2 \frac{1}{n\,(2j+1)}.\end{aligned} \tag{13}$$

By (12) and (13) we have

$$\begin{aligned}\|f - S_{n,m}f\|_2^2 &\leq \sum_{i=1}^{n}\sum_{j=m}^{\infty} B^2 \left(\frac{1}{n^{\alpha+2}} + \frac{1}{2n^2}\right)^2 \frac{1}{(2j-3)^3}\frac{1}{n(2j+1)} \\ &= B^2\left(\frac{1}{n^{\alpha+2}} + \frac{1}{2n^2}\right)^2 \sum_{j=m}^{\infty} \frac{1}{(2j-3)^3}\frac{1}{(2j+1)} \\ &\leq B^2\left(\frac{1}{n^{\alpha+2}} + \frac{1}{2n^2}\right)^2 \sum_{j=m}^{\infty} \frac{1}{(2j-3)^4} \leq B^2\left(\frac{1}{n^{\alpha+2}} + \frac{1}{2n^2}\right)^2 \frac{1}{(2m-3)^3}.\end{aligned}$$

Hence,

$$\|f - S_{n,m}f\|_2^2 \leq B\left(\frac{1}{n^{\alpha+2}} + \frac{1}{2n^2}\right)\frac{1}{(2m-3)^{\frac{3}{2}}}.$$

Therefore,

$$E_{n,m}^{(HFs)}(f) = \min \|f - S_{n,m}f\|_2 = O\left[\left(\frac{1}{n^{\alpha+2}} + \frac{1}{2n^2}\right)\frac{1}{(2m-3)^{\frac{3}{2}}}\right], \quad m \geq 2 \text{ and } 0 < \alpha \leq 1.$$

□

3 NUMERICAL EXAMPLE OF HYBRID LEGENDRE POLYNOMIALS APPROXIMATION

In this section hybrid Legendre polynomials approximation of the function

$$f(t) = \begin{cases} t^3 + t^2 + 2t + 1, & \forall t \in [0,1], \\ 0, & \text{otherwise} \end{cases}$$

for $n = 1,2$ and $m = 1,2,3$ has been explained by graphs of concerned function. $S_{n,m}$ for $n = 1,2$ and $m = 1,2,3$ are calculated and are given as

$$S_{1,1}(t) = \begin{cases} \frac{31}{12}, & 0 \le t < 1, \\ 0, & \text{otherwise}, \end{cases} \qquad S_{1,2}(t) = \begin{cases} \frac{31}{12} + \frac{39}{12}(2t-1), & 0 \le t < 1, \\ 0, & \text{otherwise}, \end{cases}$$

$$S_{1,3}(t) = \begin{cases} \frac{31}{12} + \frac{39}{12}(2t-1) + \frac{5}{24}[3(2t-1)^2 - 1], & 0 \le t < 1, \\ 0, & \text{otherwise}, \end{cases}$$

$$S_{2,1}(t) = \begin{cases} \frac{155}{96}, & 0 \le t < \frac{1}{2}, \\ \frac{341}{96}, & \frac{1}{2} \le t < 1, \\ 0, & \text{otherwise}, \end{cases} \qquad S_{2,2}(t) = \begin{cases} \frac{155}{96} + \frac{109}{160}(4t-1), & 0 \le t < \frac{1}{2}, \\ \frac{341}{96} + \frac{209}{160}(4t-3), & \frac{1}{2} \le t < 1, \\ 0, & \text{otherwise}, \end{cases}$$

$$S_{2,3}(t) = \begin{cases} \frac{155}{96} + \frac{109}{160}(4t-1) + \frac{7}{192}[3(4t-1)^2 - 1], & 0 \le t < \frac{1}{2}, \\ \frac{341}{96} + \frac{209}{160}(4t-3) + \frac{13}{192}[3(4t-3)^2 - 1], & \frac{1}{2} \le t < 1, \\ 0, & \text{otherwise}. \end{cases}$$

The graphs of $S_{n,m}$ and $f(t)$ has been plotted for $n = 1,2$ and $m = 1,2,3$ in Figures 1–6 respectively. Hybrid Legendre polynomial approximation error for different values of n and m is shown in Table 1.

n	m	$\|f - S_{n,m}\|_2$
n= 1	m=1	1.14131
	m=2	0.187295
	m=3	0.0188982
n=2	m=1	0.603409
	m=2	0.0486932
	m=3	0.00236228

Table 1. Hybrid Legendre polynomial approximation errors for different values of n and m

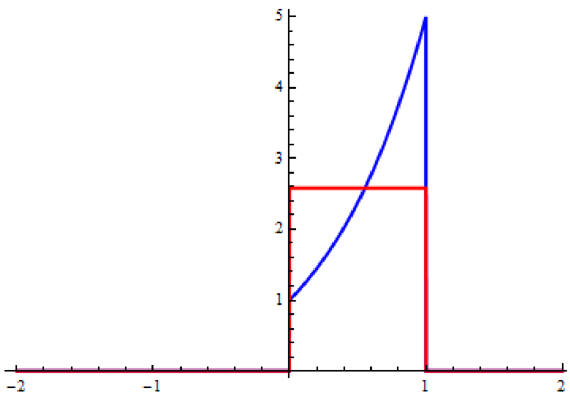

Figure 1. Graph of $S_{1,1}$ and the function $f(t)$

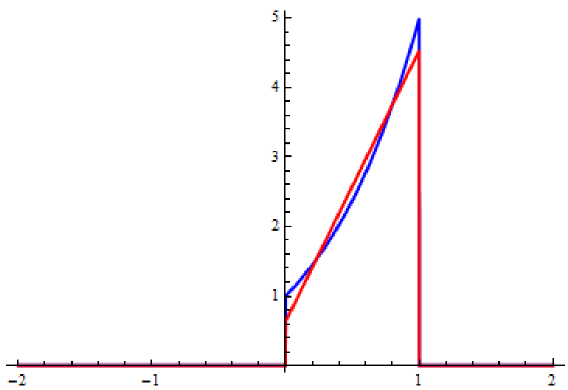

Figure 2. Graph of $S_{1,2}$ and the function $f(t)$

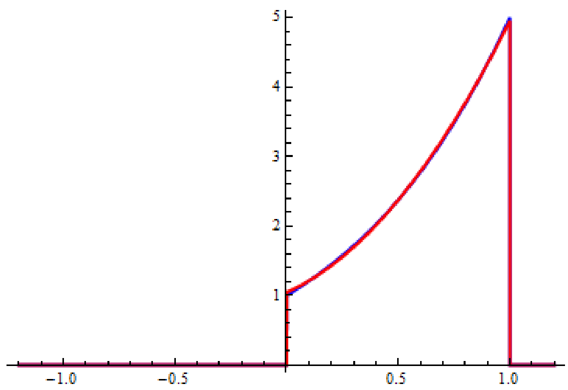

Figure 3. Graph of $S_{1,3}$ and the function $f(t)$

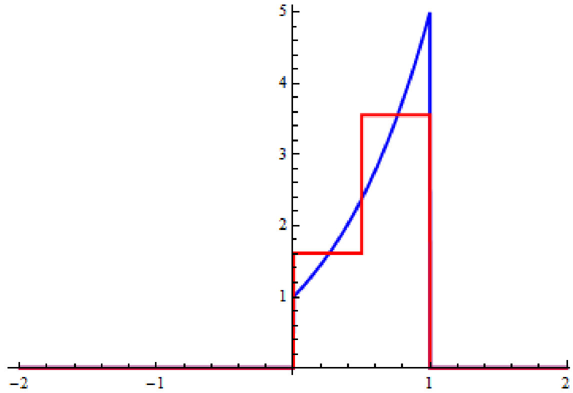

Figure 4. Graph of $S_{2,1}$ and the function $f(t)$

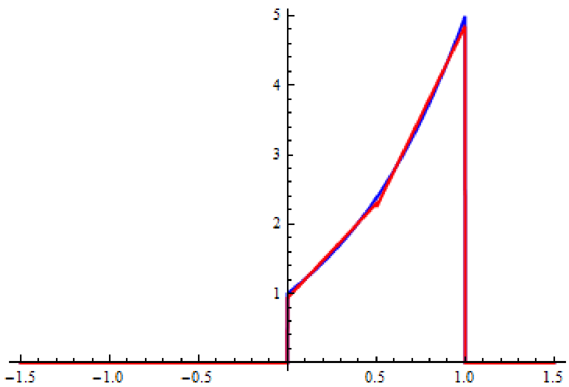

Figure 5. Graph of $S_{2,2}$ and the function $f(t)$

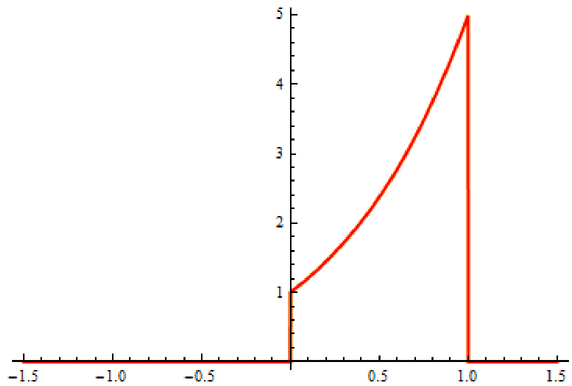

Figure 6. Graph of $S_{2,3}$ and the function $f(t)$

4 APPLICATION OF HYBRID LEGENDRE POLYNOMIALS EXPANSION

4.1 Application of hybrid Legendre polynomials expansion in real-world problems

We have used the hybrid Legendre polynomial approximation method to solve the differential equations related to the following real-world problem.

4.1.1 Radioactive decay

Radioactivity [12] is one of the effects of disruption in the nucleus of a radioactive substance. It is important to remember that radioactivity has also been used in the diagnosis of cancers through lighting in the nucleus form of the atoms to the recipient.

If $m(t)$ be the mass of a radioactive substance at time t, then (see [12])

$$\frac{dm}{dt} = -km(t), \quad m(0) = m_0, \tag{14}$$

where k is a decay constant and m_0 is the initial mass. Let us consider $k = 2$ and $m_0 = 2$, the above equation reduces to

$$\frac{dm}{dt} = -2m(t), \quad m(0) = 2. \tag{15}$$

Equation (15) is now solved using hybrid Legendre polynomials operational matrix of integration as in [6] for $n = 5$ and $m = 3$ as below.

Let

$$h(t) = [h_{10}, h_{11}, h_{12}, h_{20}, h_{21}, h_{22}, h_{30}, h_{31}, h_{32}, h_{40}, h_{41}, h_{42}, h_{50}, h_{51}, h_{52}]^T. \tag{16}$$

Here $h(t)$ be 15×1 column vector and h_{ij} for $i = 1,2,3,4,5$ and $j = 0,1,2$ are calculated as given in subsection 1.3. The integration of above vector $h(t)$ is given as

$$\int_0^t h(x)dx = Ph(t).$$

Here P is 15×15 hybrid Legendre polynomials operational matrix of integration and it is given as

$$P = \begin{bmatrix}
\frac{1}{10} & \frac{1}{10} & 0 & \frac{1}{5} & 0 & 0 & \frac{1}{5} & 0 & 0 & \frac{1}{5} & 0 & 0 & \frac{1}{5} & 0 & 0 \\
-\frac{1}{30} & 0 & \frac{1}{30} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{50} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{10} & \frac{1}{10} & 0 & \frac{1}{5} & 0 & 0 & \frac{1}{5} & 0 & 0 & \frac{1}{5} & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{30} & 0 & \frac{1}{30} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{1}{50} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{10} & \frac{1}{10} & 0 & \frac{1}{5} & 0 & 0 & \frac{1}{5} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{30} & 0 & \frac{1}{30} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{50} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{10} & \frac{1}{10} & 0 & \frac{1}{5} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{30} & 0 & \frac{1}{30} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{50} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{10} & \frac{1}{10} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{30} & 0 & \frac{1}{30} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{50} & 0
\end{bmatrix}. \tag{17}$$

Let $m(t) = N^T h(t)$, where

$$N(t) = [n_{10}, n_{11}, n_{12}, n_{20}, n_{21}, n_{22}, n_{30}, n_{31}, n_{32}, n_{40}, n_{41}, n_{42}, n_{50}, n_{51}, n_{52}]^T$$

is an unknown vector. Integrating equation (15) and using initial conditions, we observe

$$(I + 2P^T)N = 2d. \tag{18}$$

Here I be a identity matrix of order 15 and $d = [1,0,0,1,0,0,1,0,0,1,0,0,1,0,0]^T$ is a column vector of order 15×1. Equation (18) denotes the set of fifteen algebraic equations which can be solved for N. Now comparison between exact solution and approximate solution of equation (15) is given in Table 2.

t	Hybrid Legendre polynomials solution for $n = 5$, $m = 3$	Exact solution	Absolute error
0.0	1.99912	2.00000	0.00088
0.1	1.63744	1.63746	0.00002
0.2	1.34005	1.34064	0.00059
0.3	1.09761	1.09762	0.00001
0.4	0.89826	0.89866	0.00040
0.5	0.73575	0.73576	0.00001
0.6	0.60212	0.60239	0.00027
0.7	0.49319	0.49319	0.00000
0.8	0.40362	0.40379	0.00017
0.9	0.33059	0.33060	0.00001

Table 2. Comparison between approximate solution and exact solution for $k = 2$ amd $m_0 = 2$

Also, equation (14) is solved for $k = 1$ and $m_0 = 1$ and comparison between approximate solution and exact solution for $k = 1$ and $m_0 = 1$ is shown in Table 3.

t	Hybrid Legendre polynomials solution for $n = 5$, $m = 3$	Exact solution	Absolute error
0.0	0.99994	1.00000	0.00006
0.1	0.90484	0.90484	0.00000
0.2	0.81868	0.81873	0.00005
0.3	0.74082	0.74082	0.00000
0.4	0.67028	0.67032	0.00004
0.5	0.60653	0.60653	0.00000
0.6	0.54878	0.54881	0.00003
0.7	0.49659	0.49659	0.00000
0.8	0.44930	0.44933	0.00003
0.9	0.40657	0.40657	0.00000

Table 3. Comparison between approximate solution and exact solution for $k = 1$ amd $m_0 = 1$

4.2 Application of hybrid Legendre polynomials expansion in solving Hermite differential equation of order zero

Consider the Hermite differential equation of order zero (see [13])

$$y'' - 2ty' = 0 \tag{19}$$

with initial conditions

$$y(0) = y'(0) = 1. \tag{20}$$

Now we have solved the equation (19) by hybrid Legendre polynomial operational matrix of integration for $n = 5$ and $m = 3$ given by (17), which is obtained by hybrid Legendre polynomial approximation method as below.

Let

$$y''(t) = L^T h(t), \tag{21}$$

where $L = [l_{10}, l_{11}, l_{12}, l_{20}, l_{21}, l_{22}, l_{30}, l_{31}, l_{32}, l_{40}, l_{41}, l_{42}, l_{50}, l_{51}, l_{52}]^T$ is 15×1 unknown column vector and $h(t)$ is also a column vector given by (16). Now expanding $f(t) = 1$ and $g(t) = t$ by hybrid Legendre polynomials for $n = 5$ and $m = 3$, we obtain $f(t) = r^T h(t)$ and $g(t) = s^T h(t)$, where $r = [1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0]^T$ and

$$s = \left[\frac{1}{10}, \frac{1}{10}, 0, \frac{3}{10}, \frac{1}{10}, 0, \frac{5}{10}, \frac{1}{10}, 0, \frac{7}{10}, \frac{1}{10}, 0, \frac{1}{10}, \frac{9}{10}, 0\right]^T$$

are column vectors each of order 15×1. Now integrating equation (21) two times and using initial conditions given by (20), we find

$$y'(t) = L^T P h(t) + r^T h(t)$$

and

$$y(t) = L^T P^2 h(t) + r^T P h(t) + r^T h(t).$$

Approximate $s^T h h^T$ by hybrid Legendre polynomials as

$$s^T h h^T = h^T S, \tag{22}$$

where S is a square matrix of order 15 and it is given as

$$S = \begin{bmatrix}
\frac{1}{10} & \frac{1}{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{30} & \frac{1}{10} & \frac{2}{30} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{2}{50} & \frac{1}{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{3}{10} & \frac{1}{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{30} & \frac{3}{10} & \frac{2}{30} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{2}{50} & \frac{3}{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{5}{10} & \frac{1}{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{30} & \frac{5}{10} & \frac{2}{30} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{50} & \frac{5}{10} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{7}{10} & \frac{1}{10} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{30} & \frac{7}{10} & \frac{2}{30} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{50} & \frac{7}{10} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{9}{10} & \frac{1}{10} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{30} & \frac{9}{10} & \frac{2}{30} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{50} & \frac{9}{10}
\end{bmatrix}.$$

From the above we get

$$(I - 2SP^T) = 2Sr.$$

It is a system of algebraic equations which is solved for L. The exact solution of (19) is given by

$$y(t) = 1 + \int_0^t e^{x^2} dx.$$

Comparison between approximate solution and exact solution is given in Table 4.

t	Hybrid Legendre polynomials solution for $n = 5$, $m = 3$	Exact solution	Absolute error
0.0	1.000	1.000	0.000
0.1	1.101	1.100	0.001
0.2	1.204	1.203	0.001
0.3	1.311	1.309	0.002
0.4	1.425	1.422	0.003
0.5	1.548	1.545	0.003
0.6	1.686	1.680	0.006
0.7	1.840	1.833	0.007
0.8	2.019	2.009	0.010
0.9	2.229	2.215	0.014

Table 4. Comparison between approximate solution and exact solution for $n = 5$ and $m = 3$

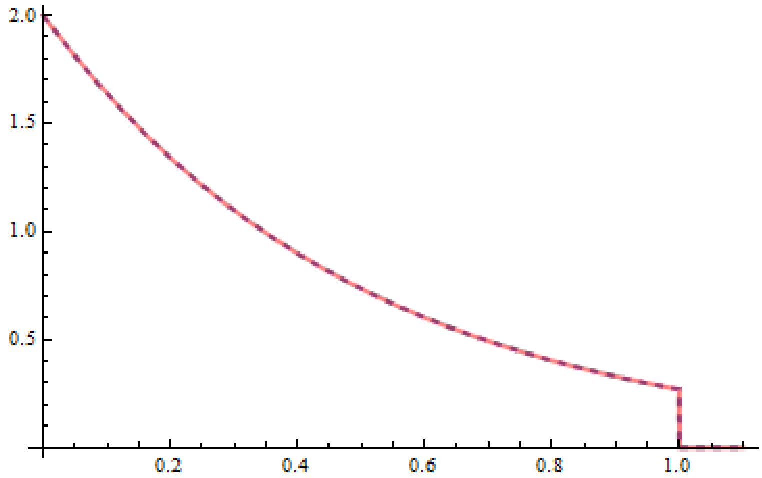

Figure 7. Graph of exact solution (dark line) and approximate solution (dashed line) of radioactive decay problem for $k = 2$ and $m_0 = 2$

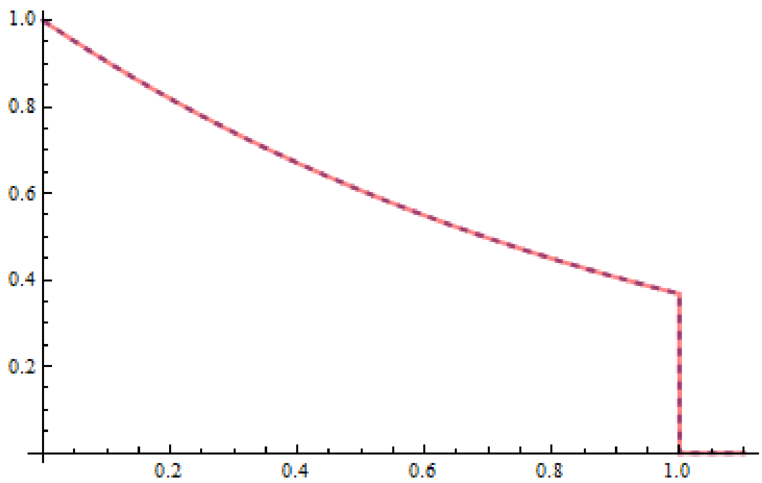

Figure 8. Graph of exact solution (dark line) and approximate solution (dashed line) of radioactive decay problem for $k = 1$ and $m_0 = 1$

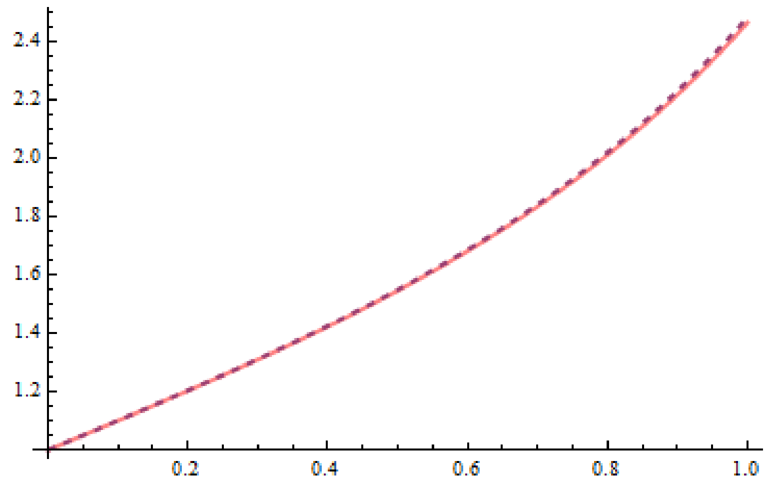

Figure 9. Graph of exact solution (dark line) and approximate solution (dashed line) of Hermite differential equation

5 CONCLUSIONS

1. Estimates of Theorems 1, 2 and 3 are given by

(i) $E_m^{(BP)}(f) = O\left[\frac{1}{m}\left(1+\frac{1}{m^\alpha}\right)\right]$;

(ii) $E_m^{(LP)}(f) = O\left[\left(1+\frac{1}{\sqrt{2\alpha+1}}\right)\frac{1}{(2m-3)^{\frac{3}{2}}}\right], \quad m \geq 2;$

(iii) $E_{n,m}^{(HFs)}(f) = O\left[\left(\frac{1}{n^{\alpha+2}}+\frac{1}{2n^2}\right)\frac{1}{(2m-3)^{\frac{3}{2}}}\right]$, where $m \geq 2,\ 0 < \alpha \leq 1$ and n is a positive integer.

Since $E_m^{(BP)}(f) \to 0$, $E_m^{(LP)}(f) \to 0$ and $E_{n,m}^{(HFs)} f \to 0$ as $m, n \to \infty$, these approximations are best possible in wavelet analysis.

2. The solution of differential equations associated with the radioactive decay problem and the solution of the Hermite differential equation of order zero by hybrid Legendre polynomials is approximately same as the exact solution. This is the significant achievement of this paper.

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У цій роботі, використовуючи блочно-імпульсні функції та гібридні поліноми Лежандра, знайдено оцінки функції f, яка має першу і другу похідні, що належать до класу $Lip_\alpha[a,b]$, де $0<\alpha\le 1$, і a, b — скінченні дійсні числа. Отримані оцінки є новими, точними та найкращими у вейвелет аналізі. Із метою пояснення обґрунтованості апроксимації функцій методом наближення гібридними поліномами Лежандра наведено приклад розв'язку задачі радіоактивного розпаду. Більше того, для пояснення важливості та застосування методики цього методу знайдено розв'язок диференціального рівняння Ерміта нульового порядку.

Ключові слова і фрази: блочно-імпульсна функція, поліном Лежандра, гібридний поліном Лежандра.

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BEDRATYUK L.[1], LUNO N.[2]

SOME PROPERTIES OF GENERALIZED HYPERGEOMETRIC APPELL POLYNOMIALS

Let $x^{(n)}$ denotes the Pochhammer symbol (rising factorial) defined by the formulas $x^{(0)} = 1$ and $x^{(n)} = x(x+1)(x+2)\cdots(x+n-1)$ for $n \geq 1$. In this paper, we present a new real-valued Appell-type polynomial family $A_n^{(k)}(m,x)$, $n, m \in \mathbb{N}_0$, $k \in \mathbb{N}$, every member of which is expressed by mean of the generalized hypergeometric function ${}_pF_q\left[\begin{matrix} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{matrix} \middle| z\right] = \sum\limits_{k=0}^{\infty} \frac{a_1^{(k)} a_2^{(k)} \ldots a_p^{(k)}}{b_1^{(k)} b_2^{(k)} \ldots b_q^{(k)}} \frac{z^k}{k!}$ as follows

$$A_n^{(k)}(m,x) = x^n{}_{k+p}F_q\left[\begin{matrix} a_1, a_2, \ldots, a_p, -\frac{n}{k}, -\frac{n-1}{k}, \ldots, -\frac{n-k+1}{k} \\ b_1, b_2, \ldots, b_q \end{matrix} \middle| \frac{m}{x^k}\right]$$

and, at the same time, the polynomials from this family are Appell-type polynomials.

The generating exponential function of this type of polynomials is firstly discovered and the proof that they are of Appell-type ones is given. We present the differential operator formal power series representation as well as an explicit formula over the standard basis, and establish a new identity for the generalized hypergeometric function. Besides, we derive the addition, the multiplication and some other formulas for this polynomial family.

Key words and phrases: Appell sequence, Appell polynomial, generalized hypergeometric polynomial, generalized hypergeometric function.

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1 INTRODUCTION

In [4], P. Appell presented polynomial sequence $\{A_n(x)\}$, $n = 0, 1, 2, \ldots$, such that $\deg A_n(x) = n$ and satisfying the identity

$$A_n'(x) = nA_{n-1}(x),$$

where $A_0(x) \neq 0$, which is called the Appell polynomial sequence.

An arbitrary Appell polynomial sequence possesses an exponential generating function

$$A(t)e^{xt} = \sum_{n=0}^{\infty} A_n(x) \frac{t^n}{n!},$$

here $A(t)$ is a formal power series

$$A(t) = a_0 + a_1 t + a_2 \frac{t^2}{2!} + \cdots + a_n \frac{t^n}{n!} + \cdots, \quad a_0 \neq 0. \tag{1}$$

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The Appell-type polynomials $A_n(x)$ are expressed in the terms of $\{a_n\}$ as follows

$$A_n(x) = \sum_{i=0}^{n} \binom{n}{i} a_{n-i} x^i.$$

The simplest example of Appell-type polynomials is the monomial sequence $\{x^n\}$, $n = 0, 1, \ldots;$ other examples are the Bernoulli, the Euler polynomials and the Hermite polynomials. For more examples one can consult [1, 11].

The Appell-type polynomials perform a large variety of features and are widely spread at the different areas of mathematics, namely, at special functions, general algebra, combinatorics and number theory. Recently, the Appell-type polynomials are of big interest. The modern researches give the alternative definitions of Appell-type polynomials and apply new approaches based, for instance, on the determinant method or in Pascal matrix method (see, e.g., [3, 16]). Consequently, many new properties of those polynomials are described and a great deal of identities involving Appell-type polynomials are obtained (see [2, 6, 7]).

Let us recall that the generalized hypergeometric function is defined as follows

$${}_pF_q\left[\begin{matrix} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{matrix} \middle| z\right] = \sum_{k=0}^{\infty} \frac{a_1^{(k)} a_2^{(k)} \ldots a_p^{(k)}}{b_1^{(k)} b_2^{(k)} \ldots b_q^{(k)}} \frac{z^k}{k!}, \tag{2}$$

where $a_1, a_2, \ldots, a_p$, $b_1, b_2, \ldots, b_p$ are complex parameters and none of b_i equals to a non-positive integer or zero, $x^{(n)}$ denotes the Pochhammer symbol (or rising factorial) defined by $x^{(n)} = x(x+1)(x+2)\cdots(x+n-1)$ for $n \geq 1$ and $x^{(0)} = 1$. Further on, we denote the generalized hypergeometric function by ${}_pF_q$ for brevity.

We note that the Gauss hypergeometric function ${}_2F_1$ and the Kummer hypergeometric function ${}_1F_1$ are the partial cases of (2).

Apart from the Appell-type polynomials, there exist some polynomial families, which admit representation via the partial cases of the generalized hypergeometric function, i.e., the Jacobi polynomials ([1])

$$P_n^{(\alpha,\beta)}(z) = \frac{(\alpha+1)^{(n)}}{n!} {}_2F_1\left[\begin{matrix} -n, n+\alpha+\beta+1 \\ \alpha+1 \end{matrix} \middle| \frac{1-z}{2}\right].$$

At the same time, there exists a number of the Appell-type polynomial families, which also admit the representation via partial cases of the Gauss hypergeometric function. It is known [1], that the Laguerre polynomials $L_n(x)$ are presented as follows

$$L_n(x) = {}_1F_1\left[\begin{matrix} -n \\ 1 \end{matrix} \middle| x\right].$$

Remarkably, the Hermite polynomials $H_n(x)$ are simply expressed in the terms of those functions ([8])

$$H_n(x) = x^n {}_2F_0\left[\begin{matrix} -\frac{n}{2}, -\frac{n-1}{2} \\ - \end{matrix} \middle| -\frac{2}{x^2}\right], \quad G(x,t) = e^{xt - \frac{1}{2}t^2}.$$

The natural way of generalisation of the Hermitte polinomials is to expand the array of ratios for another denominators, it was made in [10], the authors obtained the Gould-Hopper polynomials $g_n^m(x,h)$, with $G(x,t) = e^{xt+ht^m}$, which could be also expressed in the terms of the generalized hypergeometric function as follows

$$g_n^m(x,h) = x^n {}_mF_0 \left[\begin{matrix} -\frac{n}{m}, -\frac{n-1}{m}, \dots, -\frac{n-m+1}{m} \\ - \end{matrix} \middle| \frac{(-1)^m h m^m}{x^m} \right].$$

The aim of this paper is to find a polynomial family, which would be the Appell-type one and admit the generalized hypergeometric function representation simultaneously. Still, there exist the polynomial families, which have the needed representation, e.g., *the generalized hypergeometric polynomials* $f_n(a_i; b_j; x)$, studied at [9], such that

$$f_n(a_i; b_j; x) = {}_{p+2}F_{q+2} \left[\begin{matrix} -n, n+1, a_1, a_2, \dots, a_p \\ 1, \frac{1}{2}, b_1, b_2, \dots, b_q \end{matrix} \middle| x^n \right], \quad n \in \mathbb{N}_0,$$

and *the incomplete hypergeometric polynomials* associated with generalized incomplete hypergeometric function, studied at [13], but they both are not the Appel-type polynomials.

The difference between all mentioned classes of polynomials, depending, if they are of Appell-type or not and if they possess the generalized hypergeometric function representation or do not, has motivated the title of the paper.

Therefore, let us give the following

Definition 1. *Let $\Delta(k, -n)$ denotes the array of k ratios $-\frac{n}{k}, -\frac{n-1}{k}, \dots, -\frac{n-k+1}{k}$, $n \in \mathbb{N}_0$, $k \in \mathbb{N}$. Then we call the polynomial family*

$$A_n^{(k)}(m,x) = x^n {}_{k+p}F_q \left[\begin{matrix} a_1, a_2, \dots, a_p, \Delta(k,-n) \\ b_1, b_2, \dots, b_q \end{matrix} \middle| \frac{m}{x^k} \right], \quad n, m \in \mathbb{N}_0, \quad k \in \mathbb{N}, \tag{3}$$

where

$${}_{k+p}F_q = \sum_{i=0}^{[n/k]} \frac{\prod_{r=1}^{p} (a_r)^{(i)}}{\prod_{s=1}^{q} (b_s)^{(i)}} \prod_{j=1}^{k} \left(-\frac{n-j+1}{k} \right)^{(i)} \frac{m^i}{i! x^{ki}}, \tag{4}$$

the generalized hypergeometric Appell polynomials.

We note that if $p = 0, q = 0, k := m, m := (-1)^k h k^k$ the generalized hypergeometric Appell polynomials $A_n^{(k)}(m,x)$ become the Gould-Hopper polynomials $g_n^m(x,h)$ and if $p = 0, q = 0$, $m = -2, k = 2$ they become the Hermite polynomials $H_n(x)$ mentioned above.

The main result of this article is the following basic statement.

Theorem 1. *The generalized hypergeometric Appell polynomials $A_n^{(k)}(m,x)$ defined by definition 1 are the Appell type ones.*

2 BASIC DEFINITIONS AND NOTATION

In addition to the rising factorial we use the falling factorial, defined by $(x)_0 = 1$ and $(x)_n = x(x-1)(x-2)\cdots(x-n+1)$ for $n > 0$. In these notations, the following relations hold (see [1])

$$(x)_n = (-1)^n (-x)^{(n)}, \tag{5}$$

and the Gauss product of indexes formula (see [14]) will be written as follows

$$(-\lambda)^{(mn)} = m^{mn} \prod_{j=1}^{m} \left(-\frac{\lambda - j + 1}{m}\right)^{(n)}, \quad n \in \mathbb{N}_0. \tag{6}$$

We note that in the case, when either a or b is a non-positive integer, the generalized hypergeometric function reduces to a polynomial

$${}_pF_q \left[\begin{matrix} -m, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{matrix} \middle| z \right] = \sum_{n=0}^{\infty} (-1)^n \binom{m}{n} \frac{\prod_{j=2}^{p} {a_j}^{(n)}}{\prod_{s=1}^{q} {b_s}^{(n)}} z^n.$$

As far as we deal with the differentiation, the differentiation formula with respect to z would be useful: $\frac{d}{dx} {}_pF_q \left[\begin{matrix} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{matrix} \middle| z \right] = \frac{\prod_{j=1}^{p} a_j}{\prod_{s=1}^{q} b_s} {}_pF_q \left[\begin{matrix} a_1+1, a_2+1, \ldots, a_p+1 \\ b_1+1, b_2+1, \ldots, b_q+1 \end{matrix} \middle| z \right]$ [12].

3 Basic properties of the generalized hypergeometric Appell polynomials

3.1 Being of Appell type

Proof of Theorem 1. To prove the generalized hypergeometric Appell polynomials $A_n^{(k)}(m, x)$ are the Appell-type polynomials, it is sufficient to show that there exists a formal power series $A(t)$ such that the following relation holds

$$A(t)e^{xt} = \sum_{n=0}^{\infty} A_n^{(k)}(m, x) \frac{t^n}{n!}.$$

We set $(\gamma)^i = \left(\prod_{r=1}^{p} (a_r)^{(i)}\right) \Big/ \left(\prod_{s=1}^{q} (b_s)^{(i)}\right)$. Then from definition 2 and relations (5) and (6) it follows that

$$A_n^{(k)}(m, x) = x^n {}_{p+k}F_q \left[\begin{matrix} a_1, a_2, \ldots, a_p, \Delta(k, -n) \\ b_1, b_2, \ldots, b_q \end{matrix} \middle| \frac{m}{x^k} \right] = x^n \sum_{i=0}^{[n/k]} \frac{(\gamma)^i (-1)^{ki} (n)_{ki}}{k^{ki}} \frac{m^i}{i! x^{ki}}.$$

We choose

$$A(t) = {}_pF_q \left[\begin{matrix} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{matrix} \middle| (-1)^k m \frac{t^k}{k^k} \right]. \tag{7}$$

Using the expansion of e^{xt} into the power series and changing the product of the series by the double series, we transform the generating function as follows

$$A(t)e^{xt} = \left(\sum_{n=0}^{\infty} (\gamma)^n \frac{\left((-1)^k m \frac{t^k}{k^k}\right)^n}{n!} \right) \left(\sum_{s=0}^{\infty} \frac{(xt)^s}{s!} \right) = \sum_{n=0}^{\infty} \left(\sum_{s=0}^{\infty} (\gamma)^n (-1)^{kn} \frac{m^n x^s}{k^{kn}} \frac{t^{s+kn}}{s! n!} \right).$$

Using the infinite sums interchange formula ([5]) $\sum\limits_{n=0}^{\infty} \sum\limits_{m=0}^{\infty} a_{n,m} = \sum\limits_{p=0}^{\infty} \sum\limits_{q=0}^{p} a_{p-q,q}$ and taking into account the multiplicity of i, we have

$$\sum_{n=0}^{\infty} \sum_{s=0}^{\infty} a_{s,n} = \sum_{n=0}^{\infty} \sum_{i=0}^{[n/k]} a_{n-ki,i},$$

then

$$\sum_{n=0}^{\infty}\left(\sum_{s=0}^{\infty}(\gamma)^n\frac{m^n(-1)^{kn}x^s}{k^{kn}}\frac{t^{s+kn}}{s!n!}\right)=\sum_{n=0}^{\infty}\left(\sum_{i=0}^{[n/k]}\frac{m^i(-1)^{ki}(\gamma)^i}{k^{ki}}\frac{x^{n-ki}t^n}{(n-ki)!i!}\right)$$
$$=\sum_{n=0}^{\infty}x^n\left(\sum_{i=0}^{[n/k]}\frac{n!}{(n-ki)!i!}\frac{m^i(-1)^{ki}(\gamma)^i}{k^{ki}}\frac{1}{x^{ki}}\right)\frac{t^n}{n!}=\sum_{n=0}^{\infty}x^n\left(\sum_{i=0}^{[n/k]}(\gamma)^i\frac{m^i(-1)^{ki}(n)_{ki}}{k^{ki}}\frac{\frac{1}{x^{ki}}}{i!}\right)\frac{t^n}{n!}.$$

The inner sum is precisely equal to the generalized hypergeometric function in the form of (3) and, therefore, the relation (4) holds. This means that the generating function admit the needed representation (3).

It should be noted that there is another way to prove Theorem 1, which is to replace xt by t and m/x^k by x in [15, Problem 26, p.173].

As a consequence of Theorem1, we derive a new identity for the generalized hypergeometric function.

Corollary 1. *The following identity holds*

$$nx^{n-1}{}_{p+k}F_q\left[\begin{matrix}a_1,a_2,\ldots,a_p,\Delta(k,-n+1)\\ b_1,b_2,\ldots,b_q\end{matrix}\middle|\frac{m}{x^k}\right]=nx^{n-1}{}_{p+k}F_q\left[\begin{matrix}a_1,a_2,\ldots,a_p,\Delta(k,-n)\\ b_1,b_2,\ldots,b_q\end{matrix}\middle|\frac{m}{x^k}\right]$$
$$-km(\gamma)^1\Delta_1(k,-n)x^{n-k-1}{}_{p+k}F_q\left[\begin{matrix}a_1+1,a_2+1,\ldots,a_p+1,\Delta(k,-n+k)\\ b_1+1,b_2+1,\ldots,b_q+1\end{matrix}\middle|\frac{m}{x^k}\right],$$

where $\Delta_1(k,-n)$ *denotes the product* $\left(-\frac{n}{k}\right)\cdot\left(-\frac{n-1}{k}\right)\ldots\left(-\frac{n-k+1}{k}\right)$.

Proof. The generalized hypergeometric Appell polynomials are the Appell-type ones, hence, the identity $\frac{d}{dx}\left\{A_n^{(k)}(m,x)\right\}=nA_{n-1}^{(k)}(m,x)$ fulfils.

Representing the polynomials $A_{n-1}^{(k)}(m,x)$ in the terms of the generalized hypergeometric function according to Definition 1, we immediately obtain the left side of the corollary equality.

To obtain its right side we differentiate the hypergeometric representation of the polynomials $A_n^{(k)}(m,x)$ under the Leibnitz rule:

$$\frac{d}{dx}\left\{x^n{}_{p+k}F_q\left[\begin{matrix}a_1,a_2,\ldots,a_p,\Delta(k,-n)\\ b_1,b_2,\ldots,b_q\end{matrix}\middle|\frac{m}{x^k}\right]\right\}=nx^{n-1}{}_{p+k}F_q\left[\begin{matrix}a_1,a_2,\ldots,a_p,\Delta(k,-n)\\ b_1,b_2,\ldots,b_q\end{matrix}\middle|\frac{m}{x^k}\right]$$
$$+x^n\frac{d}{dx}\left\{{}_{p+k}F_q\left[\begin{matrix}a_1,a_2,\ldots,a_p,\Delta(k,-n)\\ b_1,b_2,\ldots,b_q\end{matrix}\middle|\frac{m}{x^k}\right]\right\}.$$

Performing the derivative of the hypergeometric function, we obtain

$$x^n\frac{d}{dx}\left\{{}_{p+k}F_q\left[\begin{matrix}a_1,a_2,\ldots,a_p,\Delta(k,-n)\\ b_1,b_2,\ldots,b_q\end{matrix}\middle|\frac{m}{x^k}\right]\right\}=x^n\left(\frac{(-1)^k(n)_k}{k^k}\frac{a_1\cdots a_p}{b_1\cdots b_q}\frac{m(-k)}{1!x^{k+1}}\right.$$
$$+\frac{(-1)^{2k}(n)_{2k}}{k^{2k}}\frac{a_1(a_1+1)\cdots a_p(a_p+1)}{b_1(b_1+1)\cdots b_q(b_q+1)}\frac{m^2(-2k)}{2!x^{2k+1}}$$
$$\left.+\frac{(-1)^{3k}(n)_{3k}}{k^{3k}}\frac{a_1(a_1+1)(a_1+2)\cdots a_p(a_p+1)(a_p+2)}{b_1(b_1+1)(b_1+2)\cdots b_q(b_q+1)(b_q+2)}\frac{m^3(-3k)}{3!x^{3k+1}}+\cdots\right)$$

$$= x^{n-k-1} mk \frac{(-1)^{k+1}(n)_k}{k^k} \frac{a_1 \cdots a_p}{b_1 \cdots b_q} \left(1 + \frac{(-1)^k (n-k)_k}{k^k} \frac{(a_1+1)\cdots(a_p+1)}{(b_1+1)\cdots(b_q+1)} \frac{m \cdot 2}{2! x^k} \right.$$
$$\left. + \frac{(-1)^{2k}(n-k)_{2k}}{k^{2k}} \frac{(a_1+1)(a_1+2)\cdots(a_p+1)(a_p+2)}{(b_1+1)(b_1+2)\cdots(b_q+1)(b_q+2)} \frac{m^2 \cdot 3}{3! x^{2k}} + \cdots \right)$$
$$= -km(\gamma)^1 \Delta_1(k,-n) x^{n-k-1} {}_{p+k}F_q \left[\begin{matrix} a_1+1, a_2+1, \ldots, a_p+1, \Delta(k, -n+k) \\ b_1+1, b_2+1, \ldots, b_q+1 \end{matrix} \middle| \frac{m}{x^k} \right],$$

that ends the proof. □

Since an arbitrary polynomial of one variable $P_n(x) \in \mathbb{C}[x]$ always permits the formal series representation

$$P_n(x) = \sum_{i=0}^{n} \alpha_i x^i,$$

we are interested in finding those representation for the generalized hypergeometric Appell polynomials.

Corollary 2. *The generalized hypergeometric Appell polynomials $A_n^{(k)}(m,x)$ possess*

(i) the standard basis $\{x^i\}_{i=0}^n$ representation

$$A_n^{(k)}(m,x) = \sum_{i=0}^{[n/k]} \frac{n!(-1)^{ki}(\gamma)^i m^i}{i! k^{ki} (n-ki)!} x^{n-ki}, \tag{8}$$

(ii) the differential operator formal power series representation

$$A_n^{(k)}(m,x) = \left(\sum_{i=0}^{[n/k]} \frac{(-1)^{ki}(\gamma)^i m^i}{i! k^{ki}} D^{ki} \right) x^n. \tag{9}$$

Proof. (i) We use an approach from [6], which is based on the idea of the connection problem.

Given the two polynomial families of Appell type $\{P_n(x)\}$ and $\{Q_n(x)\}$ with generating functions $A_1(t)$ and $A_2(t)$ respectively, the solution of its connection problem could be written as follows

$$Q_n(x) = \sum_{m=0}^{n} \frac{n!}{m!} \alpha_{n-m} P_m(x),$$

where $\frac{A_2(t)}{A_1(t)} = \sum\limits_{k=0}^{\infty} \alpha_k t^k$.

We are searching for the unknown coefficients α_k to decompose the polynomials

$$Q_n(x) = x^n, A_2(t) = 1$$

upon the polynomials $A_n^{(k)}(m,x)$ defined by (3) with generating function $A_1(t)$ defined by (7). Deriving the ratio of generating functions $A_2(t)$ and $A_1(t)$ we have

$$\frac{A_2(t)}{A_1(t)} = \sum_{r=0}^{\infty} \frac{(-1)^{kr} m^r}{k^{kr} r!} (\gamma)^r t^{kr} = \sum_{r=0}^{\infty} \alpha_{rk} t^{rk},$$

and, constructing the corresponding coefficients α_{n-m}, we obtain the needed representation.

(ii) An arbitrary Appell-type polynomial $P_n(x)$ could be also written in the symmetric form

$$P_n(x) = \sum_{i=0}^{n} \binom{n}{i} c_i x^{n-i}.$$

According to [11], the latter expression is equivalent to the following differential operator representation

$$P_n(x) = \left(\sum_{i=0}^{n} \frac{c_i}{i!} D^i \right) x^n,$$

where $D := d/dx$ is an ordinary differentiation with respect to x, consequently,

$$A_n^{(k)}(m,x) = \sum_{i=0}^{[n/k]} \binom{n}{ki} c_i x^{n-ki} = \sum_{i=0}^{[n/k]} \binom{n}{ki} \frac{(-1)^{ki}(\gamma)^i m^i (ki)!}{i! k^{ki}} x^{n-ki},$$

we deduce a differential operator formal power series representation of the generalized hypergeometric Appell polynomials of the form (9). □

Remark. *Comparing the power series (1) and operational formula (9) of the generalized hypergeometric Appell polynomials to the corresponding ones of the Gould-Hopper polynomials*

$$A(t) = e^{ht^m}, \quad g_n^m(x,h) = \left(e^{hD^m} \right) x^n,$$

it is easy to see that the latter have more compact forms.

Symmetry. Substituting the negative value of argument into the formula (8)

$$A_n^{(k)}(m,-x) = \sum_{i=0}^{[n/k]} (-1)^{n-ki} \frac{n!(-1)^{ki}(\gamma)^i m^i}{i! k^{ki} (n-ki)!} x^{n-ki},$$

we conclude that, in the case of even k, the generalized hypergeometric Appell polynomials are the even ones themselves while n is an even number, and they are the odd ones themselves while n is an odd number:

$$A_{2n}^{(2k)}(m,-x) = A_{2n}^{(2k)}(m,x), \quad A_{2n+1}^{(2k)}(m,-x) = -A_{2n+1}^{(2k)}(m,x).$$

Otherwise, for any odd k in the case of odd n, the summands standing on the even places change their signs into the opposite ones, and the same do the summands standing on the odd places in the case of even n.

3.2 Adition and multiplication formulas and other properties

Here we shall prove the following result.

Theorem 2. *The following formulas hold for the generalized hypergeometric Appell polynomials*

(i) addition formula

$$A_n^{(k)}(m,x+y) = \sum_{i=0}^{n} \binom{n}{i} y^{n-i} A_i^{(k)}(m,x) = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} A_i^{(k)}(m,y),$$

(ii) multiplication formula

$$A_n^{(k)}(m, Mx) = \sum_{i=0}^{n} \binom{n}{i} (M-1)^{n-i} x^{n-i} A_i^{(k)}(m, x),$$

(iii) indexes interchange formula

$$\sum_{i=0}^{n} \binom{n}{i} A_i^{(k_1)}(m, x) A_{n-i}^{(k_2)}(m, y) = \sum_{i=0}^{n} \binom{n}{i} A_i^{(k_2)}(m, x) A_{n-i}^{(k_1)}(m, y),$$

(iv) convolution type identity

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} A_i^{(k)}(m, x) A_{n-i}^{(k)}(m, x)$$
$$= \frac{(-1)^n m^{n/k} n!}{k^n} \sum_{i=0}^{[n/k]} \frac{a_1^{(i)} \dots a_p^{(i)}}{i! b_1^{(i)} \dots b_q^{(i)}} \frac{a_1^{(n/k-i)} \dots a_p^{(n/k-i)}}{(n/k-i)! b_1^{(n/k-i)} \dots b_q^{(n/k-i)}}.$$

Proof. The addition and the multiplication formulas hold for all Appell-type polynomial families ([11]), consequently, they hold for the generalized hypergeometric Appell polynomials as well. The indexes interchange formulas could be obtained applying methods proposed in [6] and the convolution type identity is obtained by the direct calculations at $x = 0$. □

It is worth stressing, that the polynomials $A_n^{(k)}(m, Mx)$ loose the property of being of Appell-type. Moreover, the generalized hypergeometric polynomials over the polynomials could be defined in the same manner as the generalized hypergeometric Appell polynomials:

$$A_n^{(k)}(m, f(x)) = (f(x))^n {}_{p+k}F_q \left[\begin{matrix} a_1, a_2, \dots, a_p, \Delta(k, -n) \\ b_1, b_2, \dots, b_q \end{matrix} \middle| \frac{m}{(f(x))^k} \right],$$

where $f(x) = a_0 x^p + a_1 x^{p-1} + \dots + a_p, \quad a_0 \neq 0$, which deliver us the following differentiation rule

$$\frac{d}{dx} A_n^{(k)}(m, f(x)) = n f'(x) A_{n-1}^{(k)}(m, f(x)).$$

In particular, in the case when $p = a_0 = 1$, we obtain the Appell differentiation.

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Бедратюк Л., Луньо Н. *Деякі властивості узагальнених гіпергеометричних многочленів Аппеля* // Карпатські матем. публ. — 2020. — Т.12, №1. — C. 129–137.

У цій статті ми представляємо нове сімейство многочленів типу Аппеля $\{A_n^{(k)}(m,x)\}$, $n, m \in \mathbb{N}_0$, $k \in \mathbb{N}$, кожен представник якого визначений над полем дійсних чисел і може бути представлений через узагальнену гіпергеометричну функцію

$$ {}_pF_q \left[\begin{matrix} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{matrix} \middle| z \right] = \sum_{k=0}^{\infty} \frac{a_1^{(k)} a_2^{(k)} \ldots a_p^{(k)}}{b_1^{(k)} b_2^{(k)} \ldots b_q^{(k)}} \frac{z^k}{k!}, $$

де через $x^{(n)}$ позначено символ Похгаммера (зростаючий факторіал), який визначають за формулою $x^{(n)} = x(x+1)(x+2)\cdots(x+n-1)$ для $n \geq 1$ і $x^{(0)} = 1$, у такий спосіб

$$ A_n^{(k)}(m,x) = x^n {}_{k+p}F_q \left[\begin{matrix} a_1, a_2, \ldots, a_p, -\frac{n}{k}, -\frac{n-1}{k}, \ldots, -\frac{n-k+1}{k} \\ b_1, b_2, \ldots, b_q \end{matrix} \middle| \frac{m}{x^k} \right], $$

і одночасно многочлени цього сімейства є многочленами типу Аппеля.

Для многочленів цього сімейства вперше знайдено породжуючу функцію і доведено, що вони є многочленами типу Аппеля. Знайдено розклад представників цього сімейства за стандартним базисом в замкнутій формі та у формі ряду диференціального оператора, а також нову тотожність для узагальненої гіпергеометричної функції. Крім цього, для узагальнених гіпергеометричних многочленів Аппеля встановлено формули додавання і множення аргумента та деякі інші.

Ключові слова і фрази: послідовність Аппеля, многочлен Аппеля, узагальнений гіпергеометричний многочлен, узагальнена гіпергеометрична функція.

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ASYMPTOTICS OF APPROXIMATION OF FUNCTIONS BY CONJUGATE POISSON INTEGRALS

Among the actual problems of the theory of approximation of functions one should highlight a wide range of extremal problems, in particular, studying the approximation of functional classes by various linear methods of summation of the Fourier series. In this paper, we consider the well-known Lipschitz class $\mathrm{Lip}_1\alpha$, i.e. the class of continuous 2π–periodic functions satisfying the Lipschitz condition of order α, $0 < \alpha \leq 1$, and the conjugate Poisson integral acts as the approximating operator. One of the relevant tasks at present is the possibility of finding constants for asymptotic terms of the indicated degree of smallness (the so-called Kolmogorov–Nikol'skii constants) in asymptotic distributions of approximations by the conjugate Poisson integrals of functions from the Lipschitz class in the uniform metric. In this paper, complete asymptotic expansions are obtained for the exact upper bounds of deviations of the conjugate Poisson integrals from functions from the class $\mathrm{Lip}_1\alpha$. These expansions make it possible to write down the Kolmogorov–Nikol'skii constants of the arbitrary order of smallness.

Key words and phrases: Poisson integral, asymptotic expansion, conjugate function, Kolmogorov–Nikol'skii problem.

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1 INTRODUCTION

Let C be the space of 2π-periodic continuous functions equipped with the norm $\|f\|_C = \max\limits_t |f(t)|$.

Denote by W^r any set of 2π-periodic functions with absolutely continuous derivatives up to order $(r-1)$ such that $\operatorname{ess\,sup}\limits_t \left|f^{(r)}(t)\right| \leq 1$.

The set of functions that are conjugate to those from the class W^r is denoted by $\overline{W}^r$. That is

$$\overline{W}^r = \left\{ \bar{f} \colon \bar{f}(x) = -\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+t)\cot\frac{t}{2}\,dt = -\frac{1}{2\pi}\int_0^{\pi} \psi_x(t)\cot\frac{t}{2}\,dt, \right.$$
$$\left. \psi_x(t) = f(x+t) - f(x-t), \quad f \in W^r \right\}.$$

Any $f \in C$ is contained in the class $\mathrm{Lip}_1\,\alpha$, $0 < \alpha \leq 1$, if

$$\forall\, t_1, t_2 \in \mathbb{R} \qquad |f(t_1) - f(t_2)| \leq |t_1 - t_2|^{\alpha}.$$

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Let us consider a boundary value problem (in the unit circle) for the equation $\Delta u = 0$, where Δ is the Laplace operator in polar coordinates. We can rewrite this equation as follows

$$\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial x^2} = 0, \qquad 0 \le \rho < 1, \quad -\pi \le x \le \pi. \tag{1}$$

A solution $P_\rho(f;x)$ of (1) that satisfies the boundary conditions

$$u(\rho,x)\big|_{\rho=1} = f(x), \qquad -\pi \le x \le \pi,$$

where f is a summable 2π-periodic function, is of the form

$$P_\rho(f;x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)K_\rho(t)\,dt,$$

where

$$K_\rho(t) = \frac{1}{2} + \sum_{k=1}^{\infty} \rho^k \cos kt = \frac{1-\rho^2}{2(1-2\rho\cos t+\rho^2)}.$$

The quantity $P_\rho(f;x)$ is called the Poisson integral of a function f, and, respectively, $K_\rho(t)$ is called the kernel of the Poisson integral.

In the paper, we consider the conjugate Poisson integral, i.e. the quantity of the following form

$$\overline{P}_\rho(f;x) = P_\rho(\overline{f};x) = -\frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\overline{K}_\rho(t)\,dt, \tag{2}$$

where

$$\overline{K}_\rho(t) = \sum_{k=1}^{\infty} \rho^k \sin kt = \frac{\rho\sin t}{1-2\rho\cos t+\rho^2} \tag{3}$$

is the kernel of the conjugate Poisson integral.

Let $\mathfrak{N} \subseteq C$ be a certain class of functions. According to Stepanets [12], the problem of establishment of asymptotic equalities for the quantity

$$\mathcal{E}\big(\mathfrak{N};P_\rho\big)_C = \sup_{f\in\mathfrak{N}} \big\|f(\cdot) - P_\rho(f;\cdot)\big\|_C$$

is called the Kolmogorov–Nikol'skii problem.

If we determine the explicit form of a function $\varphi(\rho)$ such that

$$\mathcal{E}\big(\mathfrak{N};P_\rho\big)_C = \varphi(\rho) + o(\varphi(\rho)) \qquad \text{as } \rho \to 1-,$$

then we say that the Kolmogorov–Nikol'skii problem for the Poisson integral P_ρ is solved on the class $\mathfrak{N}$ in the metric of the space C.

Definition 1. *A formal series $\sum\limits_{n=0}^{\infty} g_n(\rho)$ is called a complete asymptotic expansion of a function $f(\rho)$ as $\rho \to 1-$, if for an arbitrary natural number m the following equation holds*

$$f(\rho) = \sum_{n=0}^{m} g_n(\rho) + o\left(g_m(\rho)\right) \qquad \textit{as } \rho \to 1-,$$

and $\forall\, n \in N$

$$|g_{n+1}(\rho)| = o(|g_n(\rho)|) \qquad \textit{as } \rho \to 1-.$$

In what follows, this fact we denote by

$$f(\rho) \cong \sum_{n=0}^{\infty} g_n(\rho).$$

Approximation properties of the method of approximation by Poisson integrals on classes of differentiable functions are well studied. The Kolmogorov–Nikol'skii problem for the Poisson integral on the classes W^1 was solved by Natanson in [10].

Timan [14] obtained the exact values of approximative characteristics $\mathcal{E}\big(W^r; P_\rho\big)_C$. In the paper [9] Malei determined the complete asymptotic expansion of the upper bounds of deviations of Poisson integrals from functions of the class W^1. Later, this expansion was reproved by Stark [11].

The complete asymptotic expansion of the quantity $\mathcal{E}\big(W^r; P_\delta\big)_C$ in powers of $\frac{1}{\delta}$ as $\delta \to \infty$ was obtained by Baskakov [2] in the case of $r = 1, 2, 3$ and by Kharkevych, Kal'chuk [5] for any natural r. Later, the Kolmogorov–Nikol'skii problem for the Poisson integral on classes of differentiable functions was solved in works [7, 8, 15, 18–21]. Simultaneously, approximation properties of the method of approximation by Poisson integrals on classes of conjugate functions are studied not enough.

Note that the first estimates of $\mathcal{E}\big(\overline{W}^1; P_\rho\big)_C$ were obtained by Nagy [13]. Later, the general expressions that allow one to get asymptotic expansions of the quantity $\mathcal{E}\big(\overline{W}^r; P_\delta\big)_C$ in powers of $\frac{1}{\delta}$ as $\delta \to \infty$ were determined by Baskakov [1].

The present paper is an extension of the paper [6], where the corresponding results for the classes $\mathrm{Lip}_1 1$ were obtained in terms of $(1-\rho)$. In what follows, we establish a complete asymptotic expansion of the quantity

$$\mathcal{E}\big(\mathrm{Lip}_1\,\alpha; \overline{P}_\rho\big)_C = \sup_{f \in \mathrm{Lip}_1 \alpha} \big\|\bar{f}(\cdot) - \overline{P}_\rho(f;\cdot)\big\|_C, \qquad 0 < \alpha \le 1.$$

This expansion allows one to write down the Kolmogorov–Nikol'skii constants of an arbitrary order.

2 MAIN RESULTS

The following statement is true.

Theorem 1. *For* $0 < \alpha \le 1$ *the following complete asymptotic expansion holds as* $\rho \to 1-$

$$\begin{aligned}\mathcal{E}(\mathrm{Lip}_1\ \alpha; \overline{P}_\rho)_C = {} & \frac{2^{\alpha-1}}{\sin\frac{\alpha\pi}{2}}\left(\ln\frac{1}{\rho}\right)^{\alpha} - \frac{2^\alpha}{\pi}\sum_{k=0}^{\infty}(-1)^k\frac{\left(\frac{2}{\pi}\right)^{2(k+1)-\alpha}}{2k-\alpha+2}\left(\ln\frac{1}{\rho}\right)^{2k+2} \\ & + \frac{2^\alpha}{\pi\alpha}\sum_{k=0}^{\infty}(-1)^k\left(\ln\frac{1}{\rho}\right)^{2k+2}\sum_{i=1}^{\infty}\int_0^{\left(\frac{\pi}{2}\right)^\alpha}\Bigg(\left((2i-1)\pi - u^{\frac{1}{\alpha}}\right)^{-2k-3} \\ & - \left((2i-1)\pi + u^{\frac{1}{\alpha}}\right)^{-2k-3} + \left(2\pi i + u^{\frac{1}{\alpha}}\right)^{-2k-3} - \left(2\pi i - u^{\frac{1}{\alpha}}\right)^{-2k-3}\Bigg)u^{\frac{1}{\alpha}}\,\mathrm{d}u.\end{aligned} \tag{4}$$

Proof. Note first, that the kernel of the conjugate Poisson integral (3) can be rewritten as

$$\overline{K}_\rho(t) = \frac{1}{2}\cot\frac{t}{2} - \frac{1}{2}\cot\frac{t}{2}\cdot\frac{(1-\rho)^2}{1-2\rho\cos t+\rho^2}.$$

Whence, in view of 2π-periodicity of functions f, we get

$$\overline{P}_\rho(f,x) - \bar{f}(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+t)\cot\frac{t}{2}\cdot\frac{(1-\rho)^2}{1-2\rho\cos t+\rho^2}\,dt$$

$$= \frac{1}{2\pi}\int\limits_0^{\pi}(f(x+t)-f(x-t))\cot\frac{t}{2}\cdot\frac{(1-\rho)^2}{1-2\rho\cos t+\rho^2}\,dt$$

$$= \frac{1}{2\pi}\int\limits_0^{\frac{\pi}{2}}(f(x+t)-f(x-t))\cot\frac{t}{2}\cdot\frac{(1-\rho)^2}{1-2\rho\cos t+\rho^2}\,dt$$

$$-\frac{1}{2\pi}\int\limits_{\frac{\pi}{2}}^{\pi}(f(2\pi+x-t)-f(x+t))\cot\frac{t}{2}\cdot\frac{(1-\rho)^2}{1-2\rho\cos t+\rho^2}\,dt.$$

For $0\le t\le\pi$

$$\cot\frac{t}{2}\cdot\frac{(1-\rho)^2}{1-2\rho\cos t+\rho^2}\ge 0,$$

functions f belong the class $\mathrm{Lip}_1\ \alpha$, therefore it holds

$$\begin{aligned}|\overline{P}_\rho(f,x)-\bar f(x)| \le{}& \frac{2^\alpha}{2\pi}\int\limits_0^{\frac{\pi}{2}} t^\alpha\cot\frac{t}{2}\cdot\frac{(1-\rho)^2}{1-2\rho\cos t+\rho^2}\,dt\\ &+\frac{2^\alpha}{2\pi}\int\limits_{\frac{\pi}{2}}^{\pi}(\pi-t)^\alpha\cot\frac{t}{2}\cdot\frac{(1-\rho)^2}{1-2\rho\cos t+\rho^2}\,dt.\end{aligned}\tag{5}$$

Let $[g(t)]_{2\pi}$ be an odd 2π-periodic extension of the function g of the form

$$g(t)=g(\alpha,t)=\begin{cases}t^\alpha, & 0\le t\le\frac{\pi}{2},\\ (\pi-t)^\alpha, & \frac{\pi}{2}\le t\le\pi.\end{cases}\tag{6}$$

The function $f^*(t):=2^{\alpha-1}\,[g(t)]_{2\pi}$ belongs to the class $\mathrm{Lip}_1\ \alpha$, and we can see that the right hand side of (5) coincides with $\overline{P}_\rho(f^*,0)-\bar f^*(0)$. Indeed, taking into account that f^* is odd, we get

$$\begin{aligned}\overline{P}_\rho(f^*,0)-\bar f^*(0) &= \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} f^*(t)\cot\frac{t}{2}\cdot\frac{(1-\rho)^2}{1-2\rho\cos t+\rho^2}\,dt\\ &=\frac{1}{\pi}\int\limits_0^{\pi} f^*(t)\cot\frac{t}{2}\cdot\frac{(1-\rho)^2}{1-2\rho\cos t+\rho^2}\,dt.\end{aligned}\tag{7}$$

The right hand side of (7) coincides with the right hand side of (5). Hence

$$\mathcal{E}(\mathrm{Lip}_1\ \alpha;\overline{P}_\rho)_C=|\overline{P}_\rho(f^*,0)-\bar f^*(0)|.\tag{8}$$

Let us rewrite the kernel of the conjugate Poisson integral (3) in the following form

$$\overline{K}_\rho(t)=\sum_{k=1}^{\infty}e^{\ln\rho^k}\sin kt=\sum_{k=1}^{\infty}e^{-\ln\left(\frac{1}{\rho}\right)k}\sin kt.\tag{9}$$

It is known, that the Fourier cosine transform of the function $e^{-\beta t}$ takes the form [3, Ch. VII],

$$\Phi_c(u)=\frac{1}{\sqrt{2\pi}}\left\{\frac{t+u}{\beta^2+(t+u)^2}+\frac{t-u}{\beta^2+(t-u)^2}\right\}.\tag{10}$$

Further we shall need the Poisson formula [16, Ch. II]:

$$\sqrt{\gamma}\left(\frac{\Phi_c(0)}{2}+\sum_{n=1}^{\infty}\Phi_c(n\gamma)\right)=\sqrt{\omega}\left(\frac{f(0)}{2}+\sum_{n=1}^{\infty}f(n\omega)\right), \tag{11}$$

where $\omega\gamma = 2\pi$, $\omega > 0$. Setting $\omega = 1$, $\gamma = 2\pi$ in (11) and taking into account (10) with $\beta = \ln\frac{1}{\rho}$, from (9) we obtain

$$\begin{aligned}\overline{K}_\rho(t) &= \frac{1}{2}\left(e^{-\ln(\frac{1}{\rho})k}\sin kt\right)\Big|_{k=0}+\sum_{k=1}^{\infty}\left(e^{-\ln(\frac{1}{\rho})k}\sin kt\right)=\sqrt{2\pi}\left\{\frac{1}{2}\Phi_c(0)+\sum_{k=1}^{\infty}\Phi_c(2\pi k)\right\}\\ &=\frac{t}{\beta^2+t^2}+\sum_{k=1}^{\infty}\left(\frac{t+2\pi k}{\beta^2+(t+2\pi k)^2}+\frac{t-2\pi k}{\beta^2+(t-2\pi k)^2}\right).\end{aligned} \tag{12}$$

Therefore, combining (2) and (12), we can write the conjugate Poisson integral in the following equivalent form

$$\begin{aligned}\overline{P}_\rho(f^*,0) = &-\frac{1}{\pi}\int_{-\pi}^{\pi} f^*(t)\frac{t}{\beta^2+t^2}\,dt-\frac{1}{\pi}\int_{-\pi}^{\pi} f^*(t)\sum_{k=1}^{\infty}\frac{t+2\pi k}{\beta^2+(t+2\pi k)^2}\,dt\\ &-\frac{1}{\pi}\int_{-\pi}^{\pi} f^*(t)\sum_{k=1}^{\infty}\frac{t-2\pi k}{\beta^2+(t-2\pi k)^2}\,dt = I_1-I_2-I_3.\end{aligned} \tag{13}$$

Now we proceed to calculating of the term I_2:

$$\begin{aligned}I_2 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f^*(t)\sum_{k=1}^{\infty}\frac{t+2\pi k}{\beta^2+(t+2\pi k)^2}\,dt=\frac{1}{\pi}\int_{-\pi}^{\pi} f^*(t)\frac{t+2\pi}{\beta^2+(t+2\pi)^2}\,dt\\ &+\frac{1}{\pi}\int_{-\pi}^{\pi} f^*(t)\frac{t+4\pi}{\beta^2+(t+4\pi)^2}\,dt+\cdots = I_{2,1}+I_{2,2}+\cdots.\end{aligned} \tag{14}$$

Making appropriate substitutions in $I_{2,1}, I_{2,2}, \cdots$, we get

$$I_{2,1}=\frac{1}{\pi}\int_{\pi}^{3\pi} f^*(t)\frac{t}{\beta^2+t^2}\,dt, \qquad I_{2,2}=\frac{1}{\pi}\int_{3\pi}^{5\pi} f^*(t)\frac{t}{\beta^2+t^2}\,dt, \cdots .$$

Hence, from (14), we obtain

$$I_2=\frac{1}{\pi}\int_{\pi}^{+\infty} f^*(t)\frac{t}{\beta^2+t^2}\,dt. \tag{15}$$

One can verify that the term I_3 takes the following form

$$I_3=\frac{1}{\pi}\int_{-\pi}^{\pi} f^*(t)\sum_{k=1}^{\infty}\frac{t-2\pi k}{\beta^2+(t-2\pi k)^2}\,dt=\frac{1}{\pi}\int_{-\infty}^{-\pi} f^*(t)\frac{t}{\beta^2+t^2}\,dt. \tag{16}$$

Combining (13) with (15) and (16), we obtain

$$\overline{P}_\rho(f^*,0)=-\frac{1}{\pi}\int_{-\infty}^{+\infty} f^*(t)\frac{t}{\beta^2+t^2}\,dt. \tag{17}$$

It is known [12, p. 93], that a conjugate for f function can be represented in the form

$$\overline{f}(x) = -\frac{1}{\pi}\int\limits_{-\infty}^{+\infty}\frac{f(x+t)}{t}\,dt. \tag{18}$$

Therefore, from (17) and (18), taking into account that the function f^* is odd, we get

$$\begin{aligned}\overline{P}_\rho(f^*,0)-\bar{f}^*(0) &= \frac{\beta^2}{\pi}\int\limits_{-\infty}^{+\infty}\frac{f^*(t)}{t(\beta^2+t^2)}\,dt = \frac{\beta^2}{\pi}\left(\int\limits_{-\infty}^{0}+\int\limits_{0}^{+\infty}\right)\frac{f^*(t)}{t(\beta^2+t^2)}\,dt\\ &= \frac{2\beta^2}{\pi}\int\limits_{0}^{+\infty}f^*(t)\frac{1}{t(\beta^2+t^2)}\,dt.\end{aligned} \tag{19}$$

Hence, from (8) and (19), we have

$$\mathcal{E}(\mathrm{Lip}_1\ \alpha;\overline{P}_\rho)_C = \frac{2^\alpha\beta^2}{\pi}\int\limits_{0}^{+\infty}\frac{[g(t)]_{2\pi}}{t(\beta^2+t^2)}\,dt = \frac{2^\alpha\beta^2}{\pi}\left(\int\limits_{0}^{\frac{\pi}{2}}+\int\limits_{\frac{\pi}{2}}^{+\infty}\right)\frac{[g(t)]_{2\pi}}{t(\beta^2+t^2)}\,dt. \tag{20}$$

From (6) we get

$$\begin{aligned}\int\limits_{0}^{\frac{\pi}{2}}\frac{[g(t)]_{2\pi}}{t(\beta^2+t^2)}\,dt &= \int\limits_{0}^{\frac{\pi}{2}}\frac{t^\alpha}{t(t^2+\beta^2)}\,dt = \int\limits_{0}^{\frac{\pi}{2}}\frac{\left(\frac{t}{\beta}\right)^{\alpha-1}}{\left(\frac{t}{\beta}\right)^2+1}\beta^{\alpha-2}\,\mathrm{d}\left(\frac{t}{\beta}\right)\\ &= \beta^{\alpha-2}\left(\int\limits_{0}^{+\infty}-\int\limits_{\frac{\pi}{2\beta}}^{+\infty}\right)\frac{u^{\alpha-1}}{1+u^2}\,\mathrm{d}u.\end{aligned} \tag{21}$$

According to [4, p. 306]

$$\int\limits_{0}^{\infty}\frac{u^{\alpha-1}}{1+u^2}\,\mathrm{d}u = \frac{\pi}{2}\mathrm{cosec}\frac{\pi\alpha}{2}. \tag{22}$$

Let us make transformations in the second integral from the right-hand side of (21), applying geometric series

$$\begin{aligned}\int\limits_{\frac{\pi}{2\beta}}^{\infty}\frac{u^{\alpha-1}}{1+u^2}\,\mathrm{d}u &= \int\limits_{\frac{\pi}{2\beta}}^{\infty}\frac{1}{u^{3-\alpha}}\cdot\frac{1}{1-\left(-\frac{1}{u^2}\right)}\,\mathrm{d}u = \int\limits_{\frac{\pi}{2\beta}}^{\infty}\sum_{k=0}^{\infty}(-1)^k\frac{1}{u^{2k+3-\alpha}}\,\mathrm{d}u\\ &= \sum_{k=0}^{\infty}(-1)^k\frac{\left(\frac{2\beta}{\pi}\right)^{2k+2-\alpha}}{2k+2-\alpha}.\end{aligned} \tag{23}$$

Combining formulas (21), (22) and (23), we obtain

$$\int\limits_{0}^{\frac{\pi}{2}}\frac{[g(t)]_{2\pi}}{t(\beta^2+t^2)}\,dt = \left(\ln\frac{1}{\rho}\right)^{\alpha-2}\left(\frac{\pi}{2\sin\frac{\pi\alpha}{2}}-\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+2-\alpha}\left(\frac{2\ln\frac{1}{\rho}}{\pi}\right)^{2k+2-\alpha}\right). \tag{24}$$

Then, we use geometric series for calculating the second integral from the right-hand side of (20):

$$\int\limits_{\frac{\pi}{2}}^{+\infty} \frac{[g(t)]_{2\pi}}{t(\beta^2+t^2)}\,dt = \int\limits_{\frac{\pi}{2}}^{+\infty} \frac{[g(t)]_{2\pi}}{t^3\left(1+\frac{\beta^2}{t^2}\right)}\,dt = \sum_{k=0}^{\infty}(-1)^k\left(\ln\frac{1}{\rho}\right)^{2k}\int\limits_{\frac{\pi}{2}}^{+\infty}\frac{[g(t)]_{2\pi}}{t^{3+2k}}\,dt. \tag{25}$$

From (24) and (25), we get

$$\begin{aligned}\mathcal{E}(\mathrm{Lip}_1\,\alpha;\overline{P}_\rho)_C &= \left(2\ln\frac{1}{\rho}\right)^{\alpha}\frac{1}{2\sin\frac{\alpha\pi}{2}}\\ &+\frac{2^\alpha}{\pi}\sum_{k=0}^{\infty}(-1)^k\left(\ln\frac{1}{\rho}\right)^{2(k+1)}\left\{\int\limits_{\frac{\pi}{2}}^{+\infty}\frac{[g(t)]_{2\pi}}{t^{3+2k}}\,dt-\frac{\left(\frac{2}{\pi}\right)^{2k+2-\alpha}}{2k+2-\alpha}\right\}\,dt.\end{aligned}$$

For the function $[g(t)]_{2\pi}$ on $\left[\frac{\pi}{2};+\infty\right)$ the following relations hold

$$[g(t)]_{2\pi}=\begin{cases}(-t+(2i-1)\pi)^{\alpha}, & t\in\left[\frac{\pi}{2}+4(i-1)\frac{\pi}{2};\ \frac{\pi}{2}+(4i-3)\frac{\pi}{2}\right],\\ -(t-(2i-1)\pi)^{\alpha}, & t\in\left[\frac{\pi}{2}+(4i-3)\frac{\pi}{2};\ \frac{\pi}{2}+(4i-2)\frac{\pi}{2}\right],\\ -(-t+2\pi i)^{\alpha}, & t\in\left[\frac{\pi}{2}+(4i-2)\frac{\pi}{2};\ \frac{\pi}{2}+(4i-1)\frac{\pi}{2}\right],\\ (t-2\pi i)^{\alpha}, & t\in\left[\frac{\pi}{2}+(4i-1)\frac{\pi}{2};\ \frac{\pi}{2}+4i\frac{\pi}{2}\right],\end{cases}$$

where $i=1,2,\ldots$. Splitting the integral $\int\limits_{\frac{\pi}{2}}^{+\infty}\frac{[g(t)]_{2\pi}}{t^{3+2k}}\,dt$ into the sum of integrals and making corresponding substitutions in each of them, we get (4).

The theorem is proved. $\square$

Results of the theorem give us an opportunity to write down the Kolmogorov–Nikil'skii constants of an arbitrary order in asymptotic expansions in terms of $\ln\frac{1}{\rho}$ as $\rho\to 1-$.

Let us consider the class of functions $\mathrm{Lip}_1\,1$. The following statement holds.

Corollary 1. *The complete asymptotic expansion holds as $\rho\to 1-$*

$$\begin{aligned}\mathcal{E}(\mathrm{Lip}_1\,1;\overline{P}_\rho)_C &= \ln\frac{1}{\rho}-\frac{1}{\pi}\left(\ln\frac{1}{\rho}\right)^2+\sum_{k=1}^{\infty}\frac{(-1)^k}{(2k+1)(k+1)}\\ &\times\left(\sum_{i=1}^{\infty}\left(\frac{1}{(4i-1)^{2k+1}}-\frac{1}{(4i+1)^{2k+1}}\right)-1\right)\left(\frac{2}{\pi}\ln\frac{1}{\rho}\right)^{2k+2}.\end{aligned}\tag{26}$$

Proof. Putting $\alpha=1$ in (20), we obtain

$$\mathcal{E}(\mathrm{Lip}_1\,1;\overline{P}_\rho)_C=\frac{2\beta^2}{\pi}\int\limits_0^{+\infty}\frac{[g_1(t)]_{2\pi}}{t(\beta^2+t^2)}\,dt, \tag{27}$$

where $[g_1(t)]_{2\pi}$ is as it was introduced earlier, an odd 2π-periodic extension of the function g_1 of the form

$$g_1(t)=\begin{cases}t, & 0\le t\le\frac{\pi}{2},\\ \pi-t, & \frac{\pi}{2}\le t\le\pi.\end{cases}$$

Making transformations that are analogous to that in (21)–(25), from (27) we get

$$\mathcal{E}(\mathrm{Lip}_1\ 1; \overline{P}_\rho)_C = \ln\frac{1}{\rho} + \frac{2}{\pi}\sum_{k=0}^{\infty}(-1)^k\left(\ln\frac{1}{\rho}\right)^{2k+2} \times \left(\int_{\frac{\pi}{2}}^{+\infty}\frac{[g_1(t)]_{2\pi}}{t^{3+2k}}\,dt - \frac{1}{2k+1}\left(\frac{2}{\pi}\right)^{2k+1}\right). \tag{28}$$

To calculate the integral in the right-hand side of (28), let us write it down as a sum of the integrals on corresponding intervals. For this reason we use the following form of the function g_1:

$$[g_1(t)]_{2\pi} = \begin{cases} -t + (2i-1)\pi, & t \in \left[-\frac{\pi}{2} + (2i-1)\pi;\ -\frac{\pi}{2} + 2\pi i\right], \\ t - 2\pi i, & t \in \left[-\frac{\pi}{2} + 2\pi i;\ -\frac{\pi}{2} + (2i+1)\pi\right]. \end{cases}$$

We obtain

$$\int_{\frac{\pi}{2}}^{+\infty}\frac{[g_1(t)]_{2\pi}}{t^{3+2k}}\,dt = \sum_{i=1}^{\infty}\left(\int_{-\frac{\pi}{2}+(2i-1)\pi}^{-\frac{\pi}{2}+2\pi i}\frac{-t+(2i-1)\pi}{t^{3+2k}}\,dt + \int_{-\frac{\pi}{2}+2\pi i}^{-\frac{\pi}{2}+(2i+1)\pi}\frac{t-2\pi i}{t^{3+2k}}\,dt\right). \tag{29}$$

Having calculated the integral on the right-hand side of (29) and made corresponding transformations, we obtain

$$\int_{\frac{\pi}{2}}^{+\infty}\frac{[g_1(t)]_{2\pi}}{t^{3+2k}}\,dt = \left(\frac{2}{\pi}\right)^{2k+1}\frac{1}{(2k+1)(k+1)} \times \left(k + \sum_{i=1}^{\infty}\left(\frac{1}{(4i-1)^{2k+1}} - \frac{1}{(4i+1)^{2k+1}}\right)\right). \tag{30}$$

Taking into account (30), from (28) we get (26).

The Corollary 1 is proved. □

In the paper we have also obtained another form of the expansion (26), in terms of the generalized Riemann zeta function (the Hurwitz zeta function) (see definition, e.g., [17, Ch. XIII]). It is quite relevant because in approximation of functions by the Poisson integrals we obtain asymptotic expansions with non-explicit form of the coefficients. The Hurwitz zeta function gives a possibility to get sharp values of the Kolmogorov–Nikil'skii constants.

Corollary 2. *The complete asymptotic expansion holds as* $\rho \to 1-$

$$\mathcal{E}(\mathrm{Lip}_1\ 1; \overline{P}_\rho)_C = \ln\frac{1}{\rho} - \frac{1}{\pi}\left(\ln\frac{1}{\rho}\right)^2 + \sum_{k=1}^{\infty}\frac{(-1)^k}{(2k+1)(k+1)} \times \left(\frac{1}{4^{2k+1}}\left\{\zeta\left(2k+1;\frac{3}{4}\right) - \zeta\left(2k+1;\frac{5}{4}\right)\right\} - 1\right)\left(\frac{2}{\pi}\ln\frac{1}{\rho}\right)^{2k+2}, \tag{31}$$

where $\zeta(z;q) = \sum_{n=0}^{\infty}\frac{1}{(q+n)^z}$, $\mathrm{Re}\, z > 1$, *is the Hurwitz zeta function.*

Proof. Taking into account that

$$\sum_{i=1}^{\infty} \frac{1}{(4i-1)^{2k+1}} = \frac{1}{4^{2k+1}} \sum_{i=0}^{\infty} \frac{1}{(i+\frac{3}{4})^{2k+1}} = \frac{1}{4^{2k+1}} \zeta\left(2k+1; \frac{3}{4}\right),$$
$$\sum_{i=1}^{\infty} \frac{1}{(4i+1)^{2k+1}} = \frac{1}{4^{2k+1}} \zeta\left(2k+1; \frac{5}{4}\right),$$

from (30) we derive

$$\int_{\frac{\pi}{2}}^{+\infty} \frac{[g_1(t)]_{2\pi}}{t^{3+2k}}\, dt = \left(\frac{2}{\pi}\right)^{2k+1} \frac{1}{(2k+1)(k+1)} \times \left(k + \frac{1}{4^{2k+1}} \left(\zeta\left(2k+1; \frac{3}{4}\right) - \zeta\left(2k+1; \frac{5}{4}\right)\right)\right). \tag{32}$$

The Corollary 2 is proved. □

Note that (32) holds for $k = 1, 2, \dots$. In the case $k = 0$ the Hurwitz zeta function is not determined. Therefore the corresponding coefficient is individually calculated in the expansion (31).

Note also that Corollary 1 is a generalization of the B. Nagy result [13]. The estimation

$$\mathcal{E}(\mathrm{Lip}_1\, 1; \overline{P}_\rho)_C = \ln\frac{1}{\rho} + O\left(\left(\ln\frac{1}{\rho}\right)^2\right), \qquad \rho \to 1-$$

follows from Corollary 1 that coincides with the indicated result.

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Кальчук І.В., Харкевич Ю.І., Пожарська К.В. *Асимптотика наближення функцій спряженими інтегралами Пуассона* // Карпатські матем. публ. — 2020. — Т.12, №1. — C. 138–147.

Актуальними задачами теорії наближення функцій є розв'язання широкого кола екстремальних задач, зокрема, дослідження питань апроксимації функціональних класів різними лінійними методами підсумовування рядів Фур'є. В даній роботі розглядається відомий клас Ліпшиця $\mathrm{Lip}_1\alpha$, тобто клас неперервних 2π-періодичних функцій, що задовольняють умову Ліпшиця порядку α, $0 < \alpha \leq 1$, а в якості наближаючого оператора виступає спряжений інтеграл Пуассона. Досить актуальною задачею на даний час є можливість знаходження констант при асимптотичних доданках вказаного степеня малості (так званих констант Колмогорова–Нікольського) в асимптотичних розкладах величин наближень спряженими інтегралами Пуассона функцій з класу Ліпшиця в рівномірній метриці. В роботі отримано повні асимптотичні розклади для точних верхніх меж відхилень спряжених інтегралів Пуассона від функцій з класу $\mathrm{Lip}_1\alpha$. Дані розклади дають можливість записати константи Колмогорова–Нікольського довільного порядку малості.

Ключові слова і фрази: інтеграл Пуассона, асимптотичний розклад, спряжена функція, задача Колмогорова-Нікольського.

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APPROXIMATION OF THE NIKOL'SKII–BESOV FUNCTIONAL CLASSES BY ENTIRE FUNCTIONS OF A SPECIAL FORM

We establish the exact-order estimates for the approximation of functions from the Nikol'skii–Besov classes $S^r_{1,\theta}B(\mathbb{R}^d)$, $d \geq 1$, by entire functions of exponential type with some restrictions for their spectrum. The error of the approximation is estimated in the metric of the Lebesgue space $L_\infty(\mathbb{R}^d)$.

Key words and phrases: Nikol'skii–Besov classes, entire function of exponential type, Fourier transform.

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1 INTRODUCTION

In the paper, we continue to study the approximative characteristics of the Nikol'skii–Besov classes $S^r_{p,\theta}B(\mathbb{R}^d)$ of functions with a dominant mixed derivative in the Lebesgue spaces (see [4, 17, 18, 21, 23, 25]). We have established the order estimates of the best approximation of functions from these classes by entire functions of exponential type with a spectrum focused on the Lebesgue sets whose measure does not exceed M.

The spaces $S^r_{p,\theta}B(\mathbb{R}^d)$ were first considered by S. M. Nikol'skii [8] for $\theta = \infty$ (in this case also one can $S^r_{p,\infty}B(\mathbb{R}^d) \equiv S^r_pH(\mathbb{R}^d)$) and T. I. Amanov [1] for $1 \leq \theta < \infty$. In the classical form, the definition of these functional spaces was formulated by S. M. Nikol'skii and T. I. Amanov through mixed multiple differences and mixed multiple modules of continuity of functions. Here, the definition of the Nikol'skii–Besov spaces $S^r_{p,\theta}B(\mathbb{R}^d)$ is presented through so-called decomposition representation of the norm of elements from these spaces. Note that decomposition representation and corresponding rationing of the Nikol'skii–Besov spaces were first obtained by S. M. Nikol'skii and P. I. Lizorkin [5]. As it turned out, this decomposition norm of functions plays a key role in the studies of different approximative characteristics of the function classes. This representation is based on the application of the Fourier transform that can be defined using generalized functions (see, e.g., [2, Ch. 11], [6], [15, Ch. 2]).

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2 DEFINITION OF CLASSES OF FUNCTIONS AND APPROXIMATIVE CHARACTERISTICS

Let $\mathbb{R}^d$ be the d-dimensional Euclidean space with the elements $\boldsymbol{x} = (x_1, \ldots, x_d)$ and $(\boldsymbol{x}, \boldsymbol{y}) := x_1 y_1 + \cdots + x_d y_d$. Denote by $L_q(\mathbb{R}^d)$, $1 \le q \le \infty$, the space of all functions $f(\boldsymbol{x}) = f(x_1, \ldots, x_d)$ measurable on $\mathbb{R}^d$ with the finite norm

$$\|f\|_q := \left(\int\limits_{\mathbb{R}^d} |f(\boldsymbol{x})|^q d\boldsymbol{x} \right)^{\frac{1}{q}}, \ 1 \le q < \infty, \quad \text{and} \quad \|f\|_\infty := \operatorname*{ess\,sup}_{\boldsymbol{x} \in \mathbb{R}^d} |f(\boldsymbol{x})|.$$

Let $S = S(\mathbb{R}^d)$ be the Schwarz space of test complex-valued functions φ infinitely differentiable on $\mathbb{R}^d$ and decreasing at infinity together with their derivatives faster than any power of the function $\left(x_1^2 + \ldots + x_d^2\right)^{-\frac{1}{2}}$, considered in the appropriate topology. Let S' denote the space of linear continuous functionals on S. The elements of the space S' are generalized functions. If $f \in S'$, then $\langle f, \varphi \rangle$ denotes the value of a functional f on the test function $\varphi \in S$. Denote by $\mathfrak{F}\varphi$ and $\mathfrak{F}^{-1}\varphi$ the Fourier transform and the inverse Fourier transform of functions φ from the spaces S and S'.

For any continuous function φ on $\mathbb{R}^d$, the closure of the set of all points $\boldsymbol{x} \in \mathbb{R}^d$ such that $\varphi(\boldsymbol{x}) \neq 0$ is called the support of the function φ and denoted by $\operatorname{supp} \varphi$.

The generalized function f vanishes in an open set G when $\langle f, \varphi \rangle = 0$ for all $\varphi \in S$ and $\operatorname{supp} \varphi \subset G$. The union of all neighborhoods, where f is equal to zero, is an open set and called the null set of the generalized function f. It is denoted by G_f. The complement of the largest open set G_f to $\mathbb{R}^d$ is called the support of the generalized function f, i.e., $\operatorname{supp} f$ equals to $\bar{G}_f$, it is a closed set.

According to the formula

$$\langle f, \varphi \rangle = \int\limits_{\mathbb{R}^d} f(\boldsymbol{x}) \varphi(\boldsymbol{x}) d\boldsymbol{x}, \quad \varphi \in S, \tag{1}$$

each function $f \in L_p(\mathbb{R}^d)$, $1 \le p \le \infty$, defines a linear continuous functional on S and, therefore, is an element of S' in this sense. Hence, the Fourier transform of a function $f \in L_p(\mathbb{R}^d)$, $1 \le p \le \infty$, can be regarded as the Fourier transform of the generalized function (1).

Further, let $K_m(t) = \int_{\mathbb{R}} k_m(\lambda) e^{-2\pi i \lambda t} d\lambda$, $m \in \mathbb{Z}_+$, $K_{-1} := 0$, where

$$k_m(\lambda) = \begin{cases} 1, & |\lambda| < 2^{m-1}, \\ 2\left(1 - \frac{|\lambda|}{2^m}\right), & 2^{m-1} \le |\lambda| \le 2^m, \\ 0, & |\lambda| > 2^m, \end{cases} \qquad k_0(\lambda) = \begin{cases} 1 - |\lambda|, & 0 \le |\lambda| \le 1, \\ 0, & |\lambda| > 1. \end{cases}$$

For any vector $\boldsymbol{s} = (s_1, \ldots, s_d)$, $s_j \in \mathbb{Z}_+$, $j = \overline{1, d}$, we define

$$A_{\boldsymbol{s}}^*(\boldsymbol{x}) = \prod_{j=1}^{d} \left(K_{s_j}(x_j) - K_{s_j - 1}(x_j) \right),$$

$$A_{\boldsymbol{s}}^*(f, \boldsymbol{x}) = f(\boldsymbol{x}) * A_{\boldsymbol{s}}^*(\boldsymbol{x}) = \int\limits_{\mathbb{R}^d} f(\boldsymbol{y}) A_{\boldsymbol{s}}^*(\boldsymbol{x} - \boldsymbol{y}) d\boldsymbol{y}.$$

Also, for all $\boldsymbol{s} \in \mathbb{Z}_+^d$, consider the sets

$$Q_{2^{\boldsymbol{s}}}^* = \left\{ \boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_d) \colon \ \eta(s_j) 2^{s_j - 1} \le |\lambda_j| < 2^{s_j}, \lambda_j \in \mathbb{R}, \ j = \overline{1, d} \right\},$$

where $\eta(0) = 0$ and $\eta(t) = 1,\ t > 0$ (respectively, $Q^*_{2^s}$ for $d = 1$).

The following statement is true.

Lemma 1 (see example [4]). *Let $1 \le p \le \infty$, then for any $f \in L_p(\mathbb{R}^d)$, we have*

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{s}} A^*_{\boldsymbol{s}}(f, \boldsymbol{x})$$

and supp $\mathfrak{F} A_{\boldsymbol{s}}(f, \boldsymbol{x}) \subseteq Q^*_{2^s}$.

Note that $A^*_{\boldsymbol{s}}(f, \boldsymbol{x})$ is the analog of the de la Vallée Poussin block of sum of periodic function of several variables (see example [14]).

In the accepted notation, the spaces $S^{\boldsymbol{r}}_{p,\theta}B(\mathbb{R}^d)$, $1 \le p, \theta \le \infty$, $\boldsymbol{r} > 0$, can be defined as follows (see, e.g., [4, 16]):

$$S^{\boldsymbol{r}}_{p,\theta}B(\mathbb{R}^d) := \Big\{ f \in L_p(\mathbb{R}^d) \colon \|f\|_{S^{\boldsymbol{r}}_{p,\theta}B} < \infty \Big\},$$

where for $1 \le \theta < \infty$,

$$\|f\|_{S^{\boldsymbol{r}}_{p,\theta}B(\mathbb{R}^d)} \asymp \left(\sum_{\boldsymbol{s} \ge 0} 2^{(\boldsymbol{s},\boldsymbol{r})\theta} \|A^*_{\boldsymbol{s}}(f, \cdot)\|^\theta_p \right)^{\frac{1}{\theta}} \tag{2}$$

and for $\theta = \infty$,

$$\|f\|_{S^{\boldsymbol{r}}_{p,\infty}B(\mathbb{R}^d)} := \|f\|_{S^{\boldsymbol{r}}_{p}H(\mathbb{R}^d)} \asymp \sup_{\boldsymbol{s} \ge 0} 2^{(\boldsymbol{s},\boldsymbol{r})} \|A^*_{\boldsymbol{s}}(f, \cdot)\|_p. \tag{3}$$

Here and below, for positive quantities a and b, the notation $a \asymp b$ means that there exist positive constants C_1 and C_2 that do not depend on an essential parameter in the values a and b (e.g., C_1 and C_2 in the expressions (2) and (3) do not depend on the function f) such that $C_1 a \le b$ (in this case, we write $a \ll b$) and $C_2 a \ge b$ (in this case, we write $a \gg b$). In the present paper, all constants C_i, $i = 1, 2, \dots$, depend only on the parameters contained in the definition of the function class, the metric in which we estimate the error of approximation, and the dimension of the space $\mathbb{R}^d$. Moreover, for the vectors $\boldsymbol{a} = (a_1, \dots, a_d)$ and $\boldsymbol{b} = (b_1, \dots, b_d)$, the inequalities of the type $\boldsymbol{a} \le \boldsymbol{b}$ ($\boldsymbol{a} > \boldsymbol{b}$) are understood in the coordinate-wise: $a_j \le b_j$ ($a_j > b_j$), $j = \overline{1, d}$. We also use $\boldsymbol{t} \ge 0$ ($\boldsymbol{t} > 0$) if $t_j \ge 0$ ($t_j > 0$), $j = \overline{1, d}$, and $\boldsymbol{a} \ne \boldsymbol{b}$ if $a_i \ne b_i$ at least for one i, $i = \overline{1, d}$.

In what follows, we use the notations $S^{\boldsymbol{r}}_{p,\theta}B$ and $S^{\boldsymbol{r}}_{p}H$ ($S^{r}_{p,\theta}B$ and $S^{r}_{p}H$ for $d = 1$) instead of $S^{\boldsymbol{r}}_{p,\theta}B(\mathbb{R}^d)$ and $S^{\boldsymbol{r}}_{p}H(\mathbb{R}^d)$ respectively. We also assume that the coordinates of the vector $\boldsymbol{r} = (r_1, \dots, r_d)$ are ordered as follows $0 < r_1 = r_2 = \dots = r_\nu < r_{\nu+1} \le \dots \le r_d$. The vector $\boldsymbol{r} = (r_1, \dots, r_d)$ is associated with the vector $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_d)$, $\gamma_j = r_j / r_1$, $j = \overline{1, d}$, and the vector $\boldsymbol{\gamma}$ is, in turn, associated, with the vector $\boldsymbol{\gamma}'$, where $\gamma'_j = \gamma_j$, if $j = \overline{1, \nu}$ and $1 < \gamma'_j < \gamma_j$, $j = \overline{\nu + 1, d}$.

In addition, in the case $1 < p < \infty$, the norm of functions from the spaces $S^{\boldsymbol{r}}_{p,\theta}B(\mathbb{R}^d)$ can be defined in another form. Let $A \subset \mathbb{R}^d$ be a measurable set. Denote by χ_A a characteristic function of the set A and for $f \in L_p(\mathbb{R}^d)$, set $\delta^*_{\boldsymbol{s}}(f, \boldsymbol{x}) = \mathfrak{F}^{-1}(\chi_{Q^*_{2^s}} \cdot \mathfrak{F} f)$. The spaces $S^{\boldsymbol{r}}_{p,\theta}B$, $1 < p < \infty$, $1 \le \theta \le \infty$, $\boldsymbol{r} > 0$, can be defined as follows [5]

$$S^{\boldsymbol{r}}_{p,\theta}B := \Big\{ f \in L_p(\mathbb{R}^d) \colon \|f\|_{S^{\boldsymbol{r}}_{p,\theta}B} < \infty \Big\},$$

where

$$\|f\|_{S^{r}_{p,\theta}B} \asymp \left(\sum_{s\geq 0} 2^{(s,r)\theta} \|\delta^*_s(f,\cdot)\|^\theta_p \right)^{\frac{1}{\theta}} \tag{4}$$

for $1 \leq \theta < \infty$ and

$$\|f\|_{S^r_p H} \asymp \sup_{s\geq 0} 2^{(s,r)} \|\delta^*_s(f,\cdot)\|_p. \tag{5}$$

The class $S^r_{p,\theta}B$ is defined as a set of functions $f \in L_p(\mathbb{R}^d)$ such that $\|f\|_{S^r_{p,\theta}B} \leq 1$. We preserve the same notations for the classes $S^r_{p,\theta}B$ as for the spaces $S^r_{p,\theta}B$.

As can be seen from (2)–(5), for any $f \in S^r_{p,\theta}B$, $1 < p < \infty$, the following relation holds:

$$\|\delta^*_s(f,\cdot)\|_p \asymp \|A^*_s(f,\cdot)\|_p.$$

Now we consider the approximative characteristics of the classes $S^r_{p,\theta}B$.

Let $\mathcal{L} \subset \mathbb{Z}^d_+$ be a finite set, $\mathfrak{M} := \mathfrak{M}(\mathcal{L}) = \bigcup_{s\in\mathcal{L}} Q^*_{2^s}$. For any $f \in L_q(\mathbb{R}^d)$, $1 \leq q \leq \infty$, we put

$$S_{\mathfrak{M}}(f,x) := S_{\mathfrak{M}(\mathcal{L})}(f,x) = \sum_{s\in\mathcal{L}} \delta^*_s(f,x).$$

Since supp $S_{\mathfrak{M}}(f,x) \subseteq \mathfrak{M}$, then $S_{\mathfrak{M}}(f,x)$ is an entire function of the space $L_q(\mathbb{R}^d)$.

For $f \in L_q(\mathbb{R}^d)$ and $S^r_{p,\theta}B(\mathbb{R}^d) \subset L_q(\mathbb{R}^d)$, consider the following approximative characteristic

$$e^{\mathfrak{F}}_M(f)_q := \inf_{\mathcal{L}\colon \mathrm{mes}\,\mathfrak{M}(\mathcal{L})\leq M} \left\| f(\cdot) - S_{\mathfrak{M}(\mathcal{L})}(f,\cdot) \right\|_q$$

and

$$e^{\mathfrak{F}}_M(S^r_{p,\theta}B)_q := \sup_{f\in S^r_{p,\theta}B} e^{\mathfrak{F}}_M(f)_q. \tag{6}$$

3 Approximation of functions from classes $S^r_{1,\theta}B(\mathbb{R}^d)$ by entire functions

The following statements are true.

Theorem 1. *Let $r > 1$, $1 \leq \theta \leq \infty$ and $d = 1$. Then the following relation holds:*

$$e^{\mathfrak{F}}_M(S^r_{1,\theta}B(\mathbb{R}))_\infty \asymp M^{-r+1}. \tag{7}$$

Theorem 2. *Let $r_1 > 1$, $1 \leq \theta \leq \infty$. Then for $d \geq 2$ the following relation holds:*

$$e^{\mathfrak{F}}_M(S^r_{1,\theta}B(\mathbb{R}^d))_\infty \asymp (M^{-1}\log^{\nu-1} M)^{r_1-1}(\log^{\nu-1} M)^{1-\frac{1}{\theta}}. \tag{8}$$

The results of Theorems 1 and 2 are also new for Nikol'skii classes $S^r_1 H(\mathbb{R}^d)$, $d \geq 1$.

Let us note that in Theorem 1, the estimate $e^{\mathfrak{F}}_M(S^r_{1,\theta}B(\mathbb{R}))_\infty$ does not depend on the parameter θ unlike to the corresponding estimate in the case $d \geq 2$ (Theorem 2).

Before proving the main results, we formulate auxiliary theorem.

Theorem 3 ([1]). *Let $1 \leq p,\theta \leq \infty$, $1 \leq p \leq q \leq \infty$ and we have a vector ρ such that $\rho_j = r_j - \left(\frac{1}{p} - \frac{1}{q}\right) > 0$, $j = \overline{1,d}$. If $f \in S^r_{p,\theta}B(\mathbb{R}^d)$, then $f \in S^\rho_{q,\theta}B(\mathbb{R}^d)$ and*

$$\|f\|_{S^\rho_{q,\theta}B(\mathbb{R}^d)} \ll \|f\|_{S^r_{p,\theta}B(\mathbb{R}^d)}.$$

Proof of Theorem 1. Since $r > 1$, then by virtue of Theorem 3, there exists a number ρ, $\rho = r - 1 > 0$, such that for any function $f \in S^{r}_{1,\theta}B(\mathbb{R})$, we have $f \in S^{\rho}_{\infty,\theta}B(\mathbb{R}) \subset L_\infty(\mathbb{R})$.

First, we will get the upper estimate in (7). Recall now the definition of another approximative characteristic used in the proof of the results. For $\boldsymbol{s} \in \mathbb{Z}^d_+$, define the set Q^γ_n as follows

$$Q^\gamma_n = \bigcup_{(\boldsymbol{s},\gamma)\le n} Q^*_{2^{\boldsymbol{s}}},$$

where $n \in \mathbb{N}$. The set Q^γ_n is called a stepwise hyperbolic cross and, moreover, mes $Q^\gamma_n \asymp 2^n n^{d-1}$ (see, e.g., [5]), where mes Q^γ_n is the Lebesgue measure of the set Q^γ_n.

For $f \in L_q(\mathbb{R}^d)$, $1 \le q \le \infty$, we set

$$S_{Q^\gamma_n}(f,\boldsymbol{x}) = \sum_{(\boldsymbol{s},\gamma)\le n} \delta^*_{\boldsymbol{s}}(f,\boldsymbol{x}), \ \ \boldsymbol{x} \in \mathbb{R}^d$$

and denote

$$\mathcal{E}_{Q^\gamma_n}(f)_q = \|f(\cdot) - S_{Q^\gamma_n}(f,\cdot)\|_q \quad \text{and} \quad \mathcal{E}_{Q^\gamma_n}(S^{\boldsymbol{r}}_{p,\theta}B)_q = \sup_{f\in S^{\boldsymbol{r}}_{p,\theta}B} \mathcal{E}_{Q^\gamma_n}(f)_q. \tag{9}$$

We now specificating the definition of the quantity $\mathcal{E}_{Q^\gamma_n}(f)_q$ in the one-dimensional case.

For $d = 1$, each of the sets $Q^*_{2^s}$ is a union of the half intervals $(-2^s, -2^{s-1}]$ and $[2^{s-1}, 2^s)$, $s \in \mathbb{Z}_+$, with the corresponding modification at $s = 0$. Then the stepwise hyperbolic cross degenerates into the interval $(-2^n, 2^n)$, as the union of sets $Q^*_{2^s}$ for all $s \le n$, $s \in \mathbb{Z}_+$, namely $Q_n := Q^\gamma_n = \bigcup_{s\le n} Q^*_{2^s}$. In addition we have $|Q_n| \asymp 2^n$, where $|Q_n|$ denotes the length of the interval.

The definition of (9) for $f \in L_q(\mathbb{R})$, $1 \le q \le \infty$, can be rewritten as follows

$$\mathcal{E}_{Q_n}(f)_q = \|f(\cdot) - S_{Q_n}(f,\cdot)\|_q, \ \ \mathcal{E}_{Q_n}(S^r_{p,\theta}B)_q = \sup_{f\in S^r_{p,\theta}B} \mathcal{E}_{Q_n}(f)_q,$$

where

$$S_{Q_n}(f,x) = \sum_{s\le n} \delta^*_s(f,x).$$

From the definition of the approximative characteristics (6) and (9), it follows that the following relation holds in the case when mes $Q^\gamma_n \asymp$ mes $\mathfrak{M}$

$$e^{\mathfrak{F}}_M\big(S^{\boldsymbol{r}}_{p,\theta}B\big)_q \ll \mathcal{E}_{Q^\gamma_n}\big(S^{\boldsymbol{r}}_{p,\theta}B\big)_q. \tag{10}$$

The following statement is true.

Theorem 4 ([25]). *Let $r_1 > 1$, $1 \le \theta \le \infty$. Then the following relation holds:*

$$\mathcal{E}_{Q^\gamma_n}\big(S^{\boldsymbol{r}}_{1,\theta}B\big)_\infty \asymp 2^{-n(r_1-1)} n^{(\nu-1)\left(1-\frac{1}{\theta}\right)}. \tag{11}$$

In the case $d = 1$, the estimate (11) can be written as follows

$$\mathcal{E}_{Q_n}\big(S^r_{1,\theta}B\big)_\infty \asymp 2^{-n(r-1)}. \tag{12}$$

For a given M, we choose a number $n \in \mathbb{N}$ such that $|Q_n| \le M < |Q_{n+1}|$, i.e. $M \asymp 2^n$. Taking into account (10), from relation (12) we get the upper estimate in (7)

$$e^{\mathfrak{F}}_M\big(S^r_{1,\theta}B\big)_\infty \ll \mathcal{E}_{Q_n}\big(S^r_{1,\theta}B\big)_\infty \asymp 2^{-n(r-1)} \asymp M^{-r+1}.$$

To obtain the lower estimate in (7), for any $n \in \mathbb{N}$, consider the function

$$f_1(x) = C_3 2^{-nr} A_n^*(x), \quad C_3 > 0,$$

that is, the function f_1 consists of one "block" $A_n^*(x)$.

We give some auxiliary statements.

Lemma 2 ([25]). *Let $1 \le p < \infty$, then the estimate*

$$\|A_{\boldsymbol{s}}^*(\cdot)\|_p \asymp 2^{\|\boldsymbol{s}\|_1\left(1-\frac{1}{p}\right)}$$

holds, where $\|\boldsymbol{s}\|_1 = s_1 + \ldots + s_d$, $s_j \in \mathbb{Z}_+$, $j = \overline{1,d}$.

Lemma 3 ([25]). *The following relation*

$$\|A_{\boldsymbol{s}}^*(\cdot)\|_\infty \asymp 2^{\|\boldsymbol{s}\|_1}$$

holds, where $\|\boldsymbol{s}\|_1 = s_1 + \ldots + s_d$, $s_j \in \mathbb{Z}_+$, $j = \overline{1,d}$.

According to Lemma 2, we have $\left\|A_s^*(\cdot)\right\|_1 \asymp C_4$. Then for $1 \le \theta < \infty$,

$$\|f_1\|_{S^r_{1,\theta}B} \asymp \left(\sum_s 2^{sr\theta}\|A_s^*(f_1,\cdot)\|_1^\theta\right)^{\frac{1}{\theta}} \asymp \left(2^{nr\theta}2^{-nr\theta}\right)^{\frac{1}{\theta}} = 1$$

and

$$\|f_1\|_{S^r_{1,\infty}} \asymp \sup_s 2^{sr}\|A_s^*(f_1,\cdot)\|_1 \asymp 2^{nr}2^{-nr} = 1.$$

So, the function f_1 belongs to the class $S^r_{1,\theta}B$ for all $1 \le \theta \le \infty$.

For a given M, choosing a number $n \in \mathbb{N}$ such that $|\widetilde{Q}_n| \le 4M < |\widetilde{Q}_{n+1}|$, where $\widetilde{Q}_n = Q^*_{2^n}$, $|\widetilde{Q}_n| \asymp 2^n$, and using Lemma 3, we conclude that

$$\|f_1(\cdot) - S_{\mathfrak{M}}(f_1,\cdot)\|_\infty \ge \big|\|f_1(\cdot)\|_\infty - \|S_{\mathfrak{M}}(f_1,\cdot)\|_\infty\big| \gg 2^{-nr}(2^n - M) \gg 2^{-nr}\,2^n \asymp M^{-r+1}.$$

The lower estimate is established. Theorem 1 is proved. □

Before proving Theorem 2 we note that by Theorem 3 the condition $r_1 > 1$ ensures that there exists a vector $\boldsymbol{\rho}$, $\rho_j = r_j - 1 > 0$, $j = \overline{1,d}$, such that any function $f \in S^{\boldsymbol{r}}_{1,\theta}B(\mathbb{R}^d)$ belongs to the set $S^{\boldsymbol{\rho}}_{\infty,\theta}B(\mathbb{R}^d)$ and therefore $f \in L_\infty(\mathbb{R}^d)$. In addition, we can say that for some $1 < q_0 < \infty$, $f \in S^{\boldsymbol{\rho}}_{q_0,\theta}B$, where $\rho_j = r_j - \left(1 - \frac{1}{q_0}\right) > 0$, $j = \overline{1,d}$.

Proof of Theorem 2. The upper estimate in (8) follows from Theorem 4. Since $\operatorname{mes} Q_n^\gamma \ll 2^n n^{\nu-1}$, then for a given M, we choose a number $n \in \mathbb{N}$ such that $\operatorname{mes} Q_n^\gamma \le M < \operatorname{mes} Q_{n+1}^\gamma$, that is $M \asymp 2^n n^{\nu-1}$. Using relation (11), we have

$$e_M^{\mathfrak{F}}(S^{\boldsymbol{r}}_{1,\theta}B)_\infty \ll 2^{-n(r_1-1)}n^{(\nu-1)\left(1-\frac{1}{\theta}\right)} \asymp \left(M^{-1}\log^{\nu-1}M\right)^{r_1-1}\left(\log^{\nu-1}M\right)^{\left(1-\frac{1}{\theta}\right)}.$$

Passing to establishing the estimate from below in (8), we should note that it is sufficient to obtain it in the case $\nu = d$.

Let

$$\Theta(n) = \left\{\boldsymbol{s} = (s_1,\ldots,s_d) \in \mathbb{Z}^d\colon\ s_1 + \ldots + s_d = n\right\} \quad \text{and} \quad \widetilde{Q}_n = \bigcup_{\boldsymbol{s}\in\Theta(n)} Q^*_{2^{\boldsymbol{s}}},$$

and mes $\widetilde{Q}_n \asymp 2^n n^{\nu-1}$.

Unlike the one-dimensional case, we consider the following functions depending on the value of the parameter θ

$$f_2(\boldsymbol{x}) = C_5 2^{-nr_1} n^{-\frac{d-1}{\theta}} \sum_{\boldsymbol{s}\in\Theta(n)} A_{\boldsymbol{s}}^*(\boldsymbol{x}), \quad C_5 > 0,$$

when $1 \le \theta < \infty$, and

$$f_3(\boldsymbol{x}) = C_6 2^{-nr_1} \sum_{\boldsymbol{s}\in\Theta(n)} A_{\boldsymbol{s}}^*(\boldsymbol{x}), \quad C_6 > 0,$$

when $\theta = \infty$.

Let us show that the functions f_2 and f_3 belong to the classes $S^{\boldsymbol{r}}_{1,\theta}B$ and $S^{\boldsymbol{r}}_{1,\infty}B$ respectively. According to Lemma 2, we have $\left\|A_{\boldsymbol{s}}^*(\cdot)\right\|_1 \asymp C_7$. Then

$$\begin{aligned}\|f_2\|_{S^{\boldsymbol{r}}_{1,\theta}B} &\asymp \left(\sum_{\boldsymbol{s}\in\Theta(n)} 2^{(\boldsymbol{s},\boldsymbol{r})\theta} \|A_{\boldsymbol{s}}^*(f_2,\cdot)\|_1^\theta\right)^{\frac{1}{\theta}} \asymp 2^{-nr_1} n^{-\frac{d-1}{\theta}} \left(\sum_{\boldsymbol{s}\in\Theta(n)} 2^{(\boldsymbol{s},\boldsymbol{r})\theta} \|A_{\boldsymbol{s}}^*(\cdot)\|_1^\theta\right)^{\frac{1}{\theta}} \\ &\asymp 2^{-nr_1} n^{-\frac{d-1}{\theta}} \left(\sum_{\boldsymbol{s}\in\Theta(n)} 2^{r_1(\boldsymbol{s},\boldsymbol{1})\theta}\right)^{\frac{1}{\theta}} \ll n^{-\frac{d-1}{\theta}} \left(\sum_{\boldsymbol{s}\in\Theta(n)} 1\right)^{\frac{1}{\theta}} \ll 1.\end{aligned}$$

For the function f_3, the following estimates hold:

$$\|f_3\|_{S^{\boldsymbol{r}}_{1,\infty}} \asymp \sup_{\boldsymbol{s}\in\Theta(n)} 2^{(\boldsymbol{s},\boldsymbol{r})} \|A_{\boldsymbol{s}}^*(f_3,\cdot)\|_1 \asymp 2^{-nr_1} \sup_{\boldsymbol{s}\in\Theta(n)} 2^{(\boldsymbol{s},\boldsymbol{r})} \|A_{\boldsymbol{s}}^*(\cdot)\|_1 \asymp 2^{-nr_1} \sup_{(\boldsymbol{s},\boldsymbol{1})=n+1} 2^{(\boldsymbol{s},\boldsymbol{r})} \ll 1.$$

Further, denote by $\mathcal{L}'$ the set of vectors $\boldsymbol{s}$ such that $\boldsymbol{s} \in \Theta(n)$ and the set $\mathfrak{M} = \mathfrak{M}(\mathcal{L}') = \bigcup_{\boldsymbol{s}\in\mathcal{L}'} Q^*_{2^{\boldsymbol{s}}}$ satisfies the relation

$$\text{mes}\, \widetilde{Q}_n \le 4M < \text{mes}\, \widetilde{Q}_{n+1}, \tag{13}$$

where $M = M(n) = \text{mes}\, \mathfrak{M}$.

Lemma 4 ([25]). *The following relation holds:*

$$\left\| \sum_{(\boldsymbol{s},\boldsymbol{1})=n+1} A_{\boldsymbol{s}}^*(\cdot) \right\|_\infty \asymp 2^n n^{d-1}.$$

Using the Lemmas 3, 4 and relation (13), taking into account that mes $\widetilde{Q}_n \asymp 2^n n^{\nu-1}$, we can write

$$\begin{aligned}\|f_2(\cdot) - S_{\mathfrak{M}}(f_2,\cdot)\|_\infty &\ge \big| \|f_2(\cdot)\|_\infty - \|S_{\mathfrak{M}}(f_2,\cdot)\|_\infty \big| \\ &\gg 2^{-nr_1} n^{\frac{d-1}{\theta}} (2^n n^{d-1} - M) \gg 2^{-nr_1} n^{\frac{d-1}{\theta}} 2^n n^{d-1} \\ &= 2^{-n(r_1-1)} n^{(d-1)\left(1-\frac{1}{\theta}\right)} \asymp \left(M^{-1}\log^{d-1} M\right)^{r_1-1} \left(\log^{d-1} M\right)^{\left(1-\frac{1}{\theta}\right)}.\end{aligned}$$

Similarly in the case $\theta = \infty$, we get

$$\|f_3(\cdot) - S_{\mathfrak{M}}(f_3,\cdot)\|_\infty \gg \left(M^{-1}\log^{d-1} M\right)^{r_1-1} \log^{d-1} M.$$

The lower estimates are established. Theorem 2 is proved. □

The exact-order estimates of $e_M^{\mathfrak{F}}\big(S^{\boldsymbol{r}}_{p,\theta}B\big)_q$ are established in [21] for some other relations between parameters p, q and θ. In this article, we show that there are relations between the parameters p, q, θ such that the quantities $e_M^{\mathfrak{F}}\big(S^{\boldsymbol{r}}_{p,\theta}B\big)_q$ and $\mathcal{E}_{Q_n^\gamma}\big(S^{\boldsymbol{r}}_{p,\theta}B\big)_q$ have different orders.

The quantity (6) is a non-periodic analogue of the best orthogonal approximation and the quantity (9) corresponds to the approximation of the stepwise hyperbolic Fourier sum. The main results concerning the approximation of the Nikol'skii–Besov classes of periodic functions with a dominant mixed derivative can be found in monographs V. N. Temlyakov [14], A. S. Romanyuk [11] and D. Dũng, V. N. Temlyakov and T. Ullrich [3].

Currently, the generalizations of the Nikol'skii–Besov classes with the dominant mixed smoothness of periodic and non-periodic functions of many variables are currently being intensively studied, in particular, in the articles [7, 9, 10, 12, 13, 19, 26].

In the one-dimensional case, the Nikol'skii–Besov classes with mixed smoothness $S^{\boldsymbol{r}}_{p,\theta}B(\mathbb{R}^d)$ coincide with isotropic and anisotropic Nikol'skii–Besov classes $B^r_{p,\theta}(\mathbb{R}^d)$ and $B^{\boldsymbol{r}}_{p,\theta}(\mathbb{R}^d)$. The exact-order estimates of some approximate characteristics of these classes are established in [20, 22, 24].

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Одержано точні за порядком оцінки наближення функцій з класів Нікольського–Бєсова $S^{\mathbf{r}}_{1,\theta}B(\mathbb{R}^d)$, $d \geq 1$, за допомогою цілих функцій експоненціального типу з певними обмеженнями на їхній спектр. Похибка наближення оцінюється у метриці простору Лебега $L_\infty(\mathbb{R}^d)$.

Ключові слова і фрази: класи Нікольського–Бєсова, ціла функція експоненціального типу, перетворення Фур'є.

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ON CONVERGENCE CRITERIA FOR BRANCHED CONTINUED FRACTION

The starting point of the present paper is a result by E.A. Boltarovych (1989) on convergence regions, dealing with branched continued fraction

$$\sum_{i_1=1}^{N}\frac{a_{i(1)}}{1}+\sum_{i_2=1}^{N}\frac{a_{i(2)}}{1}+\cdots+\sum_{i_n=1}^{N}\frac{a_{i(n)}}{1}+\cdots,$$

where $|a_{i(2n-1)}|\le\alpha/N$, $i_p=\overline{1,N}$, $p=\overline{1,2n-1}$, $n\ge1$, and for each multiindex $i(2n-1)$ there is a single index j_{2n}, $1\le j_{2n}\le N$, such that $|a_{i(2n-1),j_{2n}}|\ge R$, $i_p=\overline{1,N}$, $p=\overline{1,2n-1}$, $n\ge1$, and $|a_{i(2n)}|\le r/(N-1)$, $i_{2n}\ne j_{2n}$, $i_p=\overline{1,N}$, $p=\overline{1,2n}$, $n\ge1$, where $N>1$ and α, r, R are real numbers that satisfying certain conditions. In the present paper, conditions for these regions are replaced by $\sum_{i_1=1}^{N}|a_{i(1)}|\le\alpha(1-\varepsilon)$, $\sum_{i_{2n+1}=1}^{N}|a_{i(2n+1)}|\le\alpha(1-\varepsilon)$, $i_p=\overline{1,N}$, $p=\overline{1,2n}$, $n\ge1$, and for each multiindex $i(2n-1)$ there is a single index j_{2n}, $1\le j_{2n}\le N$, such that $|a_{i(2n-1),j_{2n}}|\ge R$ and $\sum_{i_{2n}\in\{1,2,\ldots,N\}\setminus\{j_{2n}\}}|a_{i(2n)}|\le r$, $i_p=\overline{1,N}$, $p=\overline{1,2n-1}$, $n\ge1$, where ε, α, r and R are real numbers that satisfying certain conditions, and better convergence speed estimates are obtained.

Key words and phrases: convergence, convergence region, convergence speed estimate, branched continued fraction.

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1 INTRODUCTION

All known general methods of proof of convergence criteria of continued fractions are based on value-region considerations. The interplay between element regions and value regions leads to convergence region criteria, that is, results of the form: if the elements of continued fraction lie in some regions then the continued fraction converges. In addition, the relationship between element regions and value regions provides one with knowledge of the location of approximants of continued fraction whose elements lie in some convergence regions. Both of these phenomena (i.e., the convergence regions and the information about the location of approximants) are not to be found for most common infinite processes, such as series and products [15, pp. 63–78].

It is well know (see, for example, [7]) that branched continued fractions (BCF) are multidimensional generalization of continued fractions. Let N be a fixed natural number. For BCF with the complex elements

$$\sum_{i_1=1}^{N}\frac{a_{i(1)}}{1}+\sum_{i_2=1}^{N}\frac{a_{i(2)}}{1}+\cdots+\sum_{i_n=1}^{N}\frac{a_{i(n)}}{1}+\cdots,\tag{1}$$

E.A. Boltarovych [9] proved the following theorem.

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Theorem 1. *Let $N > 1$ and let there exist real numbers α, r and R such that $0 \le \alpha \le 1/4$, $0 \le r < \infty$, $R(1-\alpha) \ge (1+\alpha)(r+2-2\alpha)$,*

$$Q = \frac{\alpha(R+r)(1+\alpha)^2}{(R(1-\alpha)-r(1+\alpha)-1+\alpha^2)^2} < 1, \tag{2}$$

and such that BCF (1) *with elements $a_{i(n)}$ satisfying*

$$|a_{i(2n-1)}| \le \alpha/N,\ i_p = \overline{1,N},\ p = \overline{1,2n-1},\ n \ge 1, \tag{3}$$

and for each multiindex $i(2n-1)$ there is a single index j_{2n}, $1 \le j_{2n} \le N$, such that

$$|a_{i(2n-1),j_{2n}}| \ge R,\ i_p = \overline{1,N},\ p = \overline{1,2n-1},\ n \ge 1, \tag{4}$$

$$|a_{i(2n)}| \le r/(N-1),\ i_{2n} \ne j_{2n},\ i_p = \overline{1,N},\ p = \overline{1,2n},\ n \ge 1. \tag{5}$$

Then the BCF (1) *converges.*

This is analog of result by Leighton–Wall [13] on twin convergence regions, dealing with continued fractions. In the present paper, we shall study what happens to conditions on numbers α, r and R, and convergence speed estimates, when the conditions (3)–(5) are replaced by

$$\sum_{i_1=1}^{N} |a_{i(1)}| \le \alpha(1-\varepsilon),\ \sum_{i_{2n+1}=1}^{N} |a_{i(2n+1)}| \le \alpha(1-\varepsilon),\ i_p = \overline{1,N},\ p = \overline{1,2n},\ n \ge 1, \tag{6}$$

where $0 < \varepsilon < 1$, and

$$|a_{i(2n-1),j_{2n}}| \ge R,\quad \sum_{i_{2n}\in\{1,2,\ldots,N\}\setminus\{j_{2n}\}} |a_{i(2n)}| \le r,\ i_p = \overline{1,N},\ p = \overline{1,2n-1},\ n \ge 1. \tag{7}$$

The same type of problem of convergence regions for BCF is discussed in [2–6,14]. Application of the value regions to the study of the convergence of functional BCF may be found in [5,8,10]. Expansions of certain analytic functions in some classes of BCF are given in [1,8,11,12].

We give here a few facts (see [7]) that are used. Let $Q_{i(k)}^{(n)}$ denotes the "tails" of (1), that is $Q_{i(s)}^{(s)} = 1$, $i_p = \overline{1,N}$, $p = \overline{1,s}$, $s \ge 1$, and

$$Q_{i(k)}^{(n)} = 1 + \sum_{i_{k+1}=1}^{i_k} \frac{a_{i(k+1)}}{1} + \sum_{i_{k+2}=1}^{i_{k+1}} \frac{a_{i(k+2)}}{1} + \cdots + \sum_{i_n=1}^{i_{n-1}} \frac{a_{i(n)}}{1},$$

where $i_p = \overline{1,N}$, $p = \overline{1,k}$, $k = \overline{1,n-1}$, $n \ge 2$. It is clear that the following recurrence relations hold

$$Q_{i(k)}^{(n)} = 1 + \sum_{i_{k+1}=1}^{i_k} \frac{a_{i(k+1)}}{Q_{i(k+1)}^{(n)}},\ i_p = \overline{1,N},\ p = \overline{1,k},\ k = \overline{1,n-1},\ n \ge 2.$$

If f_n denotes the n-th approximant of (1), then $f_n = \sum_{i_1=1}^{N}(a_{i(1)}/Q_{i(1)}^{(n)})$, $n \ge 1$, and if all $Q_{i(k)}^{(n)} \ne 0$, then

$$f_m - f_n = (-1)^n \sum_{i_1=1}^{N}\sum_{i_2=1}^{N}\cdots\sum_{i_{n+1}=1}^{N} \frac{\prod_{k=1}^{n+1} a_{i(k)}}{\prod_{k=1}^{n+1} Q_{i(k)}^{(m)} \prod_{k=1}^{n} Q_{i(k)}^{(n)}},\ m > n \ge 1. \tag{8}$$

2 CONVERGENCE CRITERIA

We shall prove the auxiliary lemma.

Lemma. *Let there exist real numbers α, r and R such that*

$$0 \le \alpha < 1,\ 0 \le r < \infty,\ R(1-\alpha) \ge (1+\alpha)(r+2-2\alpha), \tag{9}$$

and such that BCF (1) *with elements $a_{i(n)}$ satisfying*

$$\sum_{i_1=1}^{N} |a_{i(1)}| \le \alpha,\ \sum_{i_{2n+1}=1}^{N} |a_{i(2n+1)}| \le \alpha,\ i_p = \overline{1,N},\ p = \overline{1,2n},\ n \ge 1, \tag{10}$$

and for each multiindex $i(2n-1)$ there is a single index j_{2n}, $1 \le j_{2n} \le N$, such that the inequalities (7) *hold. If $Q_{i(k)}^{(n)}$ denotes the "tails" of BCF* (1)*, the following inequalities hold*

$$1-\alpha \le |Q_{i(2k)}^{(n)}| \le 1+\alpha,\ i_p = \overline{1,N},\ 1 \le p \le 2k \le n,\ n \ge 2, \tag{11}$$

$$|Q_{i(2k-1)}^{(n)}| \ge \frac{R}{1+\alpha} - \frac{r}{1-\alpha} - 1 \ge 1,\ i_p = \overline{1,N},\ 1 \le p \le 2k-1 \le n-1,\ n \ge 2. \tag{12}$$

Proof. Let n be an arbitrary natural number. By induction on k for each $i(k)$ we show that the inequalities (11) and (12) are valid.

If n is even number and $k = n/2$, then for each $i(n)$ relations (11) are obvious. If n is odd number and $k = (n-1)/2$, then for arbitrary $i(n-1)$ use of relation (10) leads to

$$|Q_{i(n-1)}^{(n)}| \ge 1 - \sum_{i_n=1}^{N} |a_{i(n)}| \ge 1-\alpha \quad \text{and} \quad |Q_{i(n-1)}^{(n)}| \le 1 + \sum_{i_n=1}^{N} |a_{i(n)}| \le 1+\alpha.$$

By induction hypothesis that (11) hold for $k = r$ and for each $i(2r)$, where $2r \le n$, we prove the inequalities (12) for $k = r$ and for each $i(2r-1)$ and the inequalities (11) for $2k = 2r-2$ for each $i(2r-2)$. Indeed, use of relations (7), (9), (10) for arbitrary $i(2r-1)$ leads to

$$\begin{aligned} |Q_{i(2r-1)}^{(n)}| &= \left| 1 + \frac{a_{i(2r-1),j_{2r}}}{Q_{i(2r-1),j_{2r}}^{(n)}} + \sum_{i_{2r}\in\{1,2,\ldots,N\}\setminus\{j_{2r}\}} \frac{a_{i(2r)}}{Q_{i(2r)}^{(n)}} \right| \\ &\ge \frac{|a_{i(2r-1),j_{2r}}|}{|Q_{i(2r-1),j_{2r}}^{(n)}|} - \sum_{i_{2r}\in\{1,2,\ldots,N\}\setminus\{j_{2r}\}} \frac{|a_{i(2r)}|}{|Q_{i(2r)}^{(n)}|} - 1 \ge \frac{R}{1+\alpha} - \frac{r}{1-\alpha} - 1 \ge 1 \end{aligned}$$

and for arbitrary $i(2r-2)$

$$|Q_{i(2r-2)}^{(n)}| \ge 1 - \sum_{i_{2r-1}=1}^{N} \frac{|a_{i(2r-1)}|}{|Q_{i(2r-1)}^{(n)}|} \ge 1-\alpha \quad \text{and} \quad |Q_{i(2r-2)}^{(n)}| \le 1 + \sum_{i_{2r-1}=1}^{N} \frac{|a_{i(2r-1)}|}{|Q_{i(2r-1)}^{(n)}|} \le 1+\alpha.$$

This completes the proof of the lemma. □

Our main result is the following theorem.

Theorem 2. *Let there exist real numbers α, ε, r and R such that $0 \le \alpha < 1$, $0 < \varepsilon < 1$, $0 < r < \infty$, $R(1-\alpha) \ge (1+\alpha)(r+2-2\alpha)$ and such that BCF* (1) *with elements $a_{i(n)}$ satisfying the inequalities* (6) *and for each multiindex $i(2n-1)$ there is a single index j_{2n}, $1 \le j_{2n} \le N$, such that the inequalities* (7) *hold. Then the following statements hold.*

- *(A) The BCF* (1) *converges to a value f.*

- *(B) If f_n denotes the n-th approximant of the BCF* (1) *and*

$$q=\frac{\alpha(1+\alpha)(R(1-\alpha)+r(1+\alpha))}{(R(1-\alpha)-r(1+\alpha)-1+\alpha^2)^2}\leq 1, \tag{13}$$

then

$$|f-f_{2n}|\leq\frac{\alpha(1-\varepsilon)^{n+1}q^n}{R/(1+\alpha)-r/(1-\alpha)-1},\ n\geq 1. \tag{14}$$

- *(C) The values of the BCF* (1) *and of its approximants are in the region* $|z|\leq\alpha(1-\varepsilon)$.

Proof. At first, we prove (B). Let $m>2n+1$ and $n\geq 1$. From the formula (8) one obtains

$$\begin{aligned}|f_m-f_{2n}|&\leq\sum_{i_1=1}^{N}\sum_{i_2=1}^{N}\cdots\sum_{i_{2n+1}=1}^{N}\frac{|a_{i(1)}|}{|Q_{i(1)}^{(m)}|}\frac{\prod_{k=2}^{2n+1}|a_{i(k)}|}{\prod_{k=2}^{2n+1}|Q_{i(k)}^{(m)}|\prod_{k=1}^{2n}|Q_{i(k)}^{(2n)}|}\\&=\sum_{i_1=1}^{N}\sum_{i_2=1}^{N}\cdots\sum_{i_{2n+1}=1}^{N}\frac{|a_{i(1)}|}{|Q_{i(1)}^{(m)}|}\frac{\prod_{k=1}^{n}|a_{i(2k)}|}{\prod_{k=1}^{n}|Q_{i(2k-1)}^{(2n)}Q_{i(2k)}^{(2n)}|}\frac{\prod_{k=1}^{n}|a_{i(2k+1)}|}{\prod_{k=1}^{n}|Q_{i(2k)}^{(m)}Q_{i(2k+1)}^{(m)}|}.\end{aligned}$$

Obviously, the conditions of lemma hold. Let k be an arbitrary natural number. Applying (11) and (12) we have for arbitrary $i(2k-1)$

$$\begin{aligned}&\sum_{i_{2k}=1}^{N}\frac{|a_{i(2k)}|}{|Q_{i(2k-1)}^{(2n)}Q_{i(2k)}^{(2n)}|}\sum_{i_{2k+1}=1}^{N}\frac{|a_{i(2k+1)}|}{|Q_{i(2k)}^{(m)}Q_{i(2k+1)}^{(m)}|}\\&\qquad\leq\frac{\alpha(1-\varepsilon)}{(1-\alpha)(R/(1+\alpha)-r/(1-\alpha)-1)}\sum_{i_{2k}=1}^{N}\frac{|a_{i(2k)}|}{|Q_{i(2k-1)}^{(2n)}Q_{i(2k)}^{(2n)}|}\\&\qquad=\frac{\alpha(1-\varepsilon)}{(1-\alpha)(R/(1+\alpha)-r/(1-\alpha)-1)}\\&\qquad\quad\times\left(\frac{|a_{i(2k-1),j_{2k}}|}{|Q_{i(2k-1)}^{(2n)}Q_{i(2k-1),j_{2k}}^{(2n)}|}+\sum_{i_{2k}\in\{1,2,\ldots,N\}\backslash\{j_{2k}\}}\frac{|a_{i(2k)}|}{|Q_{i(2k-1)}^{(2n)}Q_{i(2k)}^{(2n)}|}\right).\end{aligned}$$

Since

$$\begin{aligned}\sum_{i_{2k}\in\{1,\ldots,N\}\backslash\{j_{2k}\}}\frac{|a_{i(2k)}|}{|Q_{i(2k-1)}^{(2n)}Q_{i(2k)}^{(2n)}|}&\leq\frac{1}{(1-\alpha)(R/(1+\alpha)-r/(1-\alpha)-1)}\sum_{i_{2k}\in\{1,2,\ldots,N\}\backslash\{j_{2k}\}}|a_{i(2k)}|\\&\leq\frac{r}{(1-\alpha)(R/(1+\alpha)-r/(1-\alpha)-1)}\end{aligned}$$

and

$$\begin{aligned}\frac{|a_{i(2k-1),j_{2k}}|}{|Q^{(2n)}_{i(2k-1)}Q^{(2n)}_{i(2k-1),j_{2k}}|} &= \left|\frac{a_{i(2k-1),j_{2k}}/Q^{(2n)}_{i(2k-1),j_{2k}}}{1+a_{i(2k-1),j_{2k}}/Q^{(2n)}_{i(2k-1),j_{2k}}+\sum_{i_{2k}\in\{1,2,\ldots,N\}\setminus\{j_{2k}\}}(a_{i(2k)}/Q^{(2n)}_{i(2k)})}\right|\\ &= \left|1-\frac{1+\sum_{i_{2k}\in\{1,2,\ldots,N\}\setminus\{j_{2k}\}}(a_{i(2k)}/Q^{(2n)}_{i(2k)})}{1+a_{i(2k-1),j_{2k}}/Q^{(2n)}_{i(2k-1),j_{2k}}+\sum_{i_{2k}\in\{1,2,\ldots,N\}\setminus\{j_{2k}\}}(a_{i(2k)}/Q^{(2n)}_{i(2k)})}\right|\\ &\le 1+\left|\frac{1+\sum_{i_{2k}\in\{1,2,\ldots,N\}\setminus\{j_{2k}\}}(a_{i(2k)}/Q^{(2n)}_{i(2k)})}{Q^{(2n)}_{i(2k-1)}}\right|\\ &\le 1+\frac{1+r/(1-\alpha)}{R/(1+\alpha)-r/(1-\alpha)-1}=\frac{R/(1+\alpha)}{R/(1+\alpha)-r/(1-\alpha)-1},\end{aligned}$$

then

$$\sum_{i_{2k}=1}^{N}\frac{|a_{i(2k)}|}{|Q^{(2n)}_{i(2k-1)}Q^{(2n)}_{i(2k)}|}\sum_{i_{2k+1}=1}^{N}\frac{|a_{i(2k+1)}|}{|Q^{(m)}_{i(2k)}Q^{(m)}_{i(2k+1)}|}\le\frac{\alpha(1-\varepsilon)(R/(1+\alpha)+r/(1-\alpha))}{(1-\alpha)(R/(1+\alpha)-r/(1-\alpha)-1)^2}.$$

Thus, for $m>2n+1$ and $n\ge 1$

$$|f_m-f_{2n}|\le\frac{\alpha^{n+1}(1-\varepsilon)^{n+1}(R/(1+\alpha)+r/(1-\alpha))^n}{(1-\alpha)^n(R/(1+\alpha)-r/(1-\alpha)-1)^{2n+1}}=\frac{\alpha(1-\varepsilon)^{n+1}q^n}{R/(1+\alpha)-r/(1-\alpha)-1},\tag{15}$$

where q is defined by (13). If in (15) we pass to the limit as $n\to\infty$, then from (13) it follows that BCF (1) converges. On the other hand, if in (15) we pass to the limit as $m\to\infty$, we obtain the estimate (14). This proves (B).

To prove (A) we consider the following equation

$$F_1(x)=F_2(x),\tag{16}$$

where

$$F_1(x)=\frac{x}{1-x}\left(\frac{R}{1+x}+\frac{r}{1-x}\right),\ F_2(x)=\left(\frac{R}{1+x}-\frac{r}{1-x}-1\right)^2.$$

It is clear that $F_1(0)<F_2(0)$, and $F_1(x)>0$ and $F_2(x)\ge 0$ for all $x\in(0;1)$. It follows from $F_1'(x)=R(1+x^2)/(1-x^2)^2+r(1+x)/(1-x)^3$ that $F_1'(x)>0$ for all $x\in(0;1)$. Let us write the function $F_2(x)$ in the form $F_2(x)=(x^2-(R+r)x+R-r-1)^2/(1-x^2)^2$ and consider the following equation

$$x^2-(R+r)x+R-r-1=0.\tag{17}$$

If $r>0$, then $x^*=(R+r-\sqrt{(R+r)^2-4(R-r-1)})/2$ is the only root of equation (17) on $(0;1)$ and, if $r=0$, then $x^*=1$ is the only root of (17). Now from

$$F_2'(x)=-2\frac{x^2-(R+r)x+R-r-1}{1-x^2}\left(\frac{R}{(1+x)^2}+\frac{r}{(1-x)^2}\right)$$

we have $F_2'(x)<0$ for all $x\in(0;x^*)$. It follows that there exists the only root α^* of equation (16) on $(0;x^*)$. If $0<\alpha\le\alpha^*$, then $F_1(\alpha)\le F_2(\alpha)$, that is, the condition (13) holds. In the case when $\alpha^*<\alpha<1$ we consider the following BCF

$$\sum_{i_1=1}^{N}\frac{a_{i(1)}z}{1}+\sum_{i_2=1}^{N}\frac{a_{i(2)}}{1}+\cdots\sum_{i_{2k-1}=1}^{N}\frac{a_{i(2k-1)}z}{1}+\sum_{i_{2k}=1}^{N}\frac{a_{i(2k)}}{1}+\cdots,\tag{18}$$

where $z \in \mathbb{C}$. It is clear that the elements of BCF (18) satisfy the conditions of lemma in domain $D_\varepsilon = \{z \in \mathbb{C} : \ |z| < 1/(1-\varepsilon)\}$. It follows from (11) and (12) that, if $f_n(z)$ denotes the n-th approximant of the BCF (18), for all $z \in D_\varepsilon$

$$|f_n(z)| \le \sum_{i_1=1}^{N} |a_{i(1)}z| \le \alpha(1-\varepsilon)|z| < \alpha,$$

i.e. the sequence $\{f_n(z)\}$ is uniformly bounded in the domain D_ε. If $z \in D_{\alpha^*}$, where $D_{\alpha^*} = \{z \in \mathbb{C} : \ |z| < \alpha^*/\alpha\}$, then according to the above BCF (18) converges. Obviously, $D_{\alpha^*} \subset D_\varepsilon$. Hence, by [16, Theorem 24.2, p. 108] BCF (18) converges uniformly on each compact subset of the domain D_ε, in particular, for $z = 1$. It follows that BCF (1) converges.

Finally, from

$$|f_n| \le \sum_{i_1=1}^{N} \frac{|a_{i(1)}|}{|Q_{i(1)}^{(n)}|} \le \sum_{i_1=1}^{N} |a_{i(1)}| \le \alpha(1-\varepsilon)$$

follows proof of (C). □

Remark. *If the conditions* (3)–(5) *are replaced by the conditions* (6) *and* (7)*, then the condition* (2) *is replaced by the condition* (13) *and the* $0 \le \alpha \le 1/4$ *is replaced by the* $0 \le \alpha < 1$*. It is clear that* $Q > q$*, and, thus, the estimates* (14) *are better than similar estimates obtained in the proof of Theorem 1. In addition, if* $q < 1$*, then* ε *can be zero.*

Corollary. *Let there exist real numbers* β *and* ε *such that* $0 \le \beta < 1/N$, $0 < \varepsilon < 1$, *and such that BCF* (1) *with elements* $a_{i(n)}$ *satisfying* $|a_{i(2n-1)}| \le \beta(1-\varepsilon)$, *where* $i_p = \overline{1,N}$, $p = \overline{1, 2n-1}$, $n \ge 1$, *and for each multiindex* $i(2n-1)$ *there is a single index* j_{2n}, $1 \le j_{2n} \le N$, *such that*

$$|a_{i(2n-1),j_{2n}}| \ge (1+N\beta)(2-(1+N)\beta)/(1-N\beta), \ i_p = \overline{1,N}, \ p = \overline{1,2n-1}, \ n \ge 1,$$
$$|a_{i(2n)}| \le \beta, \ i_{2n} \ne j_{2n}, \ i_p = \overline{1,N}, \ p = \overline{1,2n}, \ n \ge 1.$$

Then BCF (1) *converges, and its values and its approximants are in the region* $|z| \le N\beta(1-\varepsilon)$.

Proof. We set $\alpha = N\beta$, $r = (N-1)\beta$, $R = (1+N\beta)(2-\beta(1+N))/(1-N\beta)$. Then

$$R = \frac{1+N\beta}{1-N\beta}(2-2N\beta+(N-1)\beta) = (1+N\beta)\left(2+\frac{N-1}{1-N\beta}\beta\right) = (1+\alpha)\left(2+\frac{r}{1-\alpha}\right).$$

It follows that the conditions of Theorem 2 hold, and, therefore, the corollary is an immediate consequence of this theorem. □

3 EXAMPLE

Let β, r and R be some positive numbers. We consider the periodical BCF

$$\sum_{i_1=1}^{2} \frac{a_{i(1)}}{1} + \sum_{i_2=1}^{2} \frac{a_{i(2)}}{1} + \cdots + \sum_{i_n=1}^{2} \frac{a_{i(n)}}{1} + \cdots, \tag{19}$$

where $a_{i(1)} = \beta$, $a_{i(2n-1)} = (-1)^{i_{2n-2}-1}\beta$, $a_{i(2n-1),1} = (-1)^{i_{2n-1}-1}R$, $a_{i(2n-1),2} = (-1)^{i_{2n-1}}r$, which form by the following fractional transformation

$$s(w) = \frac{\beta}{1+\dfrac{R}{1+w}-\dfrac{r}{1-w}} + \frac{\beta}{1-\dfrac{R}{1+w}+\dfrac{r}{1-w}}.$$

It follows that BCF (19) can be converged only to the real root of the following equation

$$(w-2\beta)(1-w^2)^2 - w(R-r-w(R+r))^2 = 0. \tag{20}$$

We choose $\beta = \alpha(1-\varepsilon)/2$, $\alpha = 1/3$, $\varepsilon = 1/4$, $r = 2/3$ and $R = 5$. Then it is clear that the conditions of Theorem 2 are satisfied and the inequalities $|w| \leq 2\beta$ are valid. Thus, BCF (19) converges. On the other hand the equation (20) we write in the form

$$9(4w-1)(1-w^2)^2 - 4w(13-17w)^2 = 0. \tag{21}$$

Let $F(w) = 9(4w-1)(1-w^2)^2 - 4w(13-17w)^2$. Then $F(0) < 0$ and $F(-1/4) > 0$. Thus, on the interval $[-1/4; 0]$ there is root of the equation (21). The following recurrent formula

$$f_{k+2} = \frac{2\beta(1-f_k^2)^2}{(1-f_k^2)^2 - (R-r-f_k(R+r))^2}, \; k \geq 1,$$

with initial conditions $f_1 = 2\beta$ and $f_2 = 2\beta/(1-(R-r)^2)$ can be used to find of the above mentioned root.

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Антонова Т.М. *Про критерій збіжності для гіллястого ланцюгового дробу* // Карпатські матем. публ. — 2020. — Т.12, №1. — С. 157–164.

Основою цієї роботи є результат Є.А. Болтаровича (1989) про множини збіжності для гіллястого ланцюгового дробу

$$\sum_{i_1=1}^{N}\frac{a_{i(1)}}{1}+\sum_{i_2=1}^{N}\frac{a_{i(2)}}{1}+\cdots+\sum_{i_n=1}^{N}\frac{a_{i(n)}}{1}+\cdots,$$

де $|a_{i(2n-1)}|\le\alpha/N$, $i_p=\overline{1,N}$, $p=\overline{1,2n-1}$, $n\ge1$, і для кожного мультиіндексу $i(2n-1)$ існує єдиний індекс j_{2n}, $1\le j_{2n}\le N$, такий, що $|a_{i(2n-1),j_{2n}}|\ge R$, $i_p=\overline{1,N}$, $p=\overline{1,2n-1}$, $n\ge1$, та $|a_{i(2n)}|\le r/(N-1)$, $i_{2n}\ne j_{2n}$, $i_p=\overline{1,N}$, $p=\overline{1,2n}$, $n\ge1$, де $N>1$, α, r та R – дійсні числа, що задовольняють певні умови. У цій роботі умови для цих множин замінено на $\sum_{i_1=1}^{N}|a_{i(1)}|\le\alpha(1-\varepsilon)$, $\sum_{i_{2n+1}=1}^{N}|a_{i(2n+1)}|\le\alpha(1-\varepsilon)$, $i_p=\overline{1,N}$, $p=\overline{1,2n}$, $n\ge1$, і для кожного мультиіндексу $i(2n-1)$ існує єдиний індекс j_{2n}, $1\le j_{2n}\le N$, такий, що $|a_{i(2n-1),j_{2n}}|\ge R$ та $\sum_{i_{2n}\in\{1,2,\ldots,N\}\setminus\{j_{2n}\}}|a_{i(2n)}|\le r$, $i_p=\overline{1,N}$, $p=\overline{1,2n-1}$, $n\ge1$, де ε, α, r та R – дійсні числа, що задовольняють певні умови, і, отримано кращі оцінки швидкості збіжності для цього гіллястого ланцюгового дробу.

Ключові слова і фрази: збіжність, множина збіжності, оцінка швидкості збіжності, гіллястий ланцюговий дріб.

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CHAIKOVS'KYI A.[1], LAGODA O.[2]

BOUNDED SOLUTIONS OF A DIFFERENCE EQUATION WITH FINITE NUMBER OF JUMPS OF OPERATOR COEFFICIENT

We study the problem of existence of a unique bounded solution of a difference equation with variable operator coefficient in a Banach space. There is well known theory of such equations with constant coefficient. In that case the problem is solved in terms of spectrum of the operator coefficient. For the case of variable operator coefficient correspondent conditions are known too. But it is too hard to check the conditions for particular equations. So, it is very important to give an answer for the problem for those particular cases of variable coefficient, when correspondent conditions are easy to check. One of such cases is the case of piecewise constant operator coefficient. There are well known sufficient conditions of existence and uniqueness of bounded solution for the case of one jump. In this work, we generalize these results for the case of finite number of jumps of operator coefficient. Moreover, under additional assumption we obtained necessary and sufficient conditions of existence and uniqueness of bounded solution.

Key words and phrases: difference equation, bounded solution, Banach space.

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INTRODUCTION

Let $(X, \|\cdot\|)$ be a complex Banach space, $L(X)$ be the space of linear continuous operators in X, $I \in L(X)$ be the identity operator. Let us denote $\sigma(A)$ the spectrum of an operator $A \in L(X)$. Let us denote $S = \{z \in \mathbb{C} \,:\, |z| = 1\}$ the unit circle in the complex plane.

Let us consider the difference equation

$$x_{n+1} = A_n x_n + y_n, \quad n \in \mathbb{Z}, \tag{1}$$

where $\{A_n \mid n \in \mathbb{Z}\} \subset L(X)$, $\{y_n \mid n \in \mathbb{Z}\} \subset X$ are known sequences, $\{x_n \mid n \in \mathbb{Z}\} \subset X$ is a desired sequence. In the paper we investigate the question of existence and uniqueness of a bounded solution for the equation (1).

It is known [3, chapter 7.6] the equation (1) has a unique bounded solution $\{x_n \mid n \in \mathbb{Z}\}$ for any bounded sequence $\{y_n \mid n \in \mathbb{Z}\}$ if and only if operators sequence fulfills a condition of discrete dichotomy (analogue of exponential dichotomy, which is well known in the theory of differential equations). However, checking of discrete dichotomy conditions is very hard, so we need simpler conditions of existence and uniqueness of a bounded solution for special operators sequences.

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To formulate one of such conditions we need the following spectral decomposition. Assume $A \in L(X)$ and the condition $\sigma(A) \cap S = \varnothing$ is true. Then the spectrum of the operator A is decomposed into two parts, one of them is inside of the unit circle S, the other is outside. Using the theorem about decomposition [4, p. 445] we can derive:

1) an existence of projectors $P_-(A), P_+(A) \in L(X)$ such that

$$P_-(A) + P_+(A) = I;$$

2) decomposition of the space X to the direct sum

$$X = X_-(A) \dot{+} X_+(A), \tag{2}$$

where $X_-(A) = P_-(A)X$, $X_+(A) = P_+(A)X$ are subspaces in which corresponding operators $A_- = P_-(A)A$, $A_+ = P_+(A)A$ have spectra

$$\sigma(A) \cap \{z \in \mathbb{C} \mid |z| < 1\},\ \sigma(A) \cap \{z \in \mathbb{C} \mid |z| > 1\} \tag{3}$$

accordingly.

I.V. Gonchar and M.F. Gorodnii investigated the equation (1) in the papers [1,2] for the case of one jump of an operator coefficient. In the paper [1] the following result was proved.

Theorem 1. *Let X be a complex Banach space and G, U be some operators from $L(X)$, which satisfy the following conditions:*

1) $\sigma(G) \cap S = \varnothing$, $\sigma(U) \cap S = \varnothing$;

2) $X = X_-(G) \dot{+} X_+(U)$.

Then the difference equation

$$\begin{cases} x_{n+1} = Gx_n + y_n, & n \geq 1, \\ x_{n+1} = Ux_n + y_n, & n \leq 0, \end{cases}$$

has a unique bounded in X solution $\{x_n : n \in \mathbb{Z}\}$ for any bounded in X sequence $\{y_n : n \in \mathbb{Z}\}$.

In the paper the result of the Theorem 1 is generalized to an equation with several jumps of an operator coefficient.

1 Main results

Let us consider a special case of the equation (1) with an operator coefficient, which changes finite number of times:

$$\begin{cases} x_{n+1} = A_0 x_n + y_n, & n \leq 0, \\ x_{n+1} = A_n x_n + y_n, & 1 \leq n \leq N-1, \\ x_{n+1} = A_N x_n + y_n, & n \geq N. \end{cases} \tag{4}$$

Here N is a fixed natural number.

Assume the conditions $\sigma(A_0) \cap S = \varnothing$, $\sigma(A_N) \cap S = \varnothing$ are true. Then each of the operators A_0, A_N produce spectral decomposition of the form (2). Let us denote

$$\begin{aligned} &P_{0-} := P_-(A_0), && P_{0+} := P_+(A_0), && P_{N-} := P_-(A_N), && P_{N+} := P_+(A_N), \\ &X_{0-} := X_-(A_0), && X_{0+} := X_+(A_0), && X_{N-} := X_-(A_N), && X_{N+} := X_+(A_N). \end{aligned}$$

Remark 1. *In a degenerate case, when one of the sets in* (3) *is empty, the corresponding subspace contains zero element only, so we can omit it in the direct sum. Further we assume that all these sets are nonempty. For degenerate cases statements below are true if degenerate summands are omitted.*

Lemma 1. *Let $\sigma(A_0) \cap S = \varnothing$. Then for any bounded sequence $\{y_n : n \le 0\} \subset X$ all bounded solutions of the equation*

$$x_{n+1} = A_0 x_n + y_n, \quad n \le 0,$$

can be obtained by the formula

$$x_n = A_{0+}^{n-1} b - \sum_{k=n}^{0} A_0^{n-k-1} P_{0+} y_k + \sum_{k=-\infty}^{n-1} A_0^{n-k-1} P_{0-} y_k, \quad n \le 1, \tag{5}$$

where $b \in X_{0+}$ is an arbitrary element.

Proof. The condition $\sigma(A_{0+}) \subset \{z \in \mathbb{C} : |z| > 1\}$ implies the existence of the operator $A_{0+}^{-1} \in L(X)$ and the estimate

$$\exists C > 0\ \exists r \in (0,1)\ \ \forall n \ge 1\ \ ||A_{0+}^{-n}|| \le C r^n. \tag{6}$$

Similarly, the condition $\sigma(A_{0-}) \subset \{z \in \mathbb{C} : |z| < 1\}$ implies the estimate

$$\exists C > 0\ \exists r \in (0,1)\ \ \forall n \ge 1\ \ ||A_{0-}^{n}|| \le C r^n. \tag{7}$$

So, the defined sequence (5) is bounded for any element $b \in X_{0+}$.

Let us check that the sequence (5) is a solution of the difference equation. We have

$$\begin{aligned} A_0 x_n + y_n &= A_{0+}^{n} b - \sum_{k=n}^{0} A_0^{n-k} P_{0+} y_k + \sum_{k=-\infty}^{n-1} A_0^{n-k} P_{0-} y_k + P_{0+} y_n + P_{0-} y_n \\ &= A_{0+}^{(n+1)-1} b - \sum_{k=n+1}^{0} A_0^{(n+1)-k-1} P_{0+} y_k + \sum_{k=-\infty}^{(n+1)-1} A_0^{(n+1)-k-1} P_{0-} y_k = x_{n+1}, \quad n \le 0. \end{aligned}$$

On the other hand, if $\{z_n : n \ge N\}$ is any bounded solution and $\{x_n : n \ge N\}$ is any bounded solution of the form (5), the difference $\{r_n = z_n - x_n : n \ge N\}$, is a bounded solution of the homogeneous equation

$$r_{n+1} = A_0 r_n, \quad n \le -1.$$

From this equation we have

$$r_0 = A_0^{-n} r_n, \quad n \le -1,$$

and, using projection operator,

$$P_{0-} r_0 = A_{0-}^{-n} r_n \to \overline{0}, \quad n \to -\infty.$$

So, $r_0 \in X_{0+}$ and $r_n = A_{0+}^{n} r_0$, $n \le -1$. We obtained that solution $\{z_n : n \le 0\}$ has the form (5). This completes the proof. $\square$

Lemma 2. *Let $\sigma(A_N) \cap S = \varnothing$. Then for any bounded sequence $\{y_n : n \geq N\} \subset X$ all the bounded solutions of the equation*

$$x_{n+1} = A_N x_n + y_n, \quad n \geq N,$$

can be obtained by the formula

$$x_n = A_{N-}^{n-N} b + \sum_{k=N}^{n-1} A_N^{n-k-1} P_{N-} y_k - \sum_{k=n}^{+\infty} A_N^{n-k-1} P_{N+} y_k, \quad n \geq N, \tag{8}$$

where $b \in X_{N-}$ is an arbitrary element.

Proof. The conditions $\sigma(A_{N+}) \subset \{z \in \mathbb{C} : |z| > 1\}$ and $\sigma(A_{N-}) \subset \{z \in \mathbb{C} : |z| < 1\}$ imply the existence of the operator $A_{N+}^{-1} \in L(X)$ and estimates similar to (6) and (7). So, the sequence (8) is bounded for any element $b \in X_{N-}$.

If we put the sequence (8) to the difference equation, we obtain

$$\begin{aligned} A_N x_n + y_n &= A_{N-}^{n-N+1} b + \sum_{k=N}^{n-1} A_N^{n-k} P_{N-} y_k - \sum_{k=n}^{+\infty} A_N^{n-k} P_{N+} y_k + P_{N-} y_n + P_{N+} y_n \\ &= A_{N-}^{n+1-N} b + \sum_{k=N}^{(n+1)-1} A_N^{(n+1)-k-1} P_{N-} y_k - \sum_{k=n+1}^{+\infty} A_N^{(n+1)-k-1} P_{N+} y_k = x_{n+1}, \quad n \geq N. \end{aligned}$$

Similar to proof of previous lemma, the difference $\{r_n = z_n - x_n : n \geq N\}$ between any bounded solution $\{z_n : n \geq N\}$ and bounded solution $\{x_n : n \geq N\}$ of the form (8), is a bounded solution of the homogeneous equation

$$r_{n+1} = A_N r_n, \quad n \geq N,$$

and has a form

$$r_n = A_N^{n-N} r_N, \quad n \geq N.$$

Since

$$P_{N+} r_n = A_{N+}^{n-N} r_N, \; P_{N+} r_N = A_{N+}^{N-n} r_n \to \overline{0}, \quad n \to +\infty,$$

we have $r_N \in X_{N-}$ and $r_n = A_{N-}^{n-N} r_0$, $n \geq 0$. So any bounded solution has the form (8). The proof is completed. □

Lemma 3. *Let $N \geq 2$ and $A_{N-1} A_{N-2} \cdot \ldots \cdot A_1$ be injection. The boundary problem*

$$\begin{cases} x_{n+1} = A_n x_n + y_n, \; 1 \leq n \leq N-1, \\ P_{0-} x_1 = v, \; P_{N+} x_N = u, \end{cases} \tag{9}$$

has a unique solution $\{x_n : 1 \leq n \leq N\} \subset X$ for any $v \in X_{0-}$, $u \in X_{N+}$ and any $\{y_n : 1 \leq n \leq N-1\} \subset X$ if and only if

$$X = W \dot{+} X_{N-}, \tag{10}$$

where $W = \{A_{N-1} A_{N-2} \cdot \ldots \cdot A_1 x : x \in X_{0+}\}$.

Proof. If a solution of the problem (9) exists, then the formula

$$x_n = A_{n-1}A_{n-2}\cdot\ldots\cdot A_1x_1 + \sum_{k=1}^{n-2} A_{n-1}A_{n-2}\cdot\ldots\cdot A_{k+1}y_k + y_{n-1}, \quad 2 \le n \le N, \tag{11}$$

is true. One can check this result by induction. We have $x_2 = A_1x_1 + y_1$ and

$$\begin{aligned} A_nx_n + y_n = {} & A_nA_{n-1}A_{n-2}\cdot\ldots\cdot A_1x_1 \\ & + \sum_{k=1}^{n-2} A_nA_{n-1}A_{n-2}\cdot\ldots\cdot A_{k+1}y_k + A_ny_{n-1} + y_n = x_{n+1}, \quad 2 \le n \le N-1. \end{aligned}$$

Necessity. Let the boundary problem has a unique solution for any bounded sequence $\{y_n : 1 \le n \le N-1\} \subset X$ and boundary conditions $v \in X_{0-}$, $u \in X_{N+}$.

Let us fix an arbitrary element $f \in X$. In case $y_1 = y_2 = \ldots = y_{N-2} = \vec{0}$, $y_{N-1} = f$, $u = v = \vec{0}$ problem (9) has the unique solution. Formula (11) gives us

$$x_N = A_{N-1}A_{N-2}\cdot\ldots\cdot A_1x_1 + f$$

that is, using boundary conditions, we have $f = P_{N-}x_N + A_{N-1}A_{N-2}\cdot\ldots\cdot A_1(-P_{0+}x_1)$. This equality implies f is the sum of elements from W and X_{N-}.

To prove uniqueness of the element's decomposition let us assume by the contrary that there are nonzero elements $u_0 \in X_{0+}$, $v_0 \in X_{N-}$ such that

$$\vec{0} = A_{N-1}A_{N-2}\cdot\ldots\cdot A_1u_0 + v_0. \tag{12}$$

Boundary problem (9) in case $y_1 = y_2 = \ldots = y_{N-2} = y_{N-1} = \vec{0}$, $u = v = \vec{0}$ has unique solution $\{x_1, x_2, \ldots, x_{N-1}, x_N\}$ and

$$x_N = A_{N-1}A_{N-2}\cdot\ldots\cdot A_1x_1.$$

But adding assumption (12) we have

$$(x_N - v_0) = A_{N-1}A_{N-2}\cdot\ldots\cdot A_1(x_1 + u_0),$$

so, $\{x_1 + u_0, x_2, \ldots, x_{N-1}, x_N - v_0\}$ is another solution of the boundary problem. A contradiction.

Since f is arbitrary, the required decomposition (10) is proved.

Sufficiency. Let decomposition (10) is true. For arbitrary $v \in X_{0-}$, $u \in X_{N+}$ and $\{y_n : 1 \le n \le N-1\} \subset X$ let us denote

$$f := \sum_{k=1}^{N-2} A_{N-1}A_{N-2}\cdot\ldots\cdot A_{k+1}y_k + y_{N-1} - u + A_{N-1}A_{N-2}\cdot\ldots\cdot A_1v.$$

Due to the space decomposition we have

$$\exists!(w, b) \in W \times X_{N-} \ : \ f = w + b$$

or equivalently

$$\exists!(a, b) \in X_{0+} \times X_{N-} \ : \ f = A_{N-1}A_{N-2}\cdot\ldots\cdot A_1a + b.$$

Using the definition of f we have

$$\exists!(a,b) \in X_{0+} \times X_{N-} \ : \sum_{k=1}^{N-2} A_{N-1}A_{N-2} \cdot \ldots \cdot A_{k+1}y_k + y_{N-1} \\ = A_{N-1}A_{N-2} \cdot \ldots \cdot A_1(a-v) + (b+u). \tag{13}$$

This statement implies that the problem (9) has a solution. Indeed, we can put $x_1 = v - a$. The first boundary condition is fulfilled. Elements $x_2, \ldots, x_N$ could be obtained from (11). By comparing (11) for $n = N$ and (13) we obtain $x_N = b + u$ and the second boundary condition is fulfilled too.

Obtained solution is unique since for homogeneous boundary problem we have

$$x_N = A_{N-1}A_{N-2} \cdot \ldots \cdot A_1 x_1$$

and $x_N \in X_{N-}$, $x_1 \in X_{0+}$. But using space decomposition (10) we obtain $x_N = \vec{0}$, and using condition that operator $A_{N-1}A_{N-2} \cdot \ldots \cdot A_1$ is injective, we have $x_1 = \vec{0}$, so $x_2 = \ldots = x_{N-1} = \vec{0}$. The lemma is proved. □

Theorem 2. *Let $\sigma(A_0) \cap S = \varnothing$, $\sigma(A_N) \cap S = \varnothing$ and $A_{N-1}A_{N-2} \cdot \ldots \cdot A_1$ be an injection. Then the equation (4) has a unique bounded solution $\{x_n : n \in \mathbb{Z}\} \subset X$ for any bounded sequence $\{y_n : n \in \mathbb{Z}\} \subset X$ if and only if*

$$X = W \dot{+} X_{N-},$$

where $W = \{A_{N-1}A_{N-2} \cdot \ldots \cdot A_1 x : x \in X_{0+}\}$.

Proof. Necessity. Let the equation (4) has a unique bounded solution $\{x_n : n \in \mathbb{Z}\} \subset X$ for any bounded sequence $\{y_n : n \in \mathbb{Z}\} \subset X$.

Let $\{b_n : 1 \le n \le N-1\} \subset X$ and $u \in X_{N+}, v \in X_{0-}$ be arbitrary. We will consider bounded sequence $\{y_n : n \in \mathbb{Z}\} \subset X$, where $y_n = \overline{0}$, $n < 0$; $y_0 = v$; $y_n = b_n$, $1 \le n \le N-1$; $y_N = -A_{N+}u$; $y_n = \overline{0}$, $n > N$. For this sequence there exists a unique bounded solution $\{x_n : n \in \mathbb{Z}\} \subset X$.

By Lemma 1 the part of solution $\{x_n : n \le 1\}$ has such form that $x_1 = b + v$ where $b \in X_{0+}$. That implies

$$P_{0-}x_1 = v. \tag{14}$$

Similarly by Lemma 2 the part of solution $\{x_n : n \ge N\}$ has such form that $x_N = b + u$, where $b \in X_{N-}$, so

$$P_{N+}x_N = u. \tag{15}$$

Due to equalities (14) and (15) the sequence $\{x_n : 1 \le n \le N\}$ is a solution of the boundary problem (9).

Suppose by the contrary that boundary problem (9) has another solution $\{z_n : 1 \le n \le N\}$. Let

$$\begin{aligned} z_0 &= A_{0+}^{-1}(z_1 - y_1), \quad & z_n &= A_{0+}^{n} z_0, \quad & n &\le -1, \\ z_{N+1} &= A_N z_N + y_N, \quad & z_n &= A_{N-}^{n-N-1} z_{N+1}, \quad & n &\ge N+2. \end{aligned}$$

One can see that sequence $\{z_n : n \in \mathbb{Z}\}$ is bounded due to spectral properties of A_{0+} and A_{N-}. This sequence is a solution of (4). Indeed, for $1 \le n \le N-1$ equation is true due to boundary problem and since

$$z_0 = A_{0+}^{-1}(z_1 - y_1) \in X_{0+},\ z_{N+1} = A_N z_N + y_N = A_{N-}z_N + A_{N+}u - A_{N+}u \in X_{N-},$$

we have

$$z_1 = A_{0+}z_0 + y_1 = A_0 z_0 + y_1, \qquad z_{n+1} = A_{0+}^{n+1} z_0 = A_0 A_{0+}^n z_0 = A_0 z_n, \quad n \le -1,$$
$$z_{N+1} = A_N z_N + y_N, \qquad z_{n+1} = A_{N-}^{n-N} z_{N+1} = A_N A_{N-}^{n-N-1} z_{N+1} = A_N z_n, \quad n \ge N+1.$$

This solution is different from $\{x_n : n \in \mathbb{Z}\}$ (at least for $1 \le n \le N$). A contradiction.

Since boundary problem (9) has unique solution for any input data, Lemma 3 gives us decomposition (10).

Sufficiency. Assume that decomposition (10) is true. Let $\{y_n : n \in \mathbb{Z}\} \subset X$ be any bounded sequence. We will construct bounded solution of (4). This solution consists of three parts, described by Lemmas 1–3 (with intersections in x_1 and x_N).

By Lemma 1 for bounded sequence $\{y_n : n \le 0\} \subset X$ we have

$$x_n = A_{0+}^{n-1} b_1 - \sum_{k=n}^{0} A_0^{n-k-1} P_{0+} y_k + \sum_{k=-\infty}^{n-1} A_0^{n-k-1} P_{0-} y_k, \quad n \le 1,$$

where $b_1 \in X_{0+}$. In particular, $x_1 = b_1 + v$, where $v = \sum\limits_{k=-\infty}^{0} A_0^{-k} P_{0-} y_k \in X_{0-}$. So, $P_{0-} x_1 = v$.

Similarly, by Lemma 2 for bounded sequence $\{y_n : n \ge N\} \subset X$ we have

$$x_n = A_{N-}^{n-N} b_2 + \sum_{k=N}^{n-1} A_N^{n-k-1} P_{N-} y_k - \sum_{k=n}^{+\infty} A_N^{n-k-1} P_{N+} y_k, \quad n \ge N,$$

where $b_2 \in X_{N-}$. In particular, $x_N = b_2 + u$, where $u = -\sum\limits_{k=N}^{+\infty} A_{N+}^{N-k-1} y_k \in X_{N+}$. So, $P_{N+} x_N = u$.

By Lemma 3 the boundary problem (9) with defined above u and v has the unique solution $\{x_n : 1 \le n \le N\} \subset X$. So x_1, x_N are uniquely defined by sequence $\{y_n : n \in \mathbb{Z}\} \subset X$. That implies that $b_1 = P_{0+} x_1$, $b_2 = P_{N-} x_N$ are uniquely defined too. So the whole solution $\{x_n : n \in \mathbb{Z}\} \subset X$ is uniquely defined.

Constructed solution is a unique bounded solution of (4). □

Remark 2. *For $N = 1$ sufficiency of Theorem 2 gives us the statement of Theorem 1.*

Example 1. *Let $X = l_2$, $N = 2$,*

$$A_0 x = (x_1/2, x_2(2+1/2), x_3/4, x_4(2+1/4), x_5/6, x_6(2+1/6), \ldots),$$
$$A_1 x = (x_1 - x_2, x_1 + x_2, x_3 - x_4, x_3 + x_4, x_5 - x_6, x_5 + x_6, \ldots), A_2 = A_0.$$

Then

$$\sigma(A_0) = \sigma(A_2) = \{1/(2n), 2 + 1/(2n) \mid n \in \mathbb{N}\} \cup \{0, 2\},$$
$$X_{2-} = \{x \in l_2 \mid x_2 = x_4 = x_6 = \ldots = 0\},$$
$$X_{0+} = \{x \in l_2 \mid x_1 = x_3 = x_5 = \ldots = 0\},$$
$$W = \{x \in l_2 \mid x_1 = -x_2, x_3 = -x_4, x_5 = -x_6, \ldots\}.$$

Since $W \dot{+} X_{2-} = X$, conditions of Theorem 2 are fulfilled so for any bounded sequence y the equation (4) has a unique bounded solution.

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Чайковський А.В., Лагода О.А. *Обмежені розв'язки різницевого рівняння зі скінченною кількістю стрибків операторного коефіцієнта* // Карпатські матем. публ. — 2020. — Т.12, №1. — С. 165–172.

В роботі вивчається питання існування єдиного обмеженого розв'язку різницевого рівняння зі змінним операторним коефіцієнтом в банаховому просторі. Існує добре розвинена теорія відповідних рівнянь зі сталим коефіцієнтом, в рамках якої поставлене питання розв'язане в термінах спектру операторного коефіцієнта. Для випадку змінного операторного коефіцієнта відповідні умови також відомі, проте є дуже складними для перевірки. Тому важливим є дати відповідь на поставлене питання для тих частинних випадків змінного коефіцієнта, коли відповідні умови легко перевірити. Одним з таких випадків є рівняння з кусково-сталим операторним коефіцієнтом. Відомі достатні умови існування та єдиності обмеженого розв'язку для випадку одного стрибка. В цій роботі ці результати узагальнюються для випадку скінченного числа стрибків операторного коефіцієнта. Крім того, за додаткового припущення отримано необхідні та достатні умови існування та єдиності обмеженого розв'язку.

Ключові слова і фрази: різницеве рівняння, обмежений розв'язок, банахів простір.

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THE NONLOCAL BOUNDARY VALUE PROBLEM WITH PERTURBATIONS OF MIXED BOUNDARY CONDITIONS FOR AN ELLIPTIC EQUATION WITH CONSTANT COEFFICIENTS. II

In this paper we continue to investigate the properties of the problem with nonlocal conditions, which are multipoint perturbations of mixed boundary conditions, started in the first part. In particular, we construct a generalized transform operator, which maps the solutions of the self-adjoint boundary-value problem with mixed boundary conditions to the solutions of the investigated multipoint problem. The system of root functions $V(L)$ of operator L for multipoint problem is constructed. The conditions under which the system $V(L)$ is complete and minimal, and the conditions under which it is the Riesz basis are determined. In the case of an elliptic equation the conditions of existence and uniqueness of the solution for the problem are established.

Key words and phrases: differential equation with partial derivatives, root functions, Fourier method, method of transform operators, Riesz basis.

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1 Introduction and main results

In the papers [1–5] by the methods of the theory of transformation operators (see [12]), we studied nonself-adjoint problems with a multipoint spectrum and an infinite number of root functions (see [10]). In the one-dimensional case such problems are generated by regular but not strongly regular Birkhoff conditions (see [11]). For equations containing involution, multipoint problems were studied in the works [5–7]. In this paper we continue the study of the problem for an elliptic equations with constant coefficients with mixed conditions initiated in [1,7,8].

For our investigation we will use the following notations. Let $G := \{x := (x_1, x_2) \in \mathbb{R}^2 : 0 < x_1,\ x_2 < 1\}$, $D_1,\ D_2$ be the operators of differentiation by the variables $x_1,\ x_2$ respectively, $W_2^{2n}(G)$ be a Sobolev space with the following scalar product and norm respectively:

$$
\begin{aligned}
&(u;v)_{W_2^{2n}(G)} := (u;v)_{L_2(G)} + (D_1^{2n}u; D_1^{2n}v)_{L_2(G)} + (D_2^{2n}u; D_2^{2n}v)_{L_2(G)},\\
&\|u\|^2_{W_2^{2n}(G)} := (u;u)_{W_2^{2n}(G)},\\
&W_2^{2n}(0,1) := \{y \in AC[0,1] :\ y^{(r)} \in C[0,1],\ r = 1,2,\dots,2n-1,\ y^{(2n)} \in L_2(0,1)\},\\
&L_{s,2}(0,1) := \{y(t) \in L_2(0,1) :\ y(t) = (-1)^s y(1-t)\},\ s \in \{0,1\},
\end{aligned}
$$

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$[L_2(0,1)]$ be a set of linear and continuous operators given in the space $L_2(0,1)$.

Let us consider the multipoint problem

$$L(D)u := \sum_{p=0}^{n} a_p D_1^{2p} D_2^{2n-2p} u = f(x),\ x \in G, \tag{1}$$

$$\begin{cases} \ell_{s,1}u & := D_1^{2s-2}u|_{x_1=0} + D_1^{2s-2}u|_{x_1=1} + \ell_s^1 u = 0,\ s = 1,2,\ldots,n, \\ \ell_{n+s,1}u & := D_1^{2s-2}u|_{x_1=0} - D_1^{2s-2}u|_{x_1=1} = 0,\ s = 1,2,\ldots,n, \\ \ell_{s,2}u & := D_2^{2s-2}u|_{x_2=0} + D_2^{2s-2}u|_{x_2=1} = 0,\ s = 1,2,\ldots,n, \\ \ell_{n+s,2}u & := D_2^{2s-1}u|_{x_2=0} + D_2^{2s-1}u|_{x_2=1} + \ell_s^2 u = 0,\ s = 1,2,\ldots,n, \\ \ell_s^j u & := \sum\limits_{q=0}^{k_{s,j}} \sum\limits_{r=0}^{k_j} b_{q,r,s,j} D_j^q u(x)|_{x_j=x_{r,j}},\ s = 1,2,\ldots,n, \end{cases} \tag{2}$$

where $0 = x_{1,j} < x_{2,j} < \ldots < x_{k_j,j} = 1$, a_p, $b_{q,r,s,j} \in \mathbb{R}$, $k_{s,j} < 2n$, $k \in \mathbb{N}$, $s = 1,2,\ldots,n$, $p = 0,1,\ldots,n$, $j = 1,2$.

Let $L :\ L_2(G) \to L_2(G)$ be the operator of the problem (1)–(2), $Lu := L(D)u,\ u \in D(L)$, $D(L) := \{u \in W_2^{2n}(G) :\ \ell_{s,j}u = 0,\ s = 1,2,\ldots,2n,\ j = 1,2\}$.

Let us consider the following assumptions and theorems, that are necessary for further investigation.

Assumption P_1 : $b_{q,r,s,j} = (-1)^{q+j} b_{q,k_j-r,s,j}$, $x_{r,j} = 1 - x_{k_j-r,j}$, $q = 0,1,\ldots,k_{s,j}$, $r = 0,1,\ldots,k_j$, $s = 1,2,\ldots,n$, $j = 1,2$.

Assumption P_2 : there exists a positive number C_1 such that the inequality

$$C_1|\mu|^{2n} \leq \left| \sum_{p=0}^{n} a_p \mu_1^p \mu_2^{n-p} \right|$$

holds for $\mu := (\mu_1, \mu_2) \in \mathbb{R}^2$, $|\mu|^2 := |\mu_1|^2 + |\mu_2|^2 \to \infty$.

Assumption P_3 : $k_{s,1} \leq 2s-2$, $k_{s,2} \leq 2s-1$, $s = 1,2,\ldots,n$.

Theorem 1. *Let Assumption P_1 holds. Then, the operator L has a set of eigenvalues*

$$\sigma := \left\{ \lambda_{k,m} = (-1)^n \sum_{p=0}^{n} a_p \mu_{k,1}^p \mu_{m,2}^{n-p},\ \mu_{k,1} = \pi^2 k^2,\ \mu_{m,2} = \pi^2 (2m-1)^2,\ k,m \in \mathbb{N} \right\}, \tag{3}$$

and a system $V(L)$ of root functions, which is complete and minimal in the space $L_2(G)$.

Let Assumptions P_1–P_3 hold. Then, the operator L has the system $V(L)$, which is the Riesz basis of the space $L_2(G)$.

Theorem 2. *Let Assumptions P_1–P_3 hold. Then for an arbitrary function $f \in L_2(G)$ there exists a unique solution $u \in W_2^{2n}(G)$ of problem* (1)–(2).

Our research is structured as follows. In Section 2 we investigate the properties of the problem with self-adjoint boundary conditions. In Section 3 we study the spectral properties for nonlocal problem with nonself-adjoint boundary conditions. In Section 4 we construct a commutative group of transformation operators. Using spectral properties of multipoint problem and conditions for completeness the basis properties of the systems of eigenfunctions are established in Section 5. In Section 6 the main theorems are proved.

2 THE SELF-AJOINT PROBLEM

Let us consider for equation

$$-z^{(2)}(t) = \mu z(t), \ \ t \in (0,1), \ \mu \in \mathbb{C}, \tag{4}$$

the problem with boundary conditions

$$z^{(r)}(0) + z^{(r)}(1) = 0, \ \ r = 0,1. \tag{5}$$

Let $B_0 : L_2(0,1) \to L_2(0,1)$ be the operator of problem (4)–(5):

$$\begin{aligned} &B_0 z(t) := -z^{(2)}(t), \ \ z(t) \in D(B_0), \\ &D(B_0) := \left\{ z \in W_2^2(0,1) : \ z^{(r)}(0) + z^{(r)}(1) = 0, \ r = 0,1 \right\}, \\ &T_2 := \Big\{ \tau_{r,m,2}(t) \in L_2(0,1) : \ \tau_{0,m,2}(t) := \sqrt{2}\sin \pi(2m-1)t, \\ &\qquad\qquad \tau_{1,m,2}(t) := \sqrt{2}\cos \pi(2m-1)t, \ m \in \mathbb{N}, \ r = 0,1 \Big\}, \\ &T_{j,2} := \{\tau_{j,m,2}(t) \in L_{j,2}(0,1) : \ \ m \in \mathbb{N}\}, \ j = 0,1. \end{aligned}$$

Lemma 1. *The operator B_0 has a point spectrum*

$$\sigma(B_0) := \{\mu_{m,2} \in \mathbb{R} : \ \mu_{m,2} = \pi^2(2m-1)^2, \ m \in \mathbb{N}\}$$

and system of eigenfunctions T_2.

Proof. A direct substitution proves that the elements of system T_2 are the eigenfunctions of operator B_0, which correspond to the eigenvalues $\sigma(B_0)$.

Taking into account, that the subsystem of eigenfunctions $T_{j,2}$ of the operator B_0 is an orthonormal base of space $L_{j,2}(0,1)$, $j = 0,1$, we obtain the statement of lemma. □

We consider the spectral problem

$$L(D)u := \sum_{p=0}^{n} a_p D_1^{2p} D_2^{2n-2p} u = \lambda u(x), \ x \in G, \ \lambda \in \mathbb{C}, \tag{6}$$

$$\begin{cases} \ell_{0,s,1}u & := D_1^{2s-2}u|_{x_1=0} + D_1^{2s-2}u|_{x_1=1} = 0, \\ \ell_{0,n+s,1}u & := D_1^{2s-2}u|_{x_1=0} - D_1^{2s-2}u|_{x_1=1} = 0, \\ \ell_{0,s,2}u & := D_2^{2s-2}u|_{x_2=0} + D_2^{2s-2}u|_{x_2=1} = 0, \\ \ell_{0,n+s,2}u & := D_2^{2s-1}u|_{x_2=0} + D_2^{2s-1}u|_{x_2=1} = 0, \ s = 1,2,\ldots,n. \end{cases} \tag{7}$$

Let $L_0 : \ L_2(G) \to L_2(G)$ be the operator of the problem (6)–(7):

$$\begin{aligned} &L_0 u := L(D)u, \ u \in D(L_0), \ D(L_0) := \left\{ u \in W_2^{2n}(G) : \ \ell_{0,r,j}u = 0, \ r = 1,2,\ldots,2n, \ j = 1,2 \right\}, \\ &T_1 := \left\{ \tau_{s,k,1}(x_1) \in L_2(0,1) : \ \tau_{s,k,1}(x_1) := \sqrt{2}\sin \pi(2k-s)x_1, \ k = 1,2,\ldots, \ s = 0,1 \right\}, \\ &V(L_0) := \left\{ v_{r,s,k,m}(x,L_0) \in L_2(G) : v_{r,s,k,m}(x,L_0) := \tau_{r,k,1}(x_1)\tau_{s,m,2}(x_2), r,s = 0,1, m,k = 1,2,\ldots \right\}. \end{aligned}$$

Lemma 2. *The operator L_0 has eigenvalues* (3) *and a system of eigenfunctions $V(L_0)$.*

Proof. By direct substitution we obtain that $v_{r,s,k,m}(x, L_0) \in D(L_0)$ and

$$L_0 v_{r,s,k,m}(x, L_0) = \lambda_{k,m} v_{r,s,k,m}(x, L_0),$$
$$\lambda_{k,m} = (-1)^n \pi^{2n} \sum_{p=0}^{n} a_p k^{2p} (2m-1)^{2n-2p}, \quad k, m \in \mathbb{N}.$$

Therefore, the set of eigenvalues (3) for the operator L_0 corresponds the system of eigenfunctions $V(L_0)$. □

Theorem 3. *Let Assumption P_2 holds. Then for any function $f \in L_2(G)$ there exists a unique solution $u \in W_2^{2n}(G)$ of the problem* (6)–(7).

Proof. Let us expand the functions $f,\ u \in L_2(G)$ as a series by the system $V(L_0)$:

$$f = \sum_{r,s,k,m} f_{r,s,k,m} v_{r,s,k,m}(x, L_0),$$
$$u = \sum_{r,s,k,m} u_{r,s,k,m} v_{r,s,k,m}(x, L_0).$$

Substituting these functions into the equation (1), we obtain

$$u_{r,s,k,m} = \lambda_{k,m}^{-1} f_{r,s,k,m},\ r,\ s \in \{0,1\},\ k, m \in \mathbb{N}.$$

Consider the ratio

$$D_1^{2p} D_2^{2n-2p} u = (-1)^n \sum_{r,s,k,m} \mu_{k,1}^{p} \mu_{m,2}^{n-p} \lambda_{k,m}^{-1} f_{r,s,k,m} v_{r,s,k,m}(x, L_0),\ p = 0, 1, \ldots, n.$$

Taking into account Assumption P_2 for some $C_2 > 0$, we obtain

$$|\mu_{k,1}^{p} \mu_{m,2}^{n-p} \lambda_{k,m}^{-1}| \le C_2,\ p = 0, 1, \ldots, n,$$
$$\|D_1^{2p} D_2^{2n-2p} u\|_{L_2(G)} \le C_2 \|f\|_{L_2(G)},\ p = 0, 1, \ldots, n,$$
$$\|u\|_{L_2(G)} \le C_2 \|f\|_{L_2(G)}.$$

Therefore, $u \in W_2^{2n}(G)$. Theorem is proved. □

For fixed $k \in \mathbb{N},\ s \in \{0,1\}$, we consider the solutions of the problem (6)–(7) as a product

$$u(x) := z(x_2)\, \tau_{s,k,1}(x_1). \tag{8}$$

To determine the unknown function $z(x_2)$ we obtain the following eigenvalues problem

$$\sum_{p=0}^{n} (-1)^p a_p \mu_{k,1}^{p} z^{(2n-2p)}(x_2) = \lambda z(x_2),\ x_2 \in (0,1),\ \lambda \in \mathbb{C}, \tag{9}$$

$$\begin{cases} l_{0,s,2} z & := z^{(2s-2)}(0) + z^{(2s-2)}(1) = 0, \\ l_{0,n+s,2} z & := z^{(2s-1)}(0) + z^{(2s-1)}(1) = 0,\ s = 1, 2, \ldots, n. \end{cases} \tag{10}$$

Let $L_{0,k} : L_2(0,1) \to L_2(0,1)$ be the operator of the problem (9)–(10)

$$L_{0,k}z := \sum_{p=0}^{n} (-1)^p a_p \mu_{k,1}^p z^{(2n-2p)}(x_2),\ z \in D\left(L_{0,k}\right),$$

$$D\left(L_{0,k}\right) := \{z \in W_2^{2n}(0,1) :\ l_{0,r,2}z = 0,\ \ r = 1,2,\ldots,2n\}.$$

The roots $\varpi_{r,k}\left(\lambda\right)$ of the equation

$$\sum_{p=0}^{n} (-1)^p a_p \varpi^{2n-2p} \mu_{k,1}^p = \lambda,$$

which is characteristic for equation (9), are chosen so that

$$\operatorname{Re}\varpi_{n,k}\left(\lambda\right) \le \operatorname{Re}\varpi_{n-1,k}\left(\lambda\right) \le \ldots \le \operatorname{Re}\varpi_{1,k}\left(\lambda\right) \le 0,\ \varpi_{n+q,k}\left(\lambda\right) = -\varpi_{n,k}\left(\lambda\right),\ q = 1,2,\ldots,n.$$

Let us determine the functions

$$\begin{cases} z_{0,q,k}\left(x_2,\lambda\right) := \frac{1}{2}(\exp\varpi_{q,k}\left(\lambda\right)x_2 + \exp\varpi_{q,k}\left(\lambda\right)(1-\ x_2)) \in L_{0,2}(0,1), \quad q = 1,\ldots,n, \\ z_{0,n+q,k}\left(x_2,\lambda\right) := \frac{1}{2}(\exp\varpi_{q,k}\left(\lambda\right)x_2 - \exp\varpi_{q,k}\left(\lambda\right)(1-x_2)) \in L_{1,2}(0,1), \quad q = 1,\ldots,n. \end{cases} \tag{11}$$

Substituting the general solution

$$z(x_2) = \sum_{r=1}^{2n} c_r z_{0,r,k}\left(x_2,\lambda\right)$$

of the equation (9) into boundary conditions (10), we obtain the equation to determine the eigenvalues for $L_{0,k}$

$$\Delta(\lambda,k) = \det(l_{0,s,2}z_{0,r,k}\left(x_2,\lambda\right))_{r,s=1}^{2n} = 0.$$

Taking into account the ratio $z_{0,pn+q,k}\left(x_2,\lambda\right) \in L_{p,2}(0,1),\ l_{0,pn+j,2} \in W_p^*,\ p \in \{0,1\},$ we obtain

$$\begin{aligned} l_{0,n+j,2}z_{0,q,k}\left(x_2,\lambda\right) &= 0,\ j,\ q = 1,2,\ldots,n, \\ l_{0,j,2}z_{0,n+q,k}\left(x_2,\lambda\right) &= 0,\ j,\ q = 1,2,\ldots,n, \\ \Delta(\lambda,k) &= \Delta_0(\lambda,k)\Delta_1(\lambda,k), \\ \Delta_p(\lambda,k) &= \det(l_{0,pn+j,2}z_{0,pn+q,k}\left(x_2,\lambda\right))_{j,q=1}^{n},\ p = 0,1, \end{aligned}$$

$$\Delta(\lambda,k) = \prod_{q=1}^{n} \varpi_q(\lambda)(1+e^{\varpi_q(\lambda)})^2 \prod_{1\le j<q\le n} \left(\varpi_j\left(\lambda\right) - \varpi_q\left(\lambda\right)\right)^2 = 0. \tag{12}$$

Let $\varpi_{1,m,k} = \imath\pi(2m-1),\ \imath := \sqrt{-1}$ are the roots of equation (12) and $\varpi_{q,m,k} = \varpi_q(\lambda_{m,k})$, $q = 2,3,\ldots,n,\ \ m = 1,2,\ldots$. By direct calculations we obtain that the operator $L_{0,k}$ has the system of eigenfunctions

$$V(L_{0,k}) := \{v_{s,m}(x_2,L_{0,k}) \in L_2(0,1) : v_{s,m}(x_2,L_{0,k}) := \tau_{s,m,2}(x_2),\ s = 0,1,\ m = 1,2,\ldots\}$$

and the set of eigenvalues $\sigma_k := \left\{\lambda_{k,m} \in \sigma :\ m \in \mathbb{N}\right\}$.

3 THE NONSELF-AJOINT PROBLEM

Let us consider the spectral problem

$$\begin{cases} -z^{(2)}(t) = \mu z(t), \ \mu \in \mathbb{C}, \ t \in (0,1), \\ l_1 z := z(0) + z(1) = 0, \\ l_2 z := z^{(1)}(0) + z^{(1)}(1) + l_2^1 z = 0, \end{cases} \tag{13}$$

where

$$l_2^1 z := b\big(z^{(1)}(0) - z^{(1)}(1)\big), \ b \in \mathbb{R}. \tag{14}$$

Let $B : L_2(0,1) \to L_2(0,1)$ be the operator of the problem (13)–(14) and $V(B)$ the system of root functions for operator B.

Taking into account the results of the papers [1, 2], we define eigenfunctions and attach functions of the operators B by formulas

$$\begin{aligned} &v_{1,m}(t,B) = \tau_{1,m,2}(t), \\ &v_{0,m}(t,B) = \Big(1 + b(2t-1)\Big)\tau_{0,m,2}(t), \ m = 1,2,\dots . \end{aligned}$$

Therefore, the operator B has the system $V(B)$ of root functions, which are related by ratio

$$Bv_{0,m}(t,B) = \mu_{m,2} v_{0,m}(t,B) + \xi_m v_{1,m}(t,B),$$

where $\xi_m = 4b\pi(2m-1), \ m = 1,2,\dots$.

Taking into account the results of the paper [2], we obtain the following statement.

Lemma 3. *The operator B has the point spectrum $\sigma(B_0)$ and the system of root functions $V(B)$, which is the Riesz basis of the space $L_2(0,1)$.*

We consider the solutions of the spectral problem (6), (2) as a product (8). To determine the unknown function $z(x_2)$ we obtain for the equation (9) the eigenvalues problem with the conditions

$$\begin{cases} l_{s,2} z := z^{(2s-2)}(0) + z^{(2s-2)}(1) = 0, \\ l_{n+s,2} z := z^{(2s-1)}(0) + z^{(2s-1)}(1) + l_{n+s} z = 0, \ s = 1,2,\dots,n, \end{cases} \tag{15}$$

where

$$l_{n+s} z := \sum_{q=0}^{k_{s,2}} \sum_{r=0}^{k_2} b_{q,r,s,2} z^{(q)}(x_{r,2}), \ s = 1,2,\dots,n. \tag{16}$$

Let L_k be the operator of the problem (9), (15)–(16):

$$\begin{aligned} L_k z &:= \sum_{p=0}^{n} (-1)^p a_p \mu_{k,1}^p z^{(2n-2p)}(x_2), \ z \in D(L_k), \\ D(L_k) &:= \{ z \in W_2^{2n}(0,1) : \ l_{r,2} z = 0, \ r = 1,2,\dots,2n \}. \end{aligned}$$

Lemma 4. *Let Assumption P_1 holds. Then the eigenvalues of the operators $L_{0,k}$ and L_k coincide.*

Proof. Substituting the general solution (11) of the equation (9) into boundary conditions (10), we obtain the equation to determine the eigenvalues of the operator L_k

$$\Delta^1(\lambda,k) = \det(l_{s,2}z_{0,p,k}(x_2,\lambda))_{p,s=1}^{2n} = 0.$$

Taking into account the relations

$$z_{0,rn+q,k}(x_2,\lambda) \in L_{r,2}(0,1),\ \ell_{0,sn+j,2} \in W_s^*,\ l_{n+j}^1 \in W_0^*,\ s,\ r \in \{0,1\},\ j \in \{1,2,\dots,n\},$$

we obtain

$$\begin{aligned}
l_{j,2}z_{0,q,k}(x_2,\lambda) &= \varpi_q^{2j-2}(\lambda)(1+e^{\varpi_q(\lambda)}),\\
l_{n+j,2}z_{0,n+q,k}(x_2,\lambda) &= \varpi_q(\lambda)^{2j-1}(1+e^{\varpi_q(\lambda)}),\\
l_{j,2}z_{0,n+q,k}(x_2,\lambda) &= l_{0,j,2}z_{0,n+q,k}(x_2,\lambda) = 0,\\
l_{n+j,2}z_{0,n+q,k}(x_2,\lambda) &= l_{0,n+j,2}z_{0,n+q,k}(x_2,\lambda),\\
l_{j,2}z_{0,q,k}(x_2,\lambda) &= l_{j,2,2}z_{0,q,k}(x_2,\lambda),\ \ q=1,2,\dots,n,\\
\Delta(\lambda,k) &= \Delta_0(\lambda,k)\Delta_1(\lambda,k),\\
\Delta_s(\lambda,k) &= \det(l_{0,sn+j,2}z_{0,sn+q,k}(x_2,\lambda))_{j,q=1}^{n},\ \ s=0,1,
\end{aligned}$$

and

$$\Delta^1(\lambda,k) = \prod_{q=1}^{n}\varpi_q(\lambda)(1+e^{\varpi_q(\lambda)})^2\prod_{1\le j<q\le n}(\varpi_j(\lambda)-\varpi_q(\lambda))^2 = 0.$$

Therefore $\Delta^1(\lambda,k) \equiv \Delta(\lambda,k)$. The lemma is proved. □

Let us consider the boundary-value problem for the equation (9)

$$\begin{cases}
l_{1,s,2}z := z^{(2s-2)}(0)+z^{(2s-2)}(1) = 0,\ s=1,2,\dots,n,\\
l_{1,n+s,2}z := z^{(2s-1)}(0)+z^{(2s-1)}(1) = 0,\ j\neq s,\ s=1,2,\dots,n,\\
l_{1,n+j,2}z := z^{(2j-1)}(0)+z^{(2j-1)}(1)+\ell_{n+j}^1 z = 0,
\end{cases} \tag{17}$$

where

$$l_{n+j}^1 z := b_j(z^{(2j-1)}(0)-z^{(2j-1)}(1)) = 0,\ \ b_j\in\mathbb{R}. \tag{18}$$

Let $L_{1,j,k} : L_2(0,1) \to L_2(0,1)$ be the operator of the problem (9), (17)–(18)

$$\begin{aligned}
L_{1,j,k}z &:= \sum_{p=0}^{n}(-1)^p a_p \mu_{k,1}^p z^{(2n-2p)}(x_2),\ z\in D\left(L_{1,j,k}\right),\\
D\left(L_{1,j,k}\right) &:= \left\{z\in W_2^{2n}(0,1):\ \ l_{1,r,2}z=0,\ \ r=1,2,\dots,2n\right\}.
\end{aligned}$$

We determine the system of functions

$$z_{n+1,m,k}(x_2) := \frac{1}{2}(1-2x_2)\sin\rho_{m,2}x_2, \tag{19}$$

$$z_{n+q,m,k}(x_2) := \frac{1}{2}(1+e^{\varpi_{q,m,k}})^{-1}(e^{\varpi_{q,m,k}x_2}-e^{\varpi_{q,m,k}(1-x_2)}),\ \ q=2,3,\dots,n, \tag{20}$$

and a square matrix of order n, whose elements are defined as follows. j-th row is determined by the functions (19), (20) and elements of other rows are determined by numbers

$$\vartheta_{1,r,m,k} = \rho_{m,2}^{1-2r} l_{1,n+r,2} z_{n+1,m,k}(x_2) = (-1)^{r-1},$$
$$\vartheta_{q,r,m,k} = \rho_{m,2}^{1-2r} l_{1,n+r,2} z_{n+q,m,k}(x_2) = \omega_{q,m,k}^{2r-1},$$

where $q = 2, 3, \ldots, n,\ r \neq j,\ \ r = 1, 2, \ldots, n$.

We denote the determinant of the resulting matrix by $z_{j,m,k}(x_2),\ m = 1, 2, \ldots$.

Let $\Delta_{j,q,m,k} := \det(\vartheta_{s,r,m,k})_{r,s=\overline{1,n}}^{r\neq j,\, s\neq q}$. Then $z_{j,m,k}(x_2) = \sum\limits_{q=1}^{n} \Delta_{j,q,m,k} z_{n+q,m,k}(x_2)$.

Remark 1. *For any fixed $k \in \mathbb{N}$ and $m \to \infty$ we obtain the relations*

$$\delta_{1,r,m,k} = \vartheta_{1,r,m,k}\rho_{m,2}^{-1} = \imath, \quad \delta_{q,r,m,k} = \vartheta_{q,r,m,k}\rho_{m,2}^{-1} = \varepsilon_q \left(1 + O(m)^{-1}\right),$$

where ε_q are the roots of equation $(-1)^n \varepsilon^{2n} = 1,\ Im\, \varepsilon_q < 0,\ q = 2, 3, \ldots, n$.

Substituting the function $z_{j,m,k}(x_2)$ into boundary conditions (17)–(18), we obtain the equalities

$$l_{1,s,2} z_{j,m,k} = 0,\ s \neq n+j,\ s = 1, 2, \ldots, 2n,\ \ m = 1, 2, \ldots,$$
$$l_{1,n+j,2} z_{j,m,k}(x_2) := c_{j,m,k},$$
$$c_{j,m,k} = \rho_{m,2}^{2j-1} Z_{m,k} \prod_{q=1}^{n} \omega_{q,m,k},\ \ m = 1, 2, \ldots,$$

where $Z_{m,k}$ is the Vandermonde determinant of order n, which is constructed by numbers $\delta_{q,r,m,k}^2,\ q = 1, 2, \ldots, n$.

Remark 2. *For an arbitrary $k \in \mathbb{N}$ the number sequence $\{Z_{m,k}\}_{m=1}^{\infty}$ as $m \to \infty$ converges to the Vandermonde determinant $Z_n\left(\varepsilon_1^2, \ldots, \varepsilon_n^2\right)$, which is constructed by numbers $\varepsilon_1^2, \ldots, \varepsilon_n^2$.*

In addition, the sequence $\{\delta_{q,r,m,k}\}_{m=1}^{\infty}$ converges to $\varepsilon_q,\ q = 1, 2, \ldots, n$.

Thus, there are positive numbers C_3, C_4 such that the following inequality holds

$$0 < C_3 \leq \left|c_{j,m,k}\right|^{-1} \rho_{m,2}^{2j-1} \leq C_4 < \infty,\ j \in \{1, 2, \ldots, n\},\ m = 1, 2, \ldots . \tag{21}$$

We determine the function $z_{1,j,m,k}(x_2)$ such that the following inequality holds

$$z_{1,j,m,k}(x_2) = z_{n+1,m,k}(x_2) + \sum_{q=2}^{n} \Delta_{j,1,m,k}^{-1} \Delta_{j,q,m,k} z_{n+q,m,k}\left(x_2\right).$$

Therefore,

$$z_{1,j,m,k}(x_2) = \Delta_{j,1,m,k}^{-1} z_{j,m,k}(x_2), \tag{22}$$

$$\ell_{1,n+j} z_{1,j,m,k}(x_2) := \chi_{j,m,k}, \quad \chi_{j,m,k} = \Delta_{j,1,m,k}^{-1} Z_{m,k} \rho_{m,2}^{2j-1} \prod_{q=1}^{n} \omega_{q,m,k},\ \ m = 1, 2, \ldots .$$

By substituting into boundary conditions (17)–(18) we conclude that the operator $L_{1,j,k}$ has eigenfunctions

$$v_{1,m}(x_2, L_{1,j,k}) := \tau_{1,m,2}(x_2),\ \ m = 1, 2, \ldots . \tag{23}$$

The root function $v_{0,m}\left(x_2, L_{1,j,k}\right)$ of operator $L_{1,j,k}$ is determined by the sum

$$v_{0,m}(x_2, L_{1,j,k}) := \tau_{0,m,2}(x_2) + \eta_{j,m,k} z_{1,j,m,k}(x_2), \quad m = 1, 2, \dots . \tag{24}$$

To determine the unknown parameters $\eta_{j,m,k}$ we substitute the expression (24) into boundary conditions (17)–(18).

Taking into account the formula (22), we obtain

$$\eta_{j,m,k} = -l^1_{n+j}\tau_{0,m,2}(l_{1,n+j,2} z_{1,j,m,k})^{-1}, \quad m = 1, 2, \dots .$$

From the definition of the determinant $\Delta_{j,1,m,k}$ we have inequality $|\Delta^{-1}_{j,1,m,k}| \le C_5$.

Therefore, taking into account the inequality $|l^1_{n+j}\tau_{0,m,2}| \le C_6 b_j \rho^{2j-1}_{m,2}$ and the estimates (21), we obtain the relations

$$|\eta_{j,m,k}| \le C_7, \quad j \in \{1, 2, \dots, n\}, \quad m \in \mathbb{N}. \tag{25}$$

Thus, the operator $L_{1,j,k}$ has the system of root functions (23)–(24).

Let us consider the operator $B_{j,k} : \ L_2(0,1) \to L_2(0,1)$, which has a point spectrum $\sigma(B_0)$ and the system of root functions

$$\begin{aligned} V(B_{j,k}) := \Big\{ v_{r,m}(x_2, B_{j,k}) \in L_2(0,1) : \ \ & v_{1,m}(x_2, B_{j,k}) := \tau_{1,m,2}(x_2), \\ & v_{0,m}(x_2, B_{j,k}) := \Big(1 + \eta_{j,m,k}(2x_2 - 1)\Big)\tau_{0,m,2}(x_2), \quad m = 1, 2, \dots \Big\}. \end{aligned}$$

Lemma 5. *The system of functions $V(B_{j,k})$ is the Riesz basis in the space $L_2(0,1)$.*

Proof. From the inequality (25) we obtain that the system $V(B_{j,k})$ is Bessel (see [10]). Therefore, the operator $R(B_{j,k})\tau_{r,m,2}(x_2) := v_{r,m}(x_2, B_{j,k})$, $r = 0, 1$, $m = 1, 2, \dots$, is continuous in $L_2(0,1)$.

In addition, the operator $S(B_{j,k}) := R(B_{j,k}) - E$ is continuous in the space $L_2(0,1)$.

Taking into account the definition of functions in $V(B_{j,k})$ we obtain

$$S(B_{j,k}) : \ L_{1,2}(0,1) \to 0, \quad S(B_{j,k}) : \ L_{0,2}(0,1) \to L_{1,2}(0,1).$$

Thus, $S^2(B_{j,k})$ and $R^{-1}(B_{j,k}) := E - R(B_{j,k}) \in [L_2(0,1)]$.

Therefore, from the Bari theorem (see [10]) the system $V(B_{j,k})$ is the Riesz basis in the space $L_2(0,1)$. ☐

Lemma 6. *Let Assumption P_1 holds. Then the operator $L_{1,j,k}$ has the system of root functions $V\left(L_{1,j,k}\right)$, which is the Riesz basis in the space $L_2(0,1)$.*

Proof. The system of functions $V(L_{1,j,k})$ is complete and minimal in space $L_2(0,1)$ because the boundary conditions (17)–(18) are regular by Birkhoff (see [11]).

We show that the systems of functions $V(B_{j,k})$ and $V(L_{1,j,k})$ are quadratically approximate in space $L_2(0,1)$.

Let us estimate the sum of the series

$$\begin{aligned} H(L_{1,j,k}; B_{j,k}) &= \sum_{s,m} \|v_{s,m}(x_2, L_{1,j,k}) - v_{s,m}(x_2, B_{j,k})\|^2_{L_2(0,1)} \\ &= \sum_{m=1}^{\infty} \|v_{0,m}(x_2, L_{1,j,k}) - v_{0,m}(x_2, B_{j,k})\|^2_{L_2(0,1)}, \end{aligned}$$

$$H(L_{1,j,k}, B_{j,k}) \le \max |\eta_{j,m,k}|^2 |\Delta^{-2}_{j,1,m,k}| \sum_m \sum_{q=2}^{n} |\Delta_{j,q,m,k}|^2 \|z_{1,q,m,k}(x_2)\|^2_{L_2(0,1)}.$$

Taking into account the choice of numbers $\varpi_{r,m,k}$, we obtain the estimate

$$H(L_{1,j,k}, B_{j,k}) < \infty.$$

Therefore, the complete and minimal system $V(L_{1,j,k}) \in L_2(0,1)$ is quadratically approximate to Riesz basis $V(B_{1,k})$.

Thus, applying the Bari theorem (see [10]), we obtain the statement of Lemma 6. □

4 TRANSFORMATION OPERATORS

Let us determine

$$z_{2,j,m,k}(x_2) := \eta_{j,m,k} z_{1,j,m,k}(x_2), \quad m = 1, 2, \dots .$$

By choosing of arbitrary sequence of real numbers $\theta = \{\theta_m\}_{m=1}^{\infty}$ we define the operator $B_{j,\theta} : L_2(0,1) \to L_2(0,1)$, which is generated by the differential expression

$$\sum_{p=0}^{n} (-1)^p a_p \mu_{k,1}^{p} z^{(2n-2p)}(x_2)$$

and has the system $V(B_{j,\theta}) := \Big\{ v_{s,m}(x_2, B_{j,\theta}) \in L_2(0,1) : \ s = 0,1, \ m = 1,2,\dots \Big\}$ of functions

$$\begin{aligned} &v_{1,m}(x_2, B_{j,\theta}) := \tau_{1,m,2}(x_2), \\ &v_{0,m}(x_2, B_{j,\theta}) := \tau_{0,m,2}(x_2) + \theta_m z_{2,j,m,k}(x_2), \ m = 1,2,\dots, \end{aligned} \tag{26}$$

which are root functions in the sense of equalities

$$B_{j,\theta} v_{1,m}(x_2, B_{j,\theta}) = \lambda_{k,m} v_{1,m}(x_2, B_{j,\theta}), \ m = 1,2,\dots, \tag{27}$$

$$B_{j,\theta} v_{0,m}(x_2, B_{j,\theta}) = \lambda_{k,m} v_{0,m}(x_2, B_{j,\theta}) + \xi_{j,k,m} v_{1,m}(x_2, B_{j,\theta}), \tag{28}$$

where $\xi_{j,k,m} = (-1)^n 4n \eta_{j,m,k} \theta_m \sum\limits_{p=0}^{n} a_p C_{2n}^{2n-2p} \mu_{k,1}^{p} \rho_{m,2}^{2n-2p-1}$, $m = 1,2,\dots$, and has the set of eigenvalues σ_k.

Let us consider the operators $R(B_{j,\theta})$, which are defined in the space $L_2(0,1)$ by

$$\begin{aligned} &R(B_{j,\theta}) := E + S(B_{j,\theta}), \\ &S(B_{j,\theta})\tau_{1,m,2}(x_2) := 0, \ S(B_{j,\theta})\tau_{0,m,2}(x_2) := \theta_m z_{2,j,m,k}(x_2), \quad m = 1,2,\dots . \end{aligned}$$

Let $Q_j(L_{0,k})$ be the set of operators $B_{j,\theta}$, which have purely point spectrum σ_k and the system of root functions (26), $\Gamma_j(L_{0,k})$ be the set of operators $R(B_{j,\theta})$.

For any B_{j,θ_1}, $B_{j,\theta_2} \in \Gamma_j(L_{0,k})$, we define on $\Gamma_j(L_{0,k})$ the commutative multiplication operation

$$R(B_{j,\theta_1})R(B_{j,\theta_2}) = E + S(B_{j,\theta_1}) + S(B_{j,\theta_2}) = R(B_{j,\theta_2})R(B_{j,\Theta_1})$$

and the inverse operator $R^{-1}(B_{j,\theta}) = E - S(B_{j,\theta})$, $B_{j,\theta} \in \Gamma_1(L_{0,k})$.

Therefore, $\Gamma_j(L_{0,k})$ is the Abelian group, which contains a subgroup $\Gamma_j(L_{0,k}) \cap [L_2(0,1)]$.

Lemma 7. *For any sequence* $\{\theta_m\}_{m=1}^{\infty} \subset \mathbb{R}$ *the system of functions* $V(B_{j,\theta})$ *is complete and minimal in* $L_2(0,1)$.

Proof. We prove on the contrary that the system of functions $V(B_{j,\theta})$ is total (complete) in the space $L_2(0,1)$.

Let us suppose that there exists a function $h = h_0 + h_1,\ h_s \in L_{s,2}(0,1)$ that is orthogonal to all elements of the system $V(B_{j,\theta})$. Taking into account, that the system $T_{1,2}$ is the orthonormal basis of space $L_{1,2}(0,1)$, we obtain $h_1 \equiv 0$.

Therefore $h \in L_{0,2}(0,1)$. Assuming the orthogonality of the function h to the elements of the system $V(B_{j,\theta})$, we have equality

$$\big(h, v_{0,m}(x_2, B_{j,\theta})\big)_{L_2(0,1)} = \big(h, \tau_{0,m,2}\big)_{L_2(0,1)} = 0,\ m = 1, 2, \ldots .$$

Taking into account that the system $T_{0,2}$ is the orthonormal basis of $L_{0,2}(0,1)$, we obtain $h \equiv 0$.

Let us prove the minimality of the system $V(B_{j,\theta})$. We determine the set of functions

$$L_{2,\theta}(0,1) := \Big\{ h = \sum_{r=0}^{1} \sum_{m=1}^{\infty} h_{r,m} \tau_{r,m,2}(x_2) \in L_2(0,1) :\ \sum_{r=0}^{1} \sum_{m=1}^{\infty} h_{r,m}^2 \theta_{m,0}^2 < \infty \Big\},$$

where $\theta_{m,0} := 1$, in the case $\theta_m = 0$, $\theta_{m,0} = \theta_m$, if $\theta_m \neq 0$, $m = 1, 2, \ldots$.

The set $L_{2,\theta}(0,1)$ is a Hilbert space with respect to the scalar product

$$(h; g)_{L_{2,\theta}(0,1)} := \sum_{r=0}^{1} \sum_{m=1}^{\infty} \theta_{m,0}^2 h_{r,m} g_{r,m}.$$

Let us consider the relations

$$\begin{aligned}
v_{0,m}(x_2, B_{j,\theta}) &= R(B_{j,\theta})\tau_{0,m,2}(x_2) = (1-\theta_m)\tau_{0,m,2}(x_2) + \theta_m v_{0,m}(x_2, L_{1,j,k}), \\
(h; R(B_{j,\theta})\tau_{0,m,2})^2_{L_2(0,1)} &\leq 4(1+\theta_m^2)(h; \tau_{0,m,2})^2_{L_2(0,1)} + 2\theta_m^2 (h; v_{0,m}(x_2, L_{1,j,k}))^2_{L_2(0,1)}, \\
(h; v_{0,m}(x_2, B_{j,\theta}))_{L_2(0,1)} &= (R^*(B_{j,\theta})h; \tau_{0,m,2})_{L_2(0,1)}, \\
(h; R(B_{j,\theta})\tau_{1,m,2})_{L_2(0,1)} &= (h; \tau_{1,m,2})_{L_2(0,1)},\ m = 1, 2, \ldots .
\end{aligned}$$

Taking into account these relations and inequality

$$(h; v_{s,m}(x_2, L_{1,j,k}))^2_{L_2(0,1)} \leq \|R^*(L_{1,j,k})\|^2_{[L_2(0,1)]} (h; \tau_{s,m,2})^2_{L_2(0,1)},\ s = 0, 1,\ m = 1, 2, \ldots,$$

we obtain the estimate

$$\|R^*(B_{j,\theta})h\|^2_{L_2(0,1)} \leq (4 + \|R^*(L_{1,j,k})\|^2_{[L_2(0,1)]}) \|h\|^2_{L_{2,\theta}(0,1)}.$$

Therefore, for conjugate operator $R^*(B_{j,\theta})$ the following inclusion holds (see [9])

$$R^*(B_{j,\theta}) \in [L_{2,\theta}(0,1); L_2(0,1)]. \tag{29}$$

So, the inverse operator exists

$$E - S^*(B_{j,\theta}) \in [L_{2,\theta}(0,1); L_2(0,1)],$$

that is, the system of functions $V(L_{1,j,k})$ has the unique biorthogonal system $W(L_{1,j,k})$. □

Lemma 8. *The system of functions $V(B_{j,\theta})$ is the Riesz basis in $L_2(0,1)$ if and only if the sequence $\{\theta_m\}_{m=1}^{\infty}$ is bounded.*

Proof. Necessity. If the system of functions $V(B_{j,\theta})$ is the Riesz basis, then it is almost normalized. From the opposite, if $|\theta_m| \to \infty$ for $m \to \infty$, then, taking into account (27)–(28), we obtain

$$\|v_{0,m}(x_2, B_{j,\theta})\|_{L_2(0,1)} = 1 + |\theta_m| \|z_{2,j,m,k}\|_{L_2(0,1)} \to \infty, \ m \to \infty.$$

Sufficiency. If the sequence θ is bounded, then the spaces $L_{2,\theta}(0,1)$ and $L_2(0,1)$ coincide. Therefore, taking into account the inclusion (29), we obtain $R(B_{j,\theta}) \in [L_2(0,1)]$. □

The set of n real sequences $\{\theta_{j,m}\}_{m=1}^{\infty}$, $j = 1,2,\dots,n$, we denote by Θ, and consider the operator B_Θ, eigenvalues of which coincide with the eigenvalues of the operator $L_{0,k}$ and eigenfunctions are defined by the equalities

$$\begin{cases} v_{1,m}(x_2, B_\Theta) = \tau_{1,m,2}(x_2), \\ v_{0,m}(x_2, B_\Theta) = \tau_{0,m,2}(x_2) + \sum\limits_{j=1}^{n} \theta_{j,m} z_{2,j,m,k}(x_2), \ m = 1,2,\dots . \end{cases} \tag{30}$$

We define the transformation operator $R(B_\Theta) := E + S(B_\Theta) : \ L_2(0,1) \to L_2(0,1)$ which maps the system of eigenfunctions $V(L_{0,k})$ of operator $L_{0,k}$ into system of functions $V(B_\Theta)$ of operator B_Θ

$$R(B_\Theta)\tau_{s,m,2}(x_2) := v_{s,m,k}(t, B_\Theta), \ s = 0,1, \ m = 1,2,\dots .$$

From the definition of operator B_Θ we obtain

$$S(B_\Theta) : \ L_{0,2}(0,1) \to L_{1,2}(0,1), \quad L_{1,2}(0,1) \to 0, \quad S^2(B_\Theta) = 0.$$

Therefore, the bounded operator $R^{-1}(B_\Theta) = E - S(B_\Theta)$ exists.

Lemma 9. *For any sequences $\{\theta_{j,m}\}_{m=1}^{\infty}$, $j = 1,2,\dots,n$, the system of eigenfunctions of operator B_Θ is complete and minimal in the space $L_2(0,1)$.*

The system of functions $V(B_\Theta)$ is the Riesz basis in the space $L_2(0,1)$ if and only if the sequences $\{\theta_{j,m}\}_{m=1}^{\infty}$, $j = 1,2,\dots,n$, are bounded.

Proof of Lemma 9 is similar to the proof of Lemma 7. □

Let $Q(L_k)$ be the set of operators B_Θ, eigenfunctions of which is defined by formulas (30), $\Gamma(L_k)$ be the set of transformation operators $R(B_\Theta)$.

Remark 3. *On the set $\Gamma(L_k)$ we can define the multiplication operation and prove that $\Gamma(L_k)$ is an Abelian group.*

5 THE NONSELF-AJOINT PROBLEM FOR A DIFFERENTIAL EQUATION OF EVEN ORDER

For equation (9) let us consider the eigenvalues problem with nonlocal conditions

$$\begin{cases} l_{2,s,2}z := z^{(2s-2)}(0) + z^{(2s-2)}(1) = 0, \ s = 1,2,\dots,n, \\ l_{2,n+j,2}z := z^{(2j-1)}(0) + z^{(2j-1)}(1) + l_{n+j}^2 z = 0, \\ l_{2,n+s,2}z := z^{(2j-1)}(0) + z^{(2j-1)}(1) = 0, \ s \neq j, \ s = 1,2,\dots,n, \end{cases} \tag{31}$$

where

$$l_{n+j}^2 z := \sum_{q=0}^{k_{s,2}} \sum_{r=0}^{k_2} b_{s,q,j,2} z^{(q)}(x_{2,r}). \tag{32}$$

Let $L_{2,j,k} : L_2(0,1) \to L_2(0,1)$ be the operator of the problem (9), (31), (32)

$$L_{2,j,k} z(x_2) := \sum_{p=0}^{n} (-1)^p a_p \mu_{k,1}^p z^{(2n-2p)}(x_2),\ z \in D\left(L_{2,j,k}\right),$$

$$D\left(L_{2,j,k}\right) := \left\{ z \in W_2^{2n}(0,1) :\ \ell_{2,r,2} z = 0,\ r = 1,2,\ldots,2n \right\},$$

and $V\left(L_{2,j,k}\right)$ be the system of root functions for operator $L_{2,j,k}$

$$R(L_{2,j,k}) :\ L_2(0,1) \to L_2(0,1),\ \ R(L_{2,j,k}) :\ V(L_{0,k}) \to V(L_{2,j,k}).$$

Lemma 10. *Let Assumption P_1 holds. Then the operator $L_{2,j,k}$ has the system of root functions $V\left(L_{2,j,k}\right)$, which is complete and minimal in the space $L_2(0,1)$.*

If Assumptions P_1, P_3 hold, then the system of functions $V\left(L_{2,j,k}\right)$ is the Riesz basis in the space $L_2(0,1)$.

Proof. Substituting function $\tau_{1,m,2}(x_2)$ into boundary conditions (31), (32) we obtain that the operator $L_{2,j,k}$ has eigenvalues

$$v_{1,m}(x_2, L_{2,j,k}) := \tau_{1,m,2}(x_2),\ m = 1,2,\ldots. \tag{33}$$

Root function $v_{0,m}\left(x_2,\ L_{2,j,k}\right)$ of operator $L_{2,j,k}$ is defined by the sum

$$v_{0,m}(x_2,\ L_{2,j,k}) := \tau_{0,m,2}(x_2) + \eta_{j,m,k}^1 z_{2,j,m,k}(x_2),\ m = 1,2,\ldots. \tag{34}$$

For determining of unknown parameters $\eta_{j,m,k}^1$ we substitute the expression (34) into boundary conditions (31), (32).

Taking into account the ratio (22) we have the equality

$$\eta_{j,m,k}^1 = -(l_{2,n+j,2} z_{2,j,m,k})^{-1} l_{n+j}^2 \tau_{0,m,2}. \tag{35}$$

Therefore, the operator $L_{2,j,k}$ has the system of root functions (33)–(35).

Remark 4. *On the contrary, as in the proof of Lemma 8, we can prove the completeness of the system $V(L_{2,j,k})$ in the space $L_2(0,1)$.*

Taking into account that $z_{2,j,m,k}(x_2) \in L_{1,2}(0,1)$, we have the inclusion $R(L_{2,j,k}) \in \Gamma(L_{0,k})$. Therefore, the system $V(L_{2,j,k})$ is minimal in the space $L_2(0,1)$.

Let Assumption P_3 holds. Then from the inequality (25) we obtain $|\eta_{j,m,k}^1| \le C_8$. Therefore, taking into account the statement of Lemma 9, we obtain that $R(L_{2,j,k}) \in \Gamma(L_{0,k}) \cap [L_2(0,1)]$.

Let us show that for the operator $R(L_k)$ Lemma 10 holds. Substituting into boundary conditions (31), (32) we obtain that the operator L_k has the eigenfunctions

$$v_{1,m}(x_2, L_k) := \tau_{1,m,2}(x_2),\ m = 1,2,\ldots.$$

Root function $v_{0,m}\left(x_2, L_k\right)$ of operator L_k is defined by the sum

$$v_{0,m}(x_2, L_k) := \tau_{0,m,2}(x_2) + \sum_{j=1}^{n} \eta_{j,m,k}^1 z_{2,j,m,k}(x_2),\ m = 1,2,\ldots,$$

where unknown parameters $\eta^1_{j,m,k}$ are defined by formula (35).

Therefore, the transformation operator $R(L_k):\ L_2(0,1)\to L_2(0,1)$

$$R(L_k)\tau_{s,m,2}(x_2) := v_{s,m}(x_2,L_k),\ s=0,1,\ m=1,2,\ldots,$$

is the element of the set $\Gamma(L_k)$. Thus, the system $V(L_k)$ is complete and minimal in the space $L_2(0,1)$.

Taking into account the ratio $R(L_k)=\prod\limits_{j=1}^{n} R(L_{2,j,k})$ and the statement of Lemma 9, in the case of Assumptions $P_1,\ P_3$ we have $R(L_k)\in[L_2(0,1)]$.

Thus, the system $V(L_k)$ is the Riesz basis in the space $L_2(0,1)$. □

Therefore, for operator L_k Lemma 10 holds.

6 PROOF OF THE MAIN RESULTS

Proof of Theorem 1. Particular cases of the operator L, when $b_{q,r,s,j}=0$, we denoted by $L^j,\ j=1,2,$ respectively.

Let $\pi_{r,k,1}$ be the orthoprojector into one-dimensional proper subspace in $L_2(0,1)$. We define the root functions of operator L^2 by

$$v_{s,r,k,m}\left(x,L^2\right) := v_{s,m}\left(x_2,L_k\right)\tau_{r,k,1}\left(x_1\right),\ s,r\in\{0,1\},\ k,m\in\mathbb{N},$$

and the transformation operator $R(L^2): L_2(G)\to L_2(G)$ by

$$R(L^2) := \sum_{r,k,m} R(L_k)\times\pi_{r,k,1},$$
$$v_{s,r,k,m}\left(x,L^2\right) := R(L^2)v_{s,r,k,m}\left(x,L_0\right),\ s,r\in\{0,1\},\ k,m\in\mathbb{N}.$$

Similarly, when Assumption P_2 holds, we can define the biorthogonal system $W(L^2)$.

Taking into account Lemma 10, we obtain that for operator L^2 Theorem 1 holds. From Theorem 1 of the paper [1] we obtain that for operator L^1 Theorem 1 holds and for transformation operator $R(L^1)$ the ratio $R(L^1)\in[L_2(G)]$ holds.

Let us define the transformation operator $R(L): L_2(G)\to L_2(G),\ R(L):=R(L^1)R(L^2)$, and the root functions of operator L

$$v_{s,r,k,m}\left(x,L\right) := R(L)v_{s,r,k,m}\left(x,L_0\right),\ s,r\in\{0,1\},\ k,m\in\mathbb{N}.$$

By direct verification we obtain that the elements of the system $V(L)$ are roots in sense

$$(L-\lambda_{k,m})v_{r,1,k,m}(x,L)=0,$$
$$(L-\lambda_{k,m})v_{r,0,k,m}(x,L)=\xi_{r,0,k,m}v_{r,1,k,m}(x,L),$$
$$\xi_{r,0,k,m}=4n(-1)^{n-1}\eta_{r,0,k,m}\rho^{2n-1}_{m,2},\ \ r,s\in\{0,1\},\ \ k,m\in\mathbb{N}.$$

Taking into account Assumption P_1, we obtain that the systems $V(L^j)$ have the unique biorthogonal systems $W(L^j)$, $j=1,2$, in the space $L_2(G)$.

Therefore, the system $V(L)$ is complete and minimal in the space $L_2(G)$.

Let Assumptions P_1–P_3 hold. Then from Theorem 3 of the paper [8] and from Lemma 10 we obtain $R(L^1)\in[W^{2n}_2(G)]$. So, Theorem 1 is proved. □

Remark 5. *There are positive numbers C_9, C_{10} such that for any function*

$$f(x) = \sum_{r,q,k,m} f_{r,q,k,m} v_{r,q,k,m}(x,L) \tag{36}$$

the inequality

$$C_9 \|f\|^2_{L_2(G)} \leq \sum_{r,q,k,m} |f_{r,q,k,m}|^2 \leq C_{10} \|f\|^2_{L_2(G)}$$

holds.

Proof of Theorem 2. It is enough to consider the case $q_{q,r,s,1} = 0$. Let the right part of the equation (1) has the expansion (36).

The solution of the problem (1)–(2) we find in the form of a series

$$u(x) = \sum_{r,q,k,m} u_{r,q,k,m} v_{r,q,k,m}(x,L).$$

Substituting these expansions into equation (1) we obtain the equalities

$$\begin{aligned} &u_{r,0,k,m} = \lambda^{-1}_{k,m} f_{r,0,k,m}, \\ &u_{r,1,k,m} = \lambda^{-1}_{k,m} f_{r,1,k,m} - \lambda^{-2}_{k,m} \xi_{r,0,k,m} f_{r,0,k,m}, \ r \in \{0,1\}, \ k,m \in \mathbb{N}. \end{aligned}$$

Let us consider the relations

$$D_1^{2n} u(x) = \sum_{r,q,k,m} \lambda^{-1}_{k,m} \mu^n_{k,1} f_{r,q,k,m} v_{r,q,k,m}(x,L).$$

Taking into account Assumption P_2, we obtain

$$\|D_1^{2n} u\|_{L_2(G)} \leq C_{11} \|f\|_{L_2(G)}.$$

Similarly from Theorem 3 of the paper [1] we obtain the inequality

$$\|D_2^{2n} u\|_{L_2(G)} \leq C_{12} \|f\|_{L_2(G)}.$$

Therefore, taking into account Theorem 3, we obtain $R(L^2) \in [W_2^{2n}(G)]$.

Thus, for the definition of the transformation operator $R(L)$ we have $R(L) \in [W_2^{2n}(G)]$. Then

$$\|u\|_{W_2^{2n}(G)} \leq C_{13} \|f\|_{L_2(G)}.$$

Theorem is proved. □

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Баранецький Я.О., Каленюк П.І., Копач М.І., Соломко А.В. *Нелокальна крайова задача зі збуреннями мішаних крайових умов для еліптичного рівняння із сталими коефіцієнтами. II* // Карпатські матем. публ. — 2020. — Т.12, №1. — С. 173–188.

У роботі продовжено розпочаті у першій частині дослідження властивостей задачі з нелокальними умовами, які є багатоточковими збуреннями мішаних крайових умов. Зокрема, побудовано узагальнений оператор перетворення, який відображає розв'язки самоспряженої крайової задачі із мішаними крайовими умовами в розв'язки багатоточкової задачі. Побудовано систему $V(L)$ кореневих функцій оператора L багатоточкової задачі. Визначено умови, при яких система $V(L)$ повна та мінімальна та умови, за яких вона є базисом Рісса. Для випадку еліптичного рівняння встановлено умови існування та єдиності розв'язку задачі.

Ключові слова і фрази: диференціальне рівняння з частинними похідними, кореневі функції, метод Фур'є, метод операторів перетворення, базис Рісса.

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NILPOTENT LIE ALGEBRAS OF DERIVATIONS WITH THE CENTER OF SMALL CORANK

Let $\mathbb{K}$ be a field of characteristic zero, A be an integral domain over $\mathbb{K}$ with the field of fractions $R = Frac(A)$, and $Der_{\mathbb{K}} A$ be the Lie algebra of all $\mathbb{K}$-derivations on A. Let $W(A) := RDer_{\mathbb{K}} A$ and L be a nilpotent subalgebra of rank n over R of the Lie algebra $W(A)$. We prove that if the center $Z = Z(L)$ is of rank $\geq n-2$ over R and $F = F(L)$ is the field of constants for L in R, then the Lie algebra FL is contained in a locally nilpotent subalgebra of $W(A)$ of rank n over R with a natural basis over the field R. It is also proved that the Lie algebra FL can be isomorphically embedded (as an abstract Lie algebra) into the triangular Lie algebra $u_n(F)$, which was studied early by other authors.

Key words and phrases: derivation, vector field, Lie algebra, nilpotent algebra, integral domain.

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INTRODUCTION

Let $\mathbb{K}$ be a field of characteristic zero, A be an integral domain over $\mathbb{K}$, and $R = \mathrm{Frac}(A)$ be its field of fractions. Recall that a $\mathbb{K}$-derivation D on A is a $\mathbb{K}$-linear operator on the vector space A satisfying the Leibniz rule $D(ab) = D(a)b + aD(b)$ for any $a, b \in A$. The set $\mathrm{Der}_{\mathbb{K}} A$ of all $\mathbb{K}$-derivations on A is a Lie algebra over $\mathbb{K}$ with the Lie bracket $[D_1, D_2] = D_1D_2 - D_2D_1$. The Lie algebra $\mathrm{Der}_{\mathbb{K}} A$ can be isomorphically embedded into the Lie algebra $\mathrm{Der}_{\mathbb{K}} R$ (any derivation D on A can be uniquely extended on R by the rule $D(a/b) = (D(a)b - aD(b))/b^2$, $a, b \in A$). We denote by $W(A)$ the subalgebra $R\,\mathrm{Der}_{\mathbb{K}} A$ of the Lie algebra $\mathrm{Der}_{\mathbb{K}} R$ (note that $W(A)$ and $\mathrm{Der}_{\mathbb{K}} R$ are Lie algebras over the field $\mathbb{K}$ but not over R). Nevertheless, $W(A)$ and $\mathrm{Der}_{\mathbb{K}} R$ are vector spaces over the field R, so one can define the rank $\mathrm{rk}_R L$ for any subalgebra L of the Lie algebra $W(A)$ by the rule $\mathrm{rk}_R L = \dim_R RL$. Every subalgebra L of the Lie algebra $W(A)$ determines its field of constants in R by

$$F = F(L) := \{r \in R \mid D(r) = 0 \ \text{ for all } D \in L\}.$$

The product $FL = \{\sum \alpha_i D_i \mid \alpha_i \in F,\ D_i \in L\}$ is a Lie algebra over the field F, this Lie algebra often has simpler structure than L itself (note that such an extension of the ground field preserves the main properties of L from the viewpoint of Lie theory).

We study nilpotent subalgebras $L \subseteq W(A)$ of rank $n \geq 3$ over R with the center $Z = Z(L)$ of rank $\geq n-2$ over R, i.e. with the center of corank ≤ 2 over R. We prove that FL is contained

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in a locally nilpotent subalgebra of $W(A)$ with a natural basis over R, similar to the standard basis of the triangular Lie algebra $U_n(F)$ (Theorem 1). As a consequence, we get an isomorphic embedding (as Lie algebras) of the Lie algebra FL over F into the triangular Lie algebra $u_n(F)$ over F (Theorem 2). These results generalize main results of the papers [8] and [9]. Note that the problem of classifying finite dimensional Lie algebras from Theorem 1 up to isomorphism is wild (i.e., it contains the hopeless problem of classifying pairs of square matrices up to similarity, see [3]). Triangular Lie algebras were studied in [1] and [2], they are locally nilpotent but not nilpotent.

We use standard notations. The ground field $\mathbb{K}$ is arbitrary of characteristic zero. If F is a subfield of a field R and $r_1, \ldots, r_k \in R$, then $F\langle r_1, \ldots, r_k\rangle$ is the set of all linear combinations of r_i with coefficients in F, it is a subspace in the F-space R, for an infinite set $\{r_1, \ldots, r_k, \ldots\}$ we use the notation $F\left\langle \{r_i\}_{i=1}^{\infty} \right\rangle$. The triangular subalgebra $u_n(\mathbb{K})$ of the Lie algebra $W_n(\mathbb{K}) := \operatorname{Der}_{\mathbb{K}} \mathbb{K}[x_1, \ldots, x_n]$ consists of all the derivations on $\mathbb{K}[x_1, \ldots, x_n]$ of the form

$$D = f_1(x_2, \ldots, x_n)\frac{\partial}{\partial x_1} + \cdots + f_{n-1}(x_n)\frac{\partial}{\partial x_{n-1}} + f_n\frac{\partial}{\partial x_1},$$

where $f_i \in \mathbb{K}[x_{i+1}, \ldots, x_n]$, $f_n \in \mathbb{K}$. If $D \in W(A)$, then $\operatorname{Ker} D$ denotes the field of constants for D in R, i.e., $\operatorname{Ker} D = \{r \in R \mid D(r) = 0\}$.

1 MAIN PROPERTIES OF NILPOTENT SUBALGEBRAS OF W(A)

We often use the next relations for derivations which are well known (see, for example [7]). Let $D_1, D_2 \in W(A)$ and $a, b \in R$. Then

1) $[aD_1, bD_2] = ab[D_1, D_2] + aD_1(b)D_2 - bD_2(a)D_1$;

2) if $a, b \in \operatorname{Ker} D_1 \cap \operatorname{Ker} D_2$, then $[aD_1, bD_2] = ab[D_1, D_2]$.

The next two lemmas contain some results about derivations and Lie algebras of derivations.

Lemma 1 ([6], Lemma 2)**.** *Let L be a subalgebra of the Lie algebra $\operatorname{Der}_{\mathbb{K}} R$ and F the field of constants for L in R. Then FL is a Lie algebra over F, and if L is abelian, nilpotent or solvable, then so is FL, respectively.*

Lemma 2 ([6], Proposition 1)**.** *Let L be a nilpotent subalgebra of the Lie algebra $W(A)$ with $\operatorname{rk}_R L < \infty$ and $F = F(L)$ the field of constants for L in R. Then*

1) FL is finite dimensional over F;

2) if $\operatorname{rk}_R L = 1$, then L is abelian and $\dim_F FL = 1$;

3) if $\operatorname{rk}_R L = 2$, then FL is either abelian with $\dim_F FL = 2$ or FL is of the form

$$FL = F\left\langle D_2, D_1, aD_1, \ldots, \frac{a^k}{k!}D_1 \right\rangle,$$

for some $D_1, D_2 \in FL$ and $a \in R$ such that $[D_1, D_2] = 0$, $D_2(a) = 1$, $D_1(a) = 0$.

Lemma 3. *Let L be a nilpotent subalgebra of the Lie algebra $W(A)$ of rank n over R with the center $Z = Z(L)$ of rank k over R. Then $I := RZ \cap L$ is an abelian ideal of L with $\mathrm{rk}_R\, I = k$.*

Proof. By Lemma 4 from [6], I is an ideal of the Lie algebra L. Let us show that I is abelian. Let us choose an arbitrary basis $D_1, \ldots, D_k$ of the center Z over R (i.e., a maximal by inclusion linearly independent over R subset of Z). One can easy to see that $D_1, \ldots, D_k$ is a basis of the ideal I as well, so we can write for each element $D \in I$

$$D = a_1 D_1 + \cdots + a_k D_k$$

for some $a_1, \ldots, a_k \in R$. Since $D_j \in Z,\ j = 1, \ldots, k$, it holds

$$[D_j, D] = [D_j, \sum_{i=1}^{k} a_i D_i] = \sum_{i=1}^{k} D_j(a_i) D_i = 0 \tag{1}$$

for $j = 1, \ldots, k$. The derivations $D_1, \ldots, D_n$ are linearly independent over the field R, hence we obtain from (1) that $D_j(a_i) = 0,\ i, j = 1, \ldots, k$. Therefore we have for each element $\overline{D} = b_1 D_1 + \ldots b_k D_k$ of the ideal I the next equalities

$$[D, \overline{D}] = [\sum_{i=1}^{k} a_i D_i, \sum_{j=1}^{k} b_j D_j] = \sum_{i,j=1}^{k} a_i b_j [D_i, D_j] = 0,$$

since $D_i(b_j) = D_j(a_i) = 0$ as mentioned above. The latter means that I is an abelian ideal. Besides, obviously $\mathrm{rk}_R\, I = k$. □

Lemma 4. *Let L be a nilpotent subalgebra of the Lie algebra $W(A)$, $Z = Z(L)$ the center of L, $I := RZ \cap L$ and F the field of constants for L in R. If for some $D \in L$ it holds $[D, FI] \subseteq FI$, $[D, FI] \neq 0$, then there exist a basis $D_1, \ldots, D_m$ of the ideal FI of the Lie algebra FL over R and $a \in R$ such that $D(a) = 1,\ D_i(a) = 0,\ i = 1, \ldots, m$. Besides, each element $\overline{D} \in FI$ is of the form $\overline{D} = f_1(a) D_1 + \cdots + f_m(a) D_m$ for some polynomials $f_i \in F_1[t]$, where F_1 is the field of constants for the subalgebra $L_1 = FI + FD$ in R.*

Proof. By Lemma 3, the intersection $I = RZ \cap L$ is an abelian ideal of the Lie algebra L and therefore FI is an abelian ideal of the Lie algebra FL. Choose a basis $D_1, \ldots, D_m$ of FI over the field R in such a way that $D_1, \ldots, D_m \in Z$. Then FZ is the center of the Lie algebra FL. Now take any basis $T_1, \ldots, T_s$ of the F-space FI (note that the Lie algebra FL is finite dimensional over the field F by [6]). Every basis element T_i can be written in the form $T_i = \sum\limits_{j=1}^{m} r_{ij} D_j$, $i = 1, \ldots, s$, for some $r_{ij} \in R$. Denote by B the subring $B = F[r_{ij}, i = 1, \ldots, s, j = 1, \ldots, m]$ of the field R generated by F and the elements r_{ij}. Since the linear operator $\mathrm{ad}\, D$ is nilpotent on the F-space FI the derivation D is locally nilpotent on the ring B. Indeed,

$$[D, T_i] = [D, \sum_{j=1}^{m} r_{ij} D_j] = \sum_{j=1}^{m} D(r_{ij}) D_j$$

and therefore

$$(\mathrm{ad}\, D)^{k_i}(T_i) = \sum_{j=1}^{m} D^{k_i}(r_{ij}) D_j = 0$$

for some natural k_i, $i = 1, \dots, s$. Denoting $\overline{k} = \max_{1 \le t \le s} k_t$, we get $D^{\overline{k}}(r_{ij}) = 0$ and therefore D is locally nilpotent on B. One can easily show that there exists an element $p \in B$ (a preslice) such that $D(p) \in \operatorname{Ker} D, D(p) \neq 0$. Then denoting $a := p/D(p)$, we have $D(a) = 1$ (such an element a is called a slice for D). The ring B is contained in the localization $B[c^{-1}]$, where $c := D(p)$ and the derivation D is locally nilpotent on $B[c^{-1}]$. Note that $B[c^{-1}] \subseteq F_1$, where F_1 is the field of constants for $L_1 = FI + FD$ in R. Besides, by Principle 11 from [4] it holds $B[c^{-1}] = B_0[a]$, where B_0 is the kernel of D in $B[c^{-1}]$. This completes the proof because $B \subseteq B[c^{-1}]$ and every element $\overline{D}$ of FI is of the form $\overline{D} = b_1 D_1 + \dots b_m D_m, b_i \in B$. ☐

Lemma 5. *Let L be a nilpotent subalgebra of the Lie algebra $W(A)$, $Z = Z(L)$ the center of L, F the field of constants of L in R and $I = RZ \cap L$. Let $\operatorname{rk}_R Z = n - 2$. Then the following statements for the Lie algebra FL/FI hold*

1) *if FL/FI is abelian, then $\dim_F FL/FI = 2$;*

2) *if FL/FI is nonabelian, then there exist elements $D_{n-1}, D_n \in FL$, $b \in R$ such that*

$$FL/FI = F \left\langle D_{n-1} + FI,\, bD_{n-1} + FI,\, \dots,\, \frac{b^k}{k!} D_{n-1} + FI,\, D_n + FI \right\rangle$$

with $k \geq 1$, $D_n(b) = 1$, $D_{n-1}(b) = 0$, $D(b) = 0$ for all $D \in FI$.

Proof. Let us choose a basis $D_1, \dots, D_{n-2}$ of the center Z over the field R and any central ideal $FD_{n-1} + FI$ of the quotient algebra FL/FI. Denote the intersection $R(I + \mathbb{K}D_{n-1}) \cap L$ by I_1. Then it is easy to see that FI_1 is an ideal of the Lie algebra FL of rank $n - 1$ over R and the Lie algebra FL/FI_1 is of dimension 1 over F (by Lemma 5 from [6]). Let us choose an arbitrary element $D_n \in FL \setminus FI_1$. Then $D_1, \dots, D_n$ is a basis of the Lie algebra FL over the field R.

Case 1. The quotient algebra FL/FI is abelian. Let us show that

$$FL/FI = F \langle D_{n-1} + FI,\, D_n + FI \rangle .$$

Indeed, let us take any elements $S_1 + FI,\ S_2 + FI$ of FL/FI and write

$$S_1 = \sum_{i=1}^{n} r_i D_i, \quad S_2 = \sum_{i=1}^{n} s_i D_i, \quad r_i,\ s_i \in R,\ i, j = 1, \dots, n.$$

From the equalities $[D_i, S_1] = [D_i, S_2] = 0$, $i = 1, \dots, n-2$ (recall that $D_i \in Z(L)$, $i = 1, \dots, n-2$) it follows that

$$D_i(r_j) = D_i(s_j) = 0,\ i = 1 \dots, n-2,\ j = 1, \dots, n. \tag{2}$$

Since $[FL, FI] \subseteq FI$ we have $[D_i, S_1], [D_i, S_2] \in FI$ for $i = n-1, n$. Taking into account the equalities (2) we derive that

$$D_i(s_j) = D_i(r_j) = 0,\ i = n-1,\ n,\ j = n-1,\ n.$$

Therefore it holds $s_i, r_i \in F$ for $i = n-1$, n and the elements $D_{n-1} + FI$, $D_n + FI$ form a basis for the abelian Lie algebra FL/FI over the field F.

Case 2. FL/FI is nonabelian. Then $\dim_F FL/FI \geq 3$ because the Lie algebra FL/FI is nilpotent. Let us show that the ideal FI_1/FI of the Lie algebra FL/FI is abelian (recall that

$I_1 = R(I + \mathbb{K}D_{n-1}) \cap L$). Since $D_{n-1} + FI$ lies in the center of the quotient algebra FL/FI we have for any element $rD_{n-1} + FI$ of the ideal FI_1/FI the following equality

$$[D_{n-1} + FI, rD_{n-1} + FI] = FI.$$

Hence $D_{n-1}(r)D_{n-1} + FI = FI$. The last equality implies $D_{n-1}(r) = 0$. But then for any elements $rD_{n-1} + FI$, $sD_{n-1} + FI$ of FI_1/FI we get

$$\begin{aligned}[rD_{n-1} + FI, sD_{n-1} + FI] &= [rD_{n-1}, sD_{n-1} + FI] \\ &= (D_{n-1}(s)r - sD_{n-1}(r))D_{n-1} + FI = FI.\end{aligned}$$

The latter means that FI_1/FI is an abelian ideal of FL/FI.

Further, the nilpotent linear operator $\operatorname{ad} D_n$ acts on the linear space FI_1/FI with $\operatorname{Ker}(\operatorname{ad} D_n) = FD_{n-1} + FI$. Indeed, let $\operatorname{ad} D_n(rD_{n-1} + FI) = FI$. Then $[D_n, rD_{n-1}] \in FI$ and therefore $D_n(r)D_{n-1} \in FI$. This relation implies $D_n(r) = 0$ and taking into account the equalities $D_i(r) = 0,\ i = 1, \dots, n-1$, we get that $r \in F$ and $\operatorname{Ker}(\operatorname{ad} D_n) = FD_{n-1} + FI$. It follows from this relation that the linear operator $\operatorname{ad} D_n$ on FI/FI_1 has only one Jordan chain and the Jordan basis can be chosen with the first element $D_{n-1} + FI$. Since $\dim FI_1/FI \geq 2$ (recall that $\dim_F FL/FI \geq 3$) the chain is of length ≥ 2. Let us take the second element of the Jordan chain in the form $bD_{n-1} + FI,\ b \in R$. Then $\operatorname{ad} D_n(bD_{n-1} + FI) = D_{n-1} + FI$ and hence $D_n(b) = 1$. The inclusion $[D_{n-1}, bD_{n-1}] \in FI$ implies the equality $D_{n-1}(b) = 0$, and analogously one can obtain $D_i(b) = 0,\ i = 1, \dots, n-2$.

If $\dim FI_1/FI \geq 3$ and $cD_{n-1} + FI$ is the third element of the Jordan chain of $\operatorname{ad} D_n$, then repeating the above considerations we get $D_n(c) = b$. Then the element $\alpha = \frac{b^2}{2!} - c \in R$ satisfies the relations $D_{n-1}(\alpha) = D_n(\alpha) = 0$ and $D_i(\alpha) = 0,\ i = 1, \dots, n-2$, since $D_i(b) = D_i(c) = 0$. Therefore, $\alpha = \frac{b^2}{2!} - c \in F$ and $c = \frac{b^2}{2!} + \alpha$. Since $\alpha D_{n-1} + FI \in \operatorname{Ker}(\operatorname{ad} D_n)$, we can take the third element of the Jordan chain in the form $\frac{b^2}{2!}D_{n-1} + FI$. Repeating the consideration one can build the needed basis of the Lie algebra FL/FI. □

Lemma 6. *Let L be a nilpotent subalgebra of $W(A)$ with the center $Z = Z(L)$ of $\operatorname{rk}_R Z = n-2$, F the field of constants for L in R and $I = RZ \cap L$. If S, T are elements of L such that $[S, T] \in I$, the rank of the subalgebra L_1 spanned by I, S, T equals n and $C_{FL}(FI) = FI$, then there exist elements $a, b \in R$ such that $S(a) = 1$, $T(a) = 0$, $S(b) = 0$, $T(b) = 1$ and $D(a) = D(b) = 0$ for each $D \in I$. Besides, every element $D \in FI$ can be written in the form $D = f_1(a,b)D_1 + \cdots + f_{n-2}(a,b)D_{n-2}$ with some polynomials $f_i(u,v) \in F[u,v]$.*

Proof. Let us choose a basis $D_1, \dots, D_{n-2}$ of Z over R. By the lemma conditions, one can easily see that $D_1, \dots, D_{n-2}, S, T$ is a basis of L over R. The ideal FI of the Lie algebra FL is abelian by Lemma 3 and $\operatorname{ad} S, \operatorname{ad} T$ are commuting linear operators on the vector space FI (over F). Take a basis $T_1, \dots, T_s$ of FI over F (recall that $\dim_F FL < \infty$ by Theorem 1 from [6]) and write $T_i = \sum_{j=1}^{n-2} r_{ij}D_j$ for some $r_{ij} \in R, i = 1, \dots, s, j = 1, \dots, n-2$. Denote by

$$B = F[r_{ij}, i = 1, \dots, s, j = 1, \dots, n-2],$$

the subring of R generated by F and all the coefficients r_{ij}. Then B is invariant under the derivations S and T, these derivations are locally nilpotent on B and linearly independent over R (by the condition $C_{FL}(FI) = FI$ of the lemma). By Lemma 4, there exists an element $a \in B[c^{-1}]$ such that

$$S(a) = 1, \quad D_i(a) = 0, \quad i = 1, \dots, n-2,$$

(here $c = S(p)$ for a preslice p for S in B). Since $c \in \operatorname{Ker} S$ and $[S, T] = 0$ one can assume without loss of generality that $T(c) \in \operatorname{Ker} T$. But then T is a locally nilpotent derivation on the subring $B[c^{-1}]$. Repeating these considerations we can find an element $b \in B[c^{-1}][d^{-1}]$ with $T(b) = 1$ (here d is a preslice for the derivation T in $B[c^{-1}]$). Denote $B_1 = B[c^{-1}, d^{-1}]$, the subring of R generated by B, c^{-1}, d^{-1}. Then using standard facts about locally nilpotent derivations (see, for example Principle 11 in [4]) one can show that $B_1 = B_0[a, b]$, where $B_0 = \operatorname{Ker} S \cap \operatorname{Ker} T$. Therefore every element h of B_1 can be written in the form $h = f(a, b)$ with $f(u, v) \in F[u, v]$. Note that

$$F = \operatorname{Ker} T \cap \operatorname{Ker} S \cap_{i=1}^{n-2} \operatorname{Ker} D_i.$$

It follows from this representation of elements of B_1 that every element of the ideal FI can be written in the form

$$D = f_1(a, b)D_1 + \cdots + f_{n-2}(a, b)D_{n-2}$$

with some polynomials $f_i(u, v) \in F[u, v]$. □

2 THE MAIN RESULTS

Theorem 1. *Let L be a nilpotent subalgebra of rank $n \geq 3$ over R from the Lie algebra $W(A)$, $Z = Z(L)$ the center of L with $\operatorname{rk}_R Z \geq n-2$, F the field of constants of L in R. Then one of the following statements holds:*

1) $\dim_F FL = n$ *and FL is either abelian or is a direct sum of a nonabelian nilpotent Lie algebra of dimension* 3 *and an abelian Lie algebra;*

2) $\dim_F FL \geq n+1$ *and FL lies in one of the locally nilpotent subalgebras L_1, L_2 of $W(A)$ of rank n over R, which have a basis $D_1, \dots, D_n$ over R satisfying the relations $[D_i, D_j] = 0$, $i, j = 1, \dots, n$, and are one of the form*

$$L_1 = F\left\langle \left\{\frac{b^i}{i!}D_1\right\}_{i=0}^{\infty}, \dots, \left\{\frac{b^i}{i!}D_{n-1}\right\}_{i=0}^{\infty}, D_n \right\rangle$$

for some $b \in R$ such that $D_i(b) = 0$, $i = 1, \dots, n-1$, and $D_n(b) = 1$,

$$L_2 = F\left\langle \left\{\frac{a^i b^j}{i!j!}D_1\right\}_{i,j=0}^{\infty}, \dots, \left\{\frac{a^i b^j}{i!j!}D_{n-2}\right\}_{i,j=0}^{\infty}, \left\{\frac{b^i}{i!}D_{n-1}\right\}_{i=0}^{\infty}, D_n \right\rangle$$

for some $a, b \in R$ such that $D_{n-1}(a) = 1$, $D_n(a) = 0$, $D_{n-1}(b) = 0$, $D_n(b) = 1$, $D_i(a) = D_i(b) = 0$, $i = 1, \dots, n-2$.

Proof. By Lemma 3, $I = RZ \cap L$ is an abelian ideal of L and therefore FI is an abelian ideal of the Lie algebra FL (here the Lie algebra FL is considered over the field F). Let $\dim_F FL = n$. It is obvious that $\dim_F M = \operatorname{rk}_R M$ for any subalgebra M of the Lie algebra FL, in particular $\dim_F FZ \geq n-2$ because of conditions of the theorem. We may restrict ourselves only on nonabelian algebras and assume $\dim_F FZ = n-2$ (in case $\dim_F FZ \geq n-1$ the Lie algebra FL is abelian). Since FL is nilpotent of nilpotency class 2, one can easily show that FL is a direct sum of a nonabelian Lie algebra of dimension 3 and an abelian algebra and satisfies the condition 1) of the theorem. So, we may assume further that $\dim_F FL \geq n+1$.

Case 1. $\mathrm{rk}_R Z = n-1$. Then FI is of codimension 1 in FL by Lemma 5 from [6]. Therefore $\dim_F FI \geq n$ because of $\dim_F FL \geq n+1$ and $\dim_F FL/FI = 1$. We obtain the strong inclusion $FZ \subsetneqq FI$ because of $\dim_F FZ = n-1$. Take a basis $D_1, \ldots, D_{n-1}$ of Z over R and an element $D_n \in FL \setminus FI$. Then $D_1, \ldots, D_n$ is a basis for FL over R and $[D_n, FI] \neq 0$. Using Lemma 4 one can easily show that FL is contained in a subalgebra of type L_1 from $W(A)$.

Case 2. $\mathrm{rk}_R Z = n-2$ and $\dim_F FI = n-2$. Then $FI = FZ$, $\dim_F FL/FI \geq 3$ and therefore by Lemma 5 the quotient algebra FL/FI is of the form

$$FL/FI = F\left\langle \left\{ \frac{b^i}{i!} D_{n-1} + FI \right\}_{i=0}^{k}, D_n + FI \right\rangle$$

for some $k \geq 1$, $b \in R$ such that $D_n(b) = 1,\ D_{n-1}(b) = 0$ and $D(b) = 0$ for each $D \in FI$.

The F-space

$$J = F\left\langle \left\{ \frac{b^i}{i!} D_1 \right\}_{i=0}^{\infty}, \ldots, \left\{ \frac{b^i}{i!} D_{n-1} \right\}_{i=0}^{\infty} \right\rangle$$

is an abelian subalgebra of $W(A)$ and $[FL, J] \subseteq J$. Therefore the sum

$$J + F\left\langle \left\{ \frac{b^i}{i!} D_{n-1} \right\}_{i=0}^{\infty}, D_n \right\rangle$$

is a subalgebra of the Lie algebra $W(A)$. If $[D_n, D_{n-1}] \neq 0$, then taking into account the relation $[D_n, D_{n-1}] \in FI$ one can write

$$[D_n, D_{n-1}] = \alpha_1 D_1 + \cdots + \alpha_{n-2} D_{n-2}$$

for some $\alpha_i \in F$ (recall that $FI = FZ$). Consider the element of $W(A)$ of the form

$$\widetilde{D}_{n-1} = D_{n-1} - \alpha_1 b D_1 - \cdots - \alpha_{n-2} b D_{n-2}.$$

Since $[D_n, \widetilde{D}_{n-1}] = 0,\ \widetilde{D}_{n-1}(b) = 0$, one can replace the element D_{n-1} with the element $\widetilde{D}_{n-1}$ and assume without loss of generality that $[D_n, D_{n-1}] = 0$. As a result we get the Lie algebra of the type L_1 from the statement of the theorem.

Case 3. $\mathrm{rk}_R Z = n-2$ and $\dim_F FI > n-2$. First, suppose $C_{FL}(FI) = FI$. Then by Lemma 6 there are a basis $D_1, \ldots, D_{n-2}$ of the ideal FI over R and elements $a, b \in R$ such that

$$D_{n-1}(a) = 1,\ D_n(a) = 0,\ D_{n-1}(b) = 0,\ D_n(b) = 1$$

and

$$D_i(a) = D_i(b) = 0,\ i = 1, \ldots, n-2,$$

and each element $D \in FI$ can be written in the form

$$D = f_1(a,b) D_1 + \cdots + f_{n-2}(a,b) D_{n-2}$$

for some polynomials $f_i(u,v) \in F[u,v]$.

Consider the F-subspace

$$J = F[a,b]D_1 + \cdots + F[a,b]D_{n-2}$$

of the Lie algebra $W(A)$. It is easy to see that J is an abelian subalgebra of $W(A)$ and $[FL, J] \subseteq J$. If $[D_n, D_{n-1}] = 0$, then it is obvious that the subalgebra $FL + J$ is of type L_2 of the theorem and $FL \subset L_1$. Let $[D_n, D_{n-1}] \neq 0$. Since $[D_n, D_{n-1}] \in FI$, it follows

$$[D_n, D_{n-1}] = h_1(a,b)D_1 + \cdots + h_{n-2}D_{n-2}$$

for some polynomials $h_i(u,v) \in F[u,v]$. Then the subalgebra J has such an element

$$T = u_1(a,b)D_1 + \ldots u_{n-2}(a,b)D_{n-2}$$

that $D_n(u_i(a,b)) = h_i(a,b), i = 1, \ldots, n-2$ (recall that $D_n(a) = 0,\ D_n(b) = 1$), and hence the element $\widetilde{D}_{n-1} = D_{n-1} - T$ satisfies the equality $[D_n, T] = 0$. Replacing D_{n-1} with $\widetilde{D}_{n-1}$ we get the needed basis of the Lie algebra $FL + J$ and see that FL can be embedded into the Lie L_2 of $W(A)$. So in case of $C_{FL}(FI) = FI$ the Lie algebra FL can be isomorphically embedded into the Lie algebra of type L_2 from the statement of the theorem.

Further, suppose $C_{FL}(FI) \neq FI$. Since $C_{FL}(FI) \supseteq FI$ one can easily show that $D_{n-1} \in C_{FL}(FI) \setminus FI$ (note that FL/FI has the unique minimal ideal $FD_{n-1} + FI$). Then $[D_{n-1}, FI] = 0$, and therefore $[D_n, FI] \neq 0$. Therefore by Lemma 4 there is an element $c \in R$ such that

$$D_n(c) = 1,\ D_{n-1}(c) = 0,\ D_i(c) = 0,\ i = 1, \ldots, n-2.$$

Moreover, each element of FI is of the form $g_1(c)D_1 + \cdots + g_{n-2}(c)D_{n-2}$ for some polynomials $g_i(u) \in F[u]$. By Lemma 5, the quotient algebra FL/FI is of the form

$$FL/FI = F\left\langle \left\{ \frac{b^i}{i!}D_{n-1} + FI \right\}_{i=0}^{k}, D_n + FI \right\rangle$$

for some $b \in R, k \geq 1$ such that $D_n(b) = 1,\ D_{n-1}(b) = 0$. But then

$$D_{n-1}(b-c) = 0,\ D_n(b-c) = 0,\ D_i(b-c) = 0,$$

and hence $b - c = \alpha$ for some $\alpha \in F$. Without loss of generality we can assume $b = c$. The locally nilpotent subalgebra

$$L_1 = F\left\langle \left\{ \frac{a^i b^j}{i!j!}D_1 \right\}_{i,j=0}^{\infty}, \ldots, \left\{ \frac{a^i b^j}{i!j!}D_{n-2} \right\}_{i,j=0}^{\infty}, \left\{ \frac{b^i}{i!}D_{n-1} \right\}_{i=0}^{\infty}, D_n \right\rangle$$

of the Lie algebra $W(A)$ contains FL and satisfies the conditions for the Lie algebra of type L_2 from the statement of the theorem, possibly except the condition $[D_n, D_{n-1}] = 0$. If $[D_n, D_{n-1}] \neq 0$, then from the inclusion $[D_n, D_{n-1}] \in FI$ it follows that

$$[D_n, D_{n-1}] = f_1(b)D_1 + \cdots + f_{n-2}(b)D_{n-2}$$

for some polynomials $f_i(u) \in F[u]$.

One can easily show that there is such an element

$$\overline{D} = h_1(b)D_1 + \cdots + h_{n-2}(b)D_{n-2} \in L_1,$$

that $[D_n, \overline{D}] = [D_n, D_{n-1}]$ (one can take antiderivations h_i for polynomials f_i, $i = 1, \ldots, n-2$). Replacing D_{n-1} with $D_{n-1} - \overline{D}$ we get the needed basis over R of the Lie algebra L_2. $\square$

Remark 1. *Any Lie algebra of dimension n over F can be realized as a Lie algebra of rank n over R by Theorem 2 from [5]. So the Lie algebra of type 1) from Theorem 1 can be chosen in any way possible.*

As a corollary we get the next statement about embedding of Lie algebras of derivations.

Theorem 2. *Let L be a nilpotent subalgebra of rank n over R of the Lie algebra $W(A)$, $Z = Z(L)$ be the center of L and F be the field of constants of L in R. If $\mathrm{rk}_R\, Z \geq n-2$, then the Lie algebra FL can be isomorphically embedded (as an abstract Lie algebra) into the triangular Lie algebra $u_n(F)$.*

Proof. First, suppose $\dim_F FL = n$. If FL is abelian, then FL is isomorphically embeddable into the Lie algebra $u_n(F)$ because the subalgebra $F\left\langle \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right\rangle$ of $u_n(F)$ is abelian of dimension n over F. So one can assume that FL is nonabelian. Then by Theorem 1, $FL = M_1 \oplus M_2$, where M_1 is an abelian Lie algebra of dimension $n-3$ over F and M_2 is nilpotent nonabelian with $\dim_F M_2 = 3$. The subalgebra $H_2 = F\left\langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_3} \right\rangle$ of the Lie algebra $u_n(F)$ is obviously isomorphic to M_2. The abelian subalgebra $H_1 = F\left\langle \frac{\partial}{\partial x_4}, \dots, \frac{\partial}{\partial x_n} \right\rangle$, $n \geq 4$, is isomorphic to the Lie algebra M_1. So $FL \simeq H_1 \oplus H_2$ is isomorphic to a subalgebra of $u_n(F)$. Note that $H_1 \oplus H_2$ is of rank n over the field $\mathbb{K}(x_1, \dots, x_n)$ of rational functions in n variables.

Next, let $\dim_F FL > n$. By Theorem 1, the Lie algebra FL lies in one of the subalgebras of types L_1 or L_2. Therefore it is sufficient to show that the subalgebras L_1, L_2 of $W(A)$ from Theorem 1 can be isomorphically embedded into the Lie algebra $u_n(F)$. In case L_1, we define a mapping φ on the basis $D_1, \dots, D_n, \left\{\frac{b^i}{i!}D_i\right\}_{i=1}^{\infty}$ of L_1 over R by the rule $\varphi(D_i) = \frac{\partial}{\partial x_i}, i = 1, \dots, n$, $\varphi(\frac{b^i}{i!}D_i) = \frac{x_n^i}{i!}\frac{\partial}{\partial x_i}, i = 1, \dots, n-1$, and then extend it on L_1 by linearity. One can easily see that the mapping φ is an isomorphic embedding of the Lie algebra L_1 into $u_n(F)$. Analogously, on L_2 we define a mapping $\psi : L_2 \to u_n(F)$ by the rule

$$\psi(D_i) = \frac{\partial}{\partial x_i}, \quad i = 1, \dots, n, \qquad \psi(\frac{a^i b^j}{i!j!}D_k) = \frac{x_{n-1}^i x_n^j}{i!j!}\frac{\partial}{\partial x_k}, \quad k = 1, \dots, n-2$$

$$\psi(\frac{b^i}{i!}D_{n-1}) = \frac{x_n^i}{i!}\frac{\partial}{\partial x_{n-1}}, \quad i \geq 1, j \geq 1,$$

and further by linearity. Then ψ is an isomorphic embedding of the Lie algebra L_2 into the Lie algebra $u_n(F)$. □

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Нехай $\mathbb{K}$ — поле характеристики нуль, A — область цілісності над $\mathbb{K}$ з полем часток $R = Frac(A)$, і $Der_{\mathbb{K}}A$ — алгебра Лі $\mathbb{K}$-диференціювань A. Нехай $W(A) := RDer_{\mathbb{K}}A$ і L — нільпотентна підалгебра рангу n над R Лі алгебри $W(A)$. Ми показуємо, що якщо центр $Z = Z(L)$ має ранг $\geq n-2$ над R і $F = F(L)$ — поле констант алгебри Лі L в R, то алгебра Лі FL міститься в локально нільпотентній підалгебрі рангу n над R з природнім базисом над полем R. Також доводиться, що Лі алгебра FL може бути ізоморфно вкладена (як абстрактна Лі алгебра) в трикутну алгебру Лі $u_n(F)$, що була досліджена раніше іншими авторами.

Ключові слова і фрази: диференціювання, векторне поле, алгебра Лі, нільпотентна алгебра, область цілісності.

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LOCAL NEARRINGS ON FINITE NON-ABELIAN 2-GENERATED p-GROUPS

It is proved that for $p > 2$ every finite non-metacyclic 2-generated p-group of nilpotency class 2 with cyclic commutator subgroup is the additive group of a local nearring and in particular of a nearring with identity. It is also shown that the subgroup of all non-invertible elements of this nearring is of index p in its additive group.

Key words and phrases: finite p-group, local nearring.

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INTRODUCTION

Nearrings are generalizations of associative rings in the sense that with respect to the addition they need not be commutative and only one distributive law is assumed. In this paper the concept "nearring" means a left distributive nearring with a multiplicative identity. The reader is referred to the books by Meldrum [6] or Pilz [8] for terminology, definitions and basic facts concerning nearrings.

Following [3], the nearring with identity will be called local, if the set of all non-invertible elements forms a subgroup of its additive group. The main results concerning local nearrings are summarized in [11].

In [4] it is shown that every non-cyclic abelian p-group of order $p^n > 4$ is the additive group of a zero-symmetric local nearring which is not a ring. As it was noted in [5], neither a generalized quaternion group nor a non-abelian group of order 8 can be the additive group of a local nearring.

Therefore the structure of the non-abelian finite p-groups which are the additive groups of local nearrings is an open problem [2].

It was proved that every non-metacyclic Miller–Moreno p-group of order $p^n > 8$ is the additive group of a local nearring and the multiplicative group of such a nearring is the group of order $p^{n-1}(p-1)$ [9]. In this paper finite non-abelian non-metacyclic 2-generated p-groups $(p > 2)$ of nilpotency class 2 with cyclic commutator subgroup are studied.

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1 PRELIMINARIES

Let G be a finite non-abelian non-metacyclic 2-generated p-group ($p > 2$) of nilpotency class 2 with cyclic commutator subgroup.

Denote by G' and $Z(G)$ the commutator subgroup and the centre of G, respectively.

Let a and b be generators for G such that $G/G' = \langle aG' \rangle \times \langle bG' \rangle$, aG' has order p^m and bG' has order p^n. Then $c = [a, b]$ generates G', c has order p^d with $1 \le d \le n \le m$, and $c \in Z(G) = \langle a^{p^m}, b^{p^n}, c \rangle$.

Suppose that $\langle a \rangle \cap G' = \langle b \rangle \cap G' = 1$. Then

$$G = \langle a,\ b,\ c | a^{p^m} = b^{p^n} = c^{p^d} = 1,\ a^b = ac,\ c^a = c^b = c \rangle$$

and each element of G can be uniquely written in the form $a^{x_1} b^{x_2} c^{x_3}$, $x_1 \in C_{p^m}$, $x_2 \in C_{p^n}$, $x_3 \in C_{p^d}$. Therefore the group G with $p > 2$ will be denoted by $G(p^m, p^n, p^d)$.

Lemma 1. *For any natural numbers k and l the equality $[a^k, b^l] = c^{kl}$ holds.*

Proof. Since $b^{-1}ab = ac$, it follows that $b^{-l}ab^l = ac^l$. Therefore, $b^{-l}a^k b^l = (ac^l)^k = a^k c^{kl}$, thus $a^{-k} b^{-l} a^k b^l = c^{kl}$. □

Corollary 1. *Let the group $G(p^m, p^n, p^d)$ be additively written. Then for any natural numbers k and l the equalities $-ak - bl + ak + bl = c(kl)$ and $bl + ak = -c(kl) + ak + bl$ hold.*

Lemma 2. *For any natural numbers k, l and r the equality*

$$(a^k b^l)^r = a^{kr} b^{lr} c^{-kl\binom{r}{2}} \tag{1}$$

holds.

Proof. For $r = 1$, there is nothing to prove. By induction on r, we derive

$$(a^k b^l)^r = a^{kr} b^{lr} c^{-kl\binom{r}{2}}.$$

Replacing r by $r + 1$ in equality (1), we have

$$\begin{aligned}(a^k b^l)^{(r+1)} &= a^{kr} b^{lr} a^k b^l c^{-kl\binom{r}{2}} = a^{k(r+1)} b^{l(r+1)} c^{-klr} c^{-kl\binom{r}{2}} \\ &= a^{k(r+1)} b^{l(r+1)} c^{-kl(r+\binom{r}{2})} = a^{k(r+1)} b^{l(r+1)} c^{kl\binom{r+1}{2}}.\end{aligned}$$

Thus, equality (1) holds for an arbitrary r. □

Corollary 2. *Let the group $G(p^m, p^n, p^d)$ be additively written. Then for any natural numbers k, l and r the equality $(ak + bl)r = akr + blr - ckl\binom{r}{2}$ holds.*

Obviously, the exponent of $G(p^m, p^n, p^d)$ is equal to p^m for $1 \le d \le n \le m$.

Lemma 3. *If x is an element of order p^m of $G(p^m, p^n, p^d)$, then there exist generators a, b, c of this group such that $a = x$ and $a^{p^m} = b^{p^n} = c^{p^d} = 1$, $a^b = ac$, $c^a = c^b = c$.*

Proof. Indeed, for each $x \in G(p^m, p^n, p^d)$ there exist positive integers α, β and γ such that $x = a^\alpha b^\beta c^\gamma$. Thus, we have

$$x^{p^m} = (a^\alpha b^\beta c^\gamma)^{p^m} = (a^\alpha b^\beta)^{p^m} c^{\gamma p^m} = a^{\alpha p^m} b^{\beta p^m} c^{\gamma p^m - \alpha\beta\binom{p^m}{2}}$$
$$= a^{p^m \alpha} b^{p^m \beta} c^{p^m(\gamma - \alpha\beta\frac{(p^m-1)}{2})} = 1$$

by Lemma 2. Since $|a| = p^m$ and $1 \leq d \leq n \leq m$, where $m > 1$ and $p > 2$, it follows that the exponent of $G(p^m, p^n, p^d)$ equals p^m.

If

$$x^{p^{m-1}} = a^{p^{m-1}\alpha} b^{p^{m-1}\beta} c^{p^{m-1}(\gamma - \alpha\beta\frac{(p^{m-1}-1)}{2})} \neq 1,$$

then either $(\alpha, p) = 1$, or $(\beta, p) = 1$ for $m = n$, or $(\gamma, p) = 1$ for $m = n = d$. So, without loss of generality, we can assume that $(\alpha, p) = 1$. Then

$$\langle x, b\rangle = \langle a^\alpha b^\beta c^\gamma, b\rangle = \langle a^\alpha, b\rangle = \langle a, b\rangle = G$$

and

$$b^{-1}xb = b^{-1}(a^\alpha b^\beta c^\gamma)b = (ac)^\alpha b^\beta c^\gamma = (a^\alpha b^\beta c^\gamma)c^\alpha = xc^\alpha.$$

Furthermore, substituting c^α instead of c for generators x and b of $G(p^m, p^n, p^d)$, we have similar expressions as for generators a and b, thus replacing the element a by x. □

The following assertion concerning the automorphisms group of $G(p^m, p^n, p^d)$ is a direct consequence of statement (B1) [7].

Lemma 4. *Let $G = G(p^m, p^n, p^d)$ and let $Aut(G)$ be the automorphism group of G. Then the following statements hold*:

1) *if $m = n$, then $|Aut(G)| = p^{2d+4m-5}(p^2 - 1)(p - 1)$;*

2) *if $m > n$, then $|Aut(G)| = p^{2d+3n+m-2}(p - 1)^2$.*

An information about a group of automorphisms of $G(p^m, p^m, p^d)$ is given by the following lemma.

Lemma 5. *Let $G = G(p^m, p^m, p^d)$ and let there exist a subgroup A of $Aut(G)$ of order $p^{2m+d-2}(p^2 - 1)$, where m, $d > 1$ with odd p. If an element $g \in G$ of order p^m and A contains Sylow normal p-subgroup, then $G \neq g^A \cup \Phi(G)$.*

Proof. Assume that $G = g^A \cup \Phi(G)$. Then $G = (\langle a\rangle \times \langle c\rangle) \rtimes \langle b\rangle$ with generators a, b of order p^m and a central commutator $c = [a, b]$ of order p^d by the definition. Hence

$$\Phi(G) = (\langle a^p\rangle \times \langle c\rangle) \rtimes \langle b^p\rangle,$$

and thus all elements of order p^m are contained in g^A. Furthermore, $a = g^u$ for some $u \in A$, hence $g^A = a^A$, i. e. $G = a^A \cup \Phi(G)$. Since $|G| = p^{2m+d}$ and $\Phi(G) = p^{2m+d-2}$, it follows that

$$|a^A| = |G| - |\Phi(G)| = p^{2m+d-2}(p^2 - 1),$$

and so the centralizer $C_A(a)$ of a in A equals 1. In particular, $(a\langle c^p\rangle)^A = (a\langle c^p\rangle)^B = a\langle c^p\rangle$ for the normal subgroup $B = C_A(a\langle c^p\rangle)$ of order p^{d-1} in A.

Considering the factor-group $\bar{G} = G/\langle c^p\rangle$ and $\bar{A} = A/B$. Taking into consideration, that $|\bar{a}^{\bar{A}}| = p^{2m-1}(p^2 - 1)$, we have $\bar{G} = \bar{a}^{\bar{A}} \cup \Phi(\bar{G})$. Since $|\Phi(\bar{G})| = |Z(\bar{G})|$ and $xy = yx$ for all $x \in \Phi(\bar{G})$, $y \in \bar{G}$, we have $\Phi(\bar{G}) = Z(\bar{G})$. Therefore, $\bar{G}$ is a Miller–Moreno group. Since $\bar{G} = \bar{a}^{\bar{A}} \cup Z(\bar{G})$, the latter equality is impossible by [9, Lemma 7]. This contradiction completes the proof. □

2 Nearrings with identity on group $G(p^m, p^n, p^d)$

First recall some basic concepts of the theory of nearrings.

Definition 1. *A set R with two binary operations "$+$" and "$\cdot$" is called a (left) nearring if the following statements hold*

(1) $(R, +) = R^+$ is a (not necessarily abelian) group with neutral element 0;

(2) $(R, \cdot)$ is a semigroup;

(3) $x(y+z) = xy + xz$ for all $x, y, z \in R$.

If R is a nearring, then the group R^+ is called the *additive group* of R. If in addition $0 \cdot x = 0$, then the nearring R is called *zero-symmetric* and if the semigroup $(R, \cdot)$ is a monoid, i.e. it has an identity element i, then R is a *nearring with identity i*. In the latter case the group R^* of all invertible elements of the monoid $(R, \cdot)$ is called the *multiplicative group* of R.

The following assertion is well-known.

Lemma 6. *Let R be a finite nearring with identity i. Then the exponent of R^+ is equal to the additive order of i which coincides with additive order of every element of R^*.*

As a direct consequence of Lemmas 3 and 6 we have the following corollary.

Corollary 3. *Let R be a nearring with identity i whose group R^+ is isomorphic to a group $G(p^m, p^n, p^d)$. Then $R^+ = \langle a\rangle + \langle b\rangle + \langle c\rangle$ with elements a, b and c, satisfying relations $ap^m = bp^n = cp^d = 0$, $-b + a + b = a + c$ and $-a + c + a = -b + c + b = c$ with $1 \le d \le n \le m$, where $a = i$.*

The following statement [10, Lemma 1] establishes a connection between the automorphism group of the additive group of the nearring with identity and its multiplicative group.

Lemma 7. *Let R be a nearring with identity i. Then there exists a subgroup A of the automorphism group $Aut(R^+)$ which is isomorphic to R^* and satisfying the condition $i^A = \{i^a \mid a \in A\} = R^*$.*

The subgroup A defined in Lemma 7 is called the automorphism group of the group R^+ associated with the group R^*.

The following statement [11, Theorem 54] concerns the structure of L which is the subgroup of all non-invertible elements of finite local nearring R. Let $\Phi(G)$ denote the Frattini subgroup of G.

Theorem 1. *Let R be a local nearring of order p^n and let $G(R) = R^+ \rtimes R^*$ be a group associated with R. Then $H = R^+ \rtimes (i + L)$ is a Sylow normal p-subgroup of $G(R)$ and $L = R^+ \cap \Phi(H)$. In particular, if L is non-abelian, then its center is non-cyclic.*

Considering $\Phi(R^+) \le \Phi(H)$, we have the following corollary.

Corollary 4. $\Phi(R^+) \le L = \Phi(H) \cap R^+$.

Let R be a nearring with identity i whose group R^+ is isomorphic to a group $G(p^m, p^n, p^d)$. It follows from Corollary 3 that $R^+ = \langle a\rangle + \langle b\rangle + \langle c\rangle$ with elements a, b and c, satisfying relations $ap^m = bp^n = cp^d = 0$, $-b + a + b = a + c$ and $-a + c + a = -b + c + b = c$ with $1 \le d \le n \le m$, where $a = i$ and each element $x \in R$ is uniquely written in the form $x = ax_1 + bx_2 + cx_3$ with coefficients $0 \le x_1 < p^m$, $0 \le x_2 < p^n$ and $0 \le x_3 < p^d$.

Furthermore, we can assume $xa = ax = x$ for each $x \in R$. Then there exist uniquely defined mappings $\alpha\colon R \to \mathbb{Z}_{p^m}$, $\beta\colon R \to \mathbb{Z}_{p^n}$ and $\gamma\colon R \to \mathbb{Z}_{p^d}$ such that

$$xb = a\alpha(x) + b\beta(x) + c\gamma(x). \tag{2}$$

Lemma 8. *If $x = ax_1 + bx_2 + cx_3$ and $y = ay_1 + by_2 + cy_3$ are arbitrary elements of R, then*

$$\begin{aligned} xy = a(x_1y_1 + y_2\alpha(x)) &+ b(x_2y_1 + y_2\beta(x)) \\ &+ c\Big(- x_1x_2\binom{y_1}{2} - \binom{y_2}{2}\alpha(x)\beta(x) - x_2y_1y_2\alpha(x) \\ &\qquad + x_3y_1 + y_2\gamma(x) + x_1y_3\beta(x) - x_2y_3\alpha(x)\Big), \end{aligned}$$

where mappings $\alpha\colon R \to \mathbb{Z}_{p^m}$, $\beta\colon R \to \mathbb{Z}_{p^n}$ and $\gamma\colon R \to \mathbb{Z}_{p^d}$ satisfy the conditions

(0) $\alpha(0) \equiv 0 \pmod{p^m}$, $\beta(0) \equiv 0 \pmod{p^n}$ *and* $\gamma(0) \equiv 0 \pmod{p^d}$ *if and only if the nearring R is zero-symmetric;*

(1) $\alpha(xy) \equiv x_1\alpha(y) + \alpha(x)\beta(y) \pmod{p^m}$;

(2) $\beta(xy) \equiv x_2\alpha(y) + \beta(x)\beta(y) \pmod{p^n}$;

(3) $\gamma(xy) \equiv -x_1x_2\binom{\alpha(y)}{2} - \alpha(x)\beta(x)\binom{\beta(y)}{2} - x_2\alpha(x)\alpha(y)\beta(y)$
$+x_3\alpha(y) + \gamma(x)\beta(y) + x_1\beta(x)\gamma(y) - x_2\alpha(x)\gamma(y) \pmod{p^d}$.

Proof. If R is a zero-symmetric nearring, then

$$0 = 0 \cdot b = a\alpha(0) + b\beta(0) + c\gamma(0),$$

thus $\alpha(0) \equiv 0 \pmod{p^m}$, $\beta(0) \equiv 0 \pmod{p^n}$ and $\gamma(0) \equiv 0 \pmod{p^d}$. On the other hand, if the last congruences hold, then $0 \cdot b = a \cdot 0 + b \cdot 0 + c \cdot 0 = 0$. Since a is the multiplicative identity in R, we have $0 \cdot a = a \cdot 0 = 0$. Moreover, from the equality $c = -a - b + a + b$ and the left distributive law it follows that $0 \cdot c = -0 \cdot a - 0 \cdot b + 0 \cdot a + 0 \cdot b = 0$, hence

$$0 \cdot x = 0 \cdot (ax_1 + bx_2 + cx_3) = (0 \cdot a)x_1 + (0 \cdot b)x_2 + (0 \cdot c)x_3 = 0.$$

This proves statement (0).

Next, using (2) and Corollary 1, we obtain

$$\begin{aligned} xc = -xa - xb + xa + xb &= -cx_3 - bx_2 - ax_1 - c\gamma(x) - b\beta(x) - a\alpha(x) \\ &\quad + ax_1 + bx_2 + cx_3 + a\alpha(x) + b\beta(x) + c\gamma(x) \\ &= -bx_2 - ax_1 - b\beta(x) - a\alpha(x) + ax_1 + bx_2 + a\alpha(x) + b\beta(x) \\ &= -bx_2 + cx_1\beta(x) - b\beta(x) - ax_1 - a(\alpha(x) - x_1) + bx_2 + a\alpha(x) + b\beta(x) \\ &= cx_1\beta(x) - b(x_2 + \beta(x)) - a\alpha(x) + bx_2 + a\alpha(x) + b\beta(x) \end{aligned}$$

$$= cx_1\beta(x) - b(x_2 + \beta(x)) - a\alpha(x) - cx_2\alpha(x) + a\alpha(x) + bx_2 + b\beta(x)$$
$$= c(x_1\beta(x) - x_2\alpha(x)) - b(x_2 + \beta(x)) + bx_2 + b\beta(x) = c(x_1\beta(x) - x_2\alpha(x)).$$

Therefore

$$xy = (ax_1 + bx_2 + cx_3)y_1 + (a\alpha(x) + b\beta(x) + c\gamma(x))y_2 + (cx_1\beta(x) - x_2\alpha(x))y_3.$$

Corollary 2 implies that

$$(ax_1 + bx_2)y_1 = ax_1y_1 + bx_2y_1 - cx_1x_2\binom{y_1}{2},$$
$$(a\alpha(x) + b\beta(x))y_2 = ay_2\alpha(x) + by_2\beta(x) - c\binom{y_2}{2}\alpha(x)\beta(x)$$

and

$$bx_2y_1 + ay_2\alpha(x) = ay_2\alpha(x) + bx_2y_1 - cx_2y_1y_2\alpha(x).$$

By the left distributive law, we have

$$xy = a(x_1y_1 + y_2\alpha(x)) + b(x_2y_1 + y_2\beta(x)) + c\Big(- x_1x_2\binom{y_1}{2}$$
$$- \binom{y_2}{2}\alpha(x)\beta(x) - x_2y_1y_2\alpha(x) + x_3y_1 + y_2\gamma(x) + x_1y_3\beta(x) - x_2y_3\alpha(x)\Big).$$

Finally, the associativity of multiplication for all $x, y \in R$ implies that

1) $(xy)b = x(yb)$.

Thus

2) $(xy)b = a\alpha(xy) + b\beta(xy) + c\gamma(xy)$

and $yb = a\alpha(y) + b\beta(y) + c\gamma(y)$ by formula (2). Substituting the last expression in the right part of equality 1), we get

3) $x(yb) = a(x_1\alpha(y) + \alpha(x)\beta(y)) + b(x_2\alpha(y) + \beta(x)\beta(y))$
$+ c(-x_1x_2\binom{\alpha(y)}{2} - \alpha(x)\beta(x)\binom{\beta(y)}{2} - x_2\alpha(x)\alpha(y)\beta(y)$
$+ x_3\alpha(y) + \gamma(x)\beta(y) + x_1\beta(x)\gamma(y) - x_2\alpha(x)\gamma(y))$.

Comparing the coefficients a, b and c in 2) and 3) by equality 1), we derive statements (1)–(3) of the lemma. □

3 LOCAL NEARRINGS ON GROUP $G(p^m, p^n, p^d)$

Let R be a local nearring with identity i, whose group R^+ is isomorphic to the group $G(p^m, p^n, p^d)$. Then $R^+ = \langle a\rangle + \langle b\rangle + \langle c\rangle$ with elements a, b and c, satisfying relations $ap^m = bp^n = cp^d = 0, -b + a + b = a + c$ and $-a + c + a = -b + c + b = c$ with $1 \leq d \leq n \leq m$, where $a = i$ and each element $x \in R$ is uniquely written in the form $x = ax_1 + bx_2 + cx_3$ with coefficients $0 \leq x_1 < p^m$, $0 \leq x_2 < p^n$ and $0 \leq x_3 < p^d$.

We show that the set L of all non-invertible elements of R is a subgroup of index p in R^+.

Theorem 2. *The following statements hold*

1) $L = \langle a \cdot p \rangle + \langle b \rangle + \langle c \rangle$ *and, in particular, the subgroup* L *is of index* p *in* R^+ *and* $|R^*| = p^{m+n+d-1}(p-1)$;

2) $x = ax_1 + bx_2 + cx_3$ *is an invertible element if and only if* $x_1 \not\equiv 0 \pmod{p}$.

Proof. Assume that $|R^+ : L| = p^t$, $t > 1$. Since $R = R^* \cup L$, it follows that

$$|R^*| = |R| - |L| = p^{m+n+d} - p^{m+n+d-t} = p^{m+n+d-t}(p^t - 1).$$

According to Lemma 7, the group R^* is isomorphic to the subgroup A of the automorphism group of R^+ and so $|R^*|$ divides $|\mathrm{Aut}(R^+)|$. According to statement 1) of Lemma 4 it is possible only if $t = 2$ and $m = n$.

Assume that $|R^+ : L| = p^2$ and $m = n$. If $d = 1$, then it is impossible because of [9, Theorem 2]. Now let $d > 1$. Since $|R^+ : \Phi(R^+)| = p^2$ and Corollary 4, we have $L = \Phi(R^+)$. Hence by Lemma 7, we get $R^+ = a^A \cup \Phi(R^+)$, which is impossible by Lemma 5. This contradiction shows that our assumption is false and so $|R^+ : L| = p$.

It is clear that R/L is a nearfield and so the factor-group R^+/L^+ is an elementary abelian p-group. Thus for $a \notin L$ we have $ap \in L$ and so $L = \langle a \cdot p \rangle + \langle b \rangle + \langle c \rangle$. Therefore $R^* = R \setminus L$ and hence

$$R^* = \{ax_1 + bx_2 + cx_3 \mid x_1 \not\equiv 0 \ (\bmod\ p\ )\}.$$

□

Applying statement (1) of Theorem 2 to Lemma 8, we get the following formula for multiplying elements $x = ax_1 + bx_2 + cx_3$ and $y = ay_1 + by_2 + cy_3$ in the local nearring R.

Corollary 5. *If* $x, y \in R$ *with* $1 \le d \le n \le m$ *and* $xb = a\alpha(x) + b\beta(x) + c\gamma(x)$, *then*

$$xy = a(x_1y_1 + y_2\alpha(x)) + b(x_2y_1 + y_2\beta(x)) + c\Big(- x_1x_2\binom{y_1}{2} - \binom{y_2}{2}\alpha(x)\beta(x) - x_2y_1y_2\alpha(x) + x_3y_1 + y_2\gamma(x) + x_1y_3\beta(x) - x_2y_3\alpha(x)\Big),$$

where mappings $\alpha\colon R \to \mathbb{Z}_{p^m}$, $\beta\colon R \to \mathbb{Z}_{p^n}$ *and* $\gamma\colon R \to \mathbb{Z}_{p^d}$ *and the following statements hold*

(0) $\alpha(0) \equiv 0 \pmod{p^m}$, $\beta(0) \equiv 0 \pmod{p^n}$ *and* $\gamma(0) \equiv 0 \pmod{p^d}$ *if and only if the nearring* R *is zero-symmetric*;

(1) $\alpha(x) \equiv 0 \pmod{p}$;

(2) *if* $\beta(x) \equiv 0 \pmod{p}$, *then* $x_1 \equiv 0 \pmod{p}$;

(3) $\alpha(xy) \equiv x_1\alpha(y) + \alpha(x)\beta(y) \pmod{p^m}$;

(4) $\beta(xy) \equiv x_2\alpha(y) + \beta(x)\beta(y) \pmod{p^n}$;

(5) $\gamma(xy) \equiv -x_1x_2\binom{\alpha(y)}{2} - \alpha(x)\beta(x)\binom{\beta(y)}{2} - x_2\alpha(x)\alpha(y)\beta(y) + x_3\alpha(y) + \gamma(x)\beta(y) + x_1\beta(x)\gamma(y) - x_2\alpha(x)\gamma(y) \pmod{p^d}$.

Proof. Indeed, statements (0), (3)–(5) repeat statements (0)–(4) of Lemma 8. Since $L = \langle a \cdot p\rangle + \langle b\rangle + \langle c\rangle$ by Theorem 2 and L is an (R, R)-subgroup in R by statement 2) [1, Lemma 3.2], it follows that $xb \in L$ and hence $\alpha(x) \equiv 0 \pmod p$, proving statement (1). Taking $y = c$, we have $xc = c(x_1\beta(x) - x_2\alpha(x))$. Thus, if $\beta(x) \equiv 0 \pmod p$, then $xc = 0 \pmod p$, and so $x \in L$. Thus $x_1 \equiv 0 \pmod p$ by Theorem 2, proving statement (2). □

The following theorem shows the conditions given in Theorem 2 are sufficient for existing of finite local nearrings on $G(p^m, p^n, p^d)$. Moreover, each group $G(p^m, p^n, p^d)$ is the additive group of a nearring with identity.

Theorem 3. *For each prime p and positive integers m, n and d with* $1 \le d \le n \le m$ *there exists a local nearring R whose additive group* R^+ *is isomorphic to the group* $G(p^m, p^n, p^d)$.

Proof. Let R be an additively written group $G(p^m, p^n, p^d)$ with generators a, b and c satisfying the relations $|a| = p^m$, $|b| = p^n$, $|c| = p^d$, $b^{-1}ab = ac$ and $a^{-1}ca = b^{-1}cb = c$. Then $G = \langle a\rangle + \langle b\rangle + \langle c\rangle$ and each element $x \in R$ is uniquely written in the form $x = ax_1 + bx_2 + cx_3$ with coefficients $0 \le x_1 < p^m$, $0 \le x_2 < p^n$ and $0 \le x_3 < p^d$. In order to define a multiplication "$\cdot$" on R in such a manner that $(R, +, \cdot)$ is a local nearring.

Assume that $1 \le d \le n \le m$ and let the mappings from Corollary 5 be defined by the congruences $\alpha(x) \equiv 0 \pmod{p^m}$, $\beta(x) \equiv x_1 \pmod{p^n}$ and $\gamma(x) \equiv 0 \pmod{p^d}$ for each $x \in G$. Then

$$x \cdot y = ax_1y_1 + b(x_2y_1 + x_1y_2) + c\Big(- x_1x_2\binom{y_1}{2} + x_3y_1 + x_1^2y_3\Big).$$

It suffices to show that the mappings $\alpha : G \to \mathbb{Z}_{p^m}$, $\beta : G \to \mathbb{Z}_{p^n}$ and $\gamma : G \to \mathbb{Z}_{p^d}$ with respect to the multiplication "$\cdot$" satisfy statements (0)–(5) of Corollary 5.

Indeed, $\alpha(0) \equiv 0 \pmod{p^m}$, $\beta(0) \equiv 0 \pmod{p^n}$ and $\gamma(0) \equiv 0 \pmod{p^d}$ by the definition. Since $0 \cdot y = a \cdot 0 + b \cdot 0 + c \cdot 0 = 0$ for each $y \in G$, this implies that a multiplication "$\cdot$" is zero-symmetric and so, proving statement (0) of Corollary 5. Indeed, we have $\alpha(x) \equiv 0 \pmod p$ and $x_1 \equiv 0 \pmod p$, if $\beta(x) \equiv 0 \pmod p$, so that statements (1) and (2) of Corollary 5 hold. Clearly, we derive statements (3)–(5) of Corollary 5. □

As corollary we have the following assertion.

Corollary 6. *Each group* $G(p^m, p^n, p^d)$ *is the additive group of a nearring with identity.*

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Раєвська І.Ю., Раєвська М.Ю. *Локальні майже-кільця на скінченних неабелевих неметациклічних 2-породжених p-групах* // Карпатські матем. публ. — 2020. — Т.12, №1. — С. 199–207.

Доведено, що для $p > 2$ кожна скінченна неметациклічна 2-породжена p-група зі ступенем нільпотентності рівним 2 з циклічним комутантом є адитивною групою деякого локального майже-кільця, зокрема, майже-кільця з одиницею. Показано, що підгрупа всіх необоротних елементів цього локального майже-кільця має індекс p в його адитивній групі.

Ключові слова і фрази: скінченна p-группа, локальне майже-кільце.

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CERTAIN RESULTS FOR A CLASS OF NONLINEAR FUNCTIONAL SPACES

In this article, we study properties of a class of functional spaces, so-called pn-spaces, which arise from investigation of nonlinear differential equations. We establish some integral inequalities to analyse the structures of the pn-spaces with the constant and variable exponent. We prove embedding theorems, which indicate the relation of these spaces with the well known classical Lebesgue and Sobolev spaces with the constant and variable exponents.

Key words and phrases: pn-space, variable exponent, integral inequality, nonlinear differential equation, embedding theorem.

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INTRODUCTION

This paper is concerned with some features of a class of functional spacesб which are emerged from investigation of nonlinear differential equations. Studying boundary value problems (BVPs) require to examine and understand the functional spacesб which are directly related with the considered problem. In other words, it is required to work on the domain of the operator generated by the addressed boundary value problem. We specify that it is better to study each BVPs on its own space. Furthermore, detailed analysis of these spaces and examining their topology, structure etc. cause to gain better results of the possed problem (for example, regularity of the solution).

The spaces generated by boundary value problems for the linear differential equations are generally linear spaces such as Sobolev spaces and different generalizations of them. Apart from boundary value problems for linear differential equations, the spaces generated by nonlinear differential equations (essentially the domain of the corresponding operator) are subsets of linear spaces and do not have linear structure. The class of spaces of this type were introduced and investigated by Soltanov in the abstract case (see, e.g. [21–26]), and also in the case of functions spaces (see, e.g. [23–30] and references therein, where various subsets of linear spaces of this type were searched). In the mentioned articles, topology of these spaces were investigated and shown that under what circumstances they are metric or pseudo-metric spaces. Starting from these features of the introduced spaces, they were defined as the class of pseudo-normed spaces or pn-spaces and the class of quasi-pseudo normed spaces or qn-spaces.

In this work, we focus on the characteristics of certain class of functional pn-spaces. Essentially, we deal with the following class of functional pn-spaces.

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Let $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ be bounded domain with sufficiently smooth boundary. In this work the class of functions $u : \Omega \longrightarrow \mathbb{R}$ of the following type will be investigated

$$S_{m,\alpha,\beta}(\Omega) := \left\{ u \in L^1(\Omega) : [u]^{\alpha+\beta}_{S_{m,\alpha,\beta}(\Omega)} < \infty \right\},$$

where

$$[u]^{\alpha+\beta}_{S_{m,\alpha,\beta}(\Omega)} := \sum_{0 \leq |k| \leq m} \left(\int_\Omega |u|^\alpha \left| D^k u \right|^\beta dx \right), \quad D = (D_1, D_2, \ldots, D_n),$$

$D_i = \frac{\partial}{\partial x_i}$, $D^k \equiv D_1^{k_1} D_2^{k_2}, \ldots, D_n^{k_n}$, $i = \overline{1,n}$, $|k| = \sum_{i=1}^{n} k_i$. Here, we only address the cases $m = 1, 2$.

It is important to note that the following subset of $L^p(\Omega)$, $p \geq 2$,

$$M := \left\{ u \in L^1(\Omega) : \sum_{i=1}^{n} \left(\int_\Omega |u|^{p-2} |D_i u|^2 dx \right) < \infty, u \mid_{\partial\Omega} = 0 \right\}$$

was arose in the article of Dubinskii earlier ([7, 8, 11]) while studying the following nonlinear problem

$$\frac{\partial u}{\partial t} - \sum_{i=1}^{n} D_i \left(|u|^{p-2} D_i u \right) = h(x,t), \quad (t,x) \in (0,T) \times \Omega,$$

$$u(0,x) = u_0(x), \quad u \Big|_{(0,T] \times \partial\Omega} = 0 .$$

Here, compact inclusion of subset M to the space $L^p(\Omega)$ and also necessary compactness theorems for analysis of the parabolic problem were proved. Later on, different new subsets of $L^1(\Omega)$ appeared in the articles of Soltanov (see, e.g. [23–25]) while studying the mixed problem for the following nonlinear equation, which is type of the Prandtl-von-Mises equation

$$\frac{\partial u}{\partial t} - |u|^\rho \frac{\partial^2 u}{\partial x^2} = h(t,x), \ \rho > 0, \quad (t,x) \in (0,T) \times \Omega. \tag{1}$$

For example, one of the emerged class in the case of $\Omega = (a,b) \subset \mathbb{R}$ can be expressed in the form

$$\left\{ u \in L^1(\Omega) : \int_\Omega |u|^\alpha \left| D^2 u \right|^\beta dx < \infty, u(a) = u(b) = 0 \right\},$$

and also as type of subsets in the form $S_{m,\alpha,\beta}(\Omega)$. Here, we specify that different problems to the equation (1) were studied under various additional conditions as well (see, e.g. [12, 14, 18, 35–37]).

Accordingly, in the papers [24, 25] etc. different classes of sets of this type were examined and it was shown that these sets are nonlinear topological spaces, moreover they are either metric or pseudo-metric spaces. Many other properties of the introduced spaces were investigated as well in these works. For instance, relations of these spaces amongst themselves and with well known functional spaces (e.g. Lebesgue or Sobolev spaces etc).

Consequently, in the mentioned works pn-spaces and qn-spaces were defined with taking into account the principal attributes of the presented spaces.

These spaces may arise from the research of the existence of smooth solution of the following differential equation

$$-\Delta u + u + |u|^p u = h(x), \ x \in \Omega \subset \mathbb{R}^n, \ n \geq 2,$$

$$\left(\frac{\partial u}{\partial \eta}+|u|^{\mu}\,u\right)\Big|_{\partial\Omega}=\psi\left(x'\right),\; x'\in\partial\Omega,\; p,\mu\geq 0,$$

which was studied by Soltanov [32]. We emphasize that equation of this form was considered by many authors, who tried to answer various questions of different problems for this equation, (see, e.g. Berestycki ve Nirenberg [3], Brezis [4], etc.). In [15], Pohozaev employed another approach for this problem that led to gaining distinct results other than [32].

This kind of nonlinear spaces are generated by the differential equations, which ensue from the mathematical models of some processes in flood mechanics. For example, we may present the nonlinear equation of type

$$\frac{\partial u}{\partial t}-|u|^{p-2}\,\Delta u=h\left(x,t\right),\; p\geq 2,$$

where this equation were studied [24,31] and [33]. Similar equations were handled by Oleynik [14], Walter [36] only using the approximation way and Tsutsumi, Ishiwata [35] focused on understanding the behavior of the solution.

In recent years, there have been an increasing interest in the study of equations with variable exponents of nonlinearities. The interest in the study of differential equations that involves variable exponents is motivated by their applications to the theory of elasticity and hydrodynamics, in particular, the models of electrorheological fluids [17] in which a substantial part of viscous energy, the thermistor problem [38], image processing [5] and modeling of non-Newtonian fluids with thermo-convective effects [2] etc.

In the most of these papers, that concern with equations, which have non standard growth, authors studied the problems, which involve $p(.)$-Laplacian type equation or equations, which fulfill monotonicity conditions, where enable to apply monotonicity methods. Unlike these works, in the articles [19, 20] investigating some properties of nonlinear spaces with variable exponent, we developed an approach based on the spaces corresponding to problem under consideration. It is necessary to note, that the questions mentioned above may arise for the problems, which have variable exponent nonlinearity. Eventually, here we also study variable exponent nonlinear spaces that are essential for the investigation of the following type of equations

$$\nabla\cdot\left[\left(|\nabla u|^{p_0(x)-2}+|u|^{p_1(x)-2}\right)\nabla u\right]=h\left(x,u\right).$$

Since we want to establish the regularity of solution of the nonlinear differential equations related with mentioned pn-spaces, thus our aim is to understand the structure and nature of these spaces better, that allows to investigate the characteristics of solutions. For this reason, in this article we prove some embedding results, which indicate the relation of these spaces between Sobolev and Lebesgue spaces. We show that these spaces are not merely subsets of Lebesgue spaces also subsets of Sobolev spaces.

This paper is organized as follows. In the next section, we give the definitions of certain type of pn-spaces with variable and constant exponents ([20, 33] and for general definition see [34]) as well as recall some basic results for these spaces and variable exponent spaces. In Section 2, we prove embedding theorems for constant exponent pn-spaces and give certain results with examples in one dimensional case. In Section 3 firstly, we establish some integral inequalities with variable exponents, which are required to prove embedding theorems of variable exponent nonlinear spaces then investigate some attributes of variable exponent pn-spaces.

1 Preliminaries

In this section in the beginning we will give the general definition of spaces that are studied here in the functional case. Let X, Y be locally convex vector topological spaces, $B \subseteq Y$ be a Banach space and $g : D(g) \subseteq X \longrightarrow Y$. Let's introduce the following subset of X

$$\mathcal{M}_{gB} \equiv \{x \in X : g(x) \in B, \operatorname{Im} g \cap B \neq \varnothing\}.$$

Definition 1. *A subset $\mathcal{M} \subseteq X$ is called a pn-space (i.e. pseudonormed space) if S is a topological space and there is a function $[\cdot]_{\mathcal{M}} : \mathcal{M} \longrightarrow \mathbb{R}^1_+ \equiv [0, \infty)$ (which is called p-norm of $\mathcal{M}$) such that*

qn) $[x]_{\mathcal{M}} \geq 0, \forall x \in \mathcal{M}$ *and* $x = 0 \Longrightarrow [x]_{\mathcal{M}} = 0$;

pn) $[x_1]_{\mathcal{M}} \neq [x_2]_{\mathcal{M}} \Longrightarrow x_1 \neq x_2$ *for* $x_1, x_2 \in \mathcal{M}$, *and* $[x]_{\mathcal{M}} = 0 \Longrightarrow x = 0$.

The following conditions are often fulfilled in the spaces $\mathcal{M}_{gB}$.

N_1) There exist a convex function $\nu : \mathbb{R}^1 \longrightarrow \overline{\mathbb{R}^1_+}$ and number $K \in (0, \infty]$ such that $[\lambda x]_{\mathcal{M}} \leq \nu(\lambda)[x]_{\mathcal{M}}$ for any $x \in \mathcal{M}$ and $\lambda \in \mathbb{R}^1$, $|\lambda| < K$, moreover, $\lim\limits_{|\lambda| \longrightarrow \lambda_j} \frac{\nu(\lambda)}{|\lambda|} = c_j$, $j = 0, 1$, where $\lambda_0 = 0$, $\lambda_1 = K$ and $c_0 = c_1 = 1$ or $c_0 = 0$, $c_1 = \infty$, i.e. if $K = \infty$ then $\lambda x \in \mathcal{M}$ for any $x \in \mathcal{M}$ and $\lambda \in \mathbb{R}^1$.

Let $g : D(g) \subseteq X \longrightarrow Y$ be such a mapping that $\mathcal{M}_{gB} \neq \varnothing$ and the following conditions are fulfilled

G_1) $g : D(g) \longleftrightarrow \operatorname{Im} g$ is a bijection and $g(0) = 0$;

G_2) there is a function $\nu : \mathbb{R}^1 \longrightarrow \overline{\mathbb{R}^1_+}$ satsfying the condition N_1 such that

$$\|g(\lambda x)\|_B \leq \nu(\lambda)\|g(x)\|_B, \ \forall x \in \mathcal{M}_{gB}, \ \forall \lambda \in R^1.$$

If the mapping g satisfies the conditions G_1 and G_2 then $\mathcal{M}_{gB}$ is a pn-space with p-norm defined in the following way: there is a one-to-one function $\psi : \mathbb{R}^1_+ \longrightarrow \mathbb{R}^1_+$, $\psi(0) = 0$, $\psi, \psi^{-1} \in C^0$ such that $[x]_{\mathcal{M}_{gB}} \equiv \psi^{-1}(\|g(x)\|_B)$. In this case $\mathcal{M}_{gB}$ is a metric space with a metric: $d_{\mathcal{M}}(x_1; x_2) \equiv \|g(x_1) - g(x_2)\|_B$. Further, we consider just such type of pn-spaces.

Definition 2. *The pn-space $\mathcal{M}_{gB}$ is called weakly complete if $g(\mathcal{M}_{gB})$ is weakly closed in B. The pn-space $\mathcal{M}_{gB}$ is "reflexive" if each bounded weakly closed subset of $\mathcal{M}_{gB}$ is weakly compact in $\mathcal{M}_{gB}$.*

It is clear that if B is a reflexive Banach space and $\mathcal{M}_{gB}$ is a weakly complete pn-space, then $\mathcal{M}_{gB}$ is "reflexive". Moreover, if B is a separable Banach space, then $\mathcal{M}_{gB}$ is separable, also. For complementary properties see, e.g. [23, 33, 34].

We now remind certain integral inequalities and facts about the functional pn-spaces with constant exponent that are concerned in this paper (for general case see [21–25] and for functional case [21, 25, 27] etc).

Let $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ be a bounded domain with Lipschitz boundary $\partial\Omega$. Throughout the paper, we denote by $|\Omega|$ the Lebesgue measure of Ω.

Lemma 1. *Let $\alpha \geq 0$, $\beta \geq 1$, $|\Omega| < \infty$ and $i = \overline{1,n}$, then for all $u \in C(\bar{\Omega}) \cap C^1(\Omega)$ the inequality*

$$\int_{\Omega} |u|^{\alpha+\beta}\, dx \leq C_1 \int_{\Omega} |u|^{\alpha} |D_i u|^{\beta}\, dx + C_2 \int_{\partial\Omega} |u|^{\alpha+\beta}\, dx'$$

is satisfied. Here, $C_1 = C_1(\alpha, \beta, |\Omega|)$, $C_2 = C_2(|\Omega|) > 0$ are constants.

Lemma 2. *Assume that $\alpha, \alpha_1 \geq 0$, $\beta \geq 1$ and $\beta > \beta_1 > 0$, $\frac{\alpha_1}{\beta_1} \geq \frac{\alpha}{\beta}$, $\alpha_1 + \beta_1 \leq \alpha + \beta$ be satisfied. Then for $u \in C(\bar{\Omega}) \cap C^1(\Omega)$*

$$\int_{\Omega} |u|^{\alpha_1} |D_i u|^{\beta_1}\, dx \leq C_3 \int_{\Omega} |u|^{\alpha} |D_i u|^{\beta}\, dx + C_4 \int_{\partial\Omega} |u|^{\alpha+\beta}\, dx' + C_5$$

holds. Here, for $r = 3, 4, 5$, $C_r = C_r(\alpha, \beta, \alpha_1, \beta_1, |\Omega|) > 0$ are constants.

Lemma 3. *Let $\alpha \geq 0$, $\beta_0 + \beta_1 \geq 2$ and $\beta_1 \geq \beta_0 \geq 0$ be fulfilled. Then for all $u \in C^1(\bar{\Omega}) \cap C^2(\Omega)$*

$$\int_{\Omega} |u|^{\alpha} |D_i u|^{\beta_0+\beta_1}\, dx \leq C_6 \int_{\Omega} |u|^{\alpha+\beta_0} \left|D_i^2 u\right|^{\beta_1} dx + C_7 \int_{\partial\Omega} (|u|^{\alpha+\beta_0+\beta_1} + |u|^{\alpha+1} |D_i u|^{\beta_0+\beta_1-1}) dx'$$

holds. Here, for $j = 6, 7$, $C_j = C_j(\alpha, \beta, \beta_0) > 0$ are constants.

Definition 3. *Let $\alpha \geq 0$, $\beta \geq 1$, $\mathbf{k} = (k_1, \ldots, k_n)$ be multi-index and $|\mathbf{k}| = \sum_{i=1}^{n} k_i$, $m \in \mathbb{Z}^+$, $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ is bounded domain with sufficiently smooth boundary (at least Lipschitz boundary)*

$$S_{m,\alpha,\beta}(\Omega) := \left\{ u \in L^1(\Omega) : [u]^{\alpha+\beta}_{S_{m,\alpha,\beta}(\Omega)} \equiv \sum_{0 \leq |\mathbf{k}| \leq m} \left(\int_{\Omega} |u|^{\alpha} \left| D^{\mathbf{k}} u \right|^{\beta} dx \right) < \infty \right\}$$

and

$$\mathring{S}_{m,\alpha,\beta}(\Omega) := S_{m,\alpha,\beta}(\Omega) \cap \left\{ D^{\mathbf{k}} u \mid_{\partial\Omega} \equiv 0,\ 0 \leq |\mathbf{k}| \leq m_0 < m \right\}.$$

We state a proposition which can be easily proved by the help of Lemmas 1–3 and Definition 3.

Proposition 1. *Assume that $\alpha \geq 0$, $\beta \geq 1$, then we have the following equivalence*

$$\mathring{S}_{1,\alpha,\beta}(\Omega) := \left\{ u \in L^1(\Omega) : [u]^{\alpha+\beta}_{S_{1,\alpha,\beta}(\Omega)} \equiv \sum_{i=1}^{n} \left(\int_{\Omega} |u|^{\alpha} |D_i u|^{\beta}\, dx \right) < \infty \right\}$$

and [1]

$$\mathring{S}_{2,\alpha,\beta}(\Omega) := \left\{ u \in L^1(\Omega) : [u]^{\alpha+\beta}_{S_{2,\alpha,\beta}(\Omega)} \equiv \sum_{i=1}^{n} \left(\int_{\Omega} |u|^{\alpha} \left| D_i^2 u \right|^{\beta} dx \right) < \infty \right\}.$$

[1] $S_{1,\alpha,\beta}(\Omega)$ is a complete metric space with the following metric

$$d_{S_{1,\alpha,\beta}}(u, v) = \left\| |u|^{\frac{\alpha}{\beta}} u - |v|^{\frac{\alpha}{\beta}} v \right\|_{W^{1,\beta}(\Omega)}, \qquad \forall u, v \in S_{1,\alpha,\beta}(\Omega).$$

Theorem 1. *Let $\alpha \geq 0$, $\beta \geq 1$, then $g : \mathbb{R} \longrightarrow \mathbb{R}$, $g(t) := |t|^{\frac{\alpha}{\beta}} t$ is an one to one correspondence from $S_{1,\alpha,\beta}(\Omega)$ onto $W^{1,\beta}(\Omega)$.*

Now, we recall some basic definitions and results about variable exponent Lebesgue and Sobolev spaces [1,6,9,10,13].

Let Ω be a Lebesgue measurable subset of $\mathbb{R}^n$ such that $|\Omega| > 0$. The function set $M(\Omega)$ denotes the family of all measurable functions $p : \Omega \longrightarrow [1, \infty]$ and the set $M_0(\Omega)$ is defined by

$$M_0(\Omega) := \left\{ p \in M(\Omega) : \ 1 \leq p^- \leq p(x) \leq p^+ < \infty, \text{ a.e. } x \in \Omega \right\},$$

where $p^- := \operatorname*{ess\,inf}_{\Omega} |p(x)|$, $p^+ := \operatorname*{ess\,sup}_{\Omega} |p(x)|$.

For $p \in M(\Omega)$, $\Omega_\infty^p \equiv \Omega_\infty \equiv \{x \in \Omega | \ p(x) = \infty\}$. On the set of all functions on Ω, define the functional σ_p and $\|.\|_p$ by

$$\sigma_p(u) \equiv \int\limits_{\Omega \setminus \Omega_\infty} |u|^{p(x)} dx + \operatorname*{ess\,sup}_{\Omega_\infty} |u(x)|$$

and

$$\|u\|_{L^{p(x)}(\Omega)} \equiv \inf \left\{ \lambda > 0 : \ \sigma_p\left(\frac{u}{\lambda}\right) \leq 1 \right\}.$$

If $p \in L^\infty(\Omega)$, then $p \in M_0(\Omega)$, $\sigma_p(u) \equiv \int\limits_{\Omega} |u|^{p(x)} dx$ and the variable exponent Lebesgue space is defined as follows

$$L^{p(x)}(\Omega) := \left\{ u : u \text{ is a measurable real-valued function such that } \sigma_p(u) < \infty \right\}.$$

If $p^- > 1$, then the space $L^{p(x)}(\Omega)$ becomes a reflexive and separable Banach space with the norm $\|.\|_{L^{p(x)}(\Omega)}$, which is so-called Luxemburg norm.

If $0 < |\Omega| < \infty$, and $p_1, \ p_2 \in M(\Omega)$, then the continuous embedding $L^{p_1(x)}(\Omega) \subset L^{p_2(x)}(\Omega)$ exists $\iff p_2(x) \leq p_1(x)$ for a.e. $x \in \Omega$.

For $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$, where $p, q \in M_0(\Omega)$ and $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$, the following inequalities be satisfied

$$\int\limits_{\Omega} |uv| \, dx \leq 2 \|u\|_{L^{p(x)}(\Omega)} \|v\|_{L^{q(x)}(\Omega)},$$

and

$$\min \left\{ \|u\|_{L^{p(x)}(\Omega)}^{p^-}, \|u\|_{L^{p(x)}(\Omega)}^{p^+} \right\} \leq \sigma_p(u) \leq \max \left\{ \|u\|_{L^{p(x)}(\Omega)}^{p^-}, \|u\|_{L^{p(x)}(\Omega)}^{p^+} \right\}.$$

Lemma 4. *Let u, $u_k \in L^{p(x)}(\Omega)$, $k = 1, 2, \ldots$. Then the following statements are equivalent to each other:*

1. $\lim_{k \to \infty} \|u_k - u\|_{L^{p(x)}(\Omega)} = 0$;
2. $\lim_{k \to \infty} \sigma_p(u_k - u) = 0$;
3. *u_k converges to u in Ω in measure and* $\lim_{k \to \infty} \sigma_p(u_k) = \sigma_p(u)$.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $p \in L^\infty(\Omega)$, then variable exponent Sobolev space is defined by

$$W^{1,\,p(x)}(\Omega) := \left\{u \in L^{p(x)}(\Omega) : \ |\nabla u| \in L^{p(x)}(\Omega)\right\}$$

and this space is a separable Banach space with the norm

$$\|u\|_{W^{1,\,p(x)}(\Omega)} \equiv \|u\|_{L^{p(x)}(\Omega)} + \|\nabla u\|_{L^{p(x)}(\Omega)}.$$

In the following discussion, we give the definition of generalized nonlinear spaces (functional pn-spaces with variable exponent) and features of them that indicate their relation with known spaces. These classes are nonlinear spaces, which are generalization of nonlinear spaces with constant exponent studied in [24] (see also references therein). We also specify that some of the results and its proofs can be found in [19, 20].

Definition 4. *Let $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ be a bounded domain with Lipschitz boundary and γ, β $\in M_0(\Omega)$. We introduce $S_{1,\gamma(x),\beta(x)}(\Omega)$, the class of functions $u : \Omega \to \mathbb{R}$, and the functional $[\,\cdot\,]_{S_{\gamma,\beta}} : S_{1,\gamma(x),\beta(x)}(\Omega) \longrightarrow \mathbb{R}_+$ as follows*

$$S_{1,\gamma(x),\beta(x)}(\Omega) := \left\{u \in L^1(\Omega) : \int_\Omega |u|^{\gamma(x)+\beta(x)}\,dx + \sum_{i=1}^n \int_\Omega |u|^{\gamma(x)}\,|D_i u|^{\beta(x)}\,dx < \infty\right\},$$

$$[u]_{S_{\gamma,\beta}} := \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{u}{\lambda}\right|^{\gamma(x)+\beta(x)} dx + \sum_{i=1}^n \left(\int_\Omega \left|\frac{|u|^{\frac{\gamma(x)}{\beta(x)}}\,D_i u}{\lambda^{\frac{\gamma(x)}{\beta(x)}+1}}\right|^{\beta(x)}\right) dx \leq 1\right\}.$$

$[\,\cdot\,]_{S_{\gamma,\beta}}$ defines a pseudo-norm on $S_{1,\gamma(x),\beta(x)}(\Omega)$, actually it can be readily verified that $[\,\cdot\,]_{S_{\gamma,\beta}}$ fulfills all axioms of pseudo-norm (see [33, 34]), i.e. $[u]_{S_{\gamma,\beta}} \geq 0$, $u = 0 \Rightarrow [u]_{S_{\gamma,\beta}} = 0$, $[u]_{S_{\gamma,\beta}} \neq [v]_{S_{\gamma,\beta}} \Rightarrow u \neq v$ and $[u]_{S_{\gamma,\beta}} = 0 \Rightarrow u = 0$.

Let $S_{1,\gamma(x),\beta(x)}(\Omega)$ be the space given in the Definition 4 and $\theta(x) \in M_0(\Omega)$, we denote $S_{1,\gamma(x),\beta(x),\theta(x)}(\Omega)$, the class of functions $u : \Omega \to \mathbb{R}$, by the following intersection

$$S_{1,\gamma(x),\beta(x),\theta(x)}(\Omega) := S_{1,\gamma(x),\beta(x)}(\Omega) \cap L^{\theta(x)}(\Omega)$$

with the pseudo-norm

$$[u]_{S_{\gamma,\beta,\theta}} := [u]_{S_{\gamma,\beta}} + \|u\|_{L^{\theta(x)}(\Omega)}, \ \ \forall u \in S_{1,\gamma(x),\beta(x),\theta(x)}(\Omega).$$

Proposition 2. *If γ, β, $\theta \in M_0(\Omega)$ and $\theta(x) \geq \gamma(x) + \beta(x) + \varepsilon_0$ a.e. $x \in \Omega$ for some $\varepsilon_0 > 0$, then we have the following equivalence*

$$S_{1,\gamma(x),\beta(x),\theta(x)}(\Omega) \equiv \left\{u \in L^1(\Omega) : R^{\gamma,\beta,\theta}(u) < \infty\right\},$$

where $R^{\gamma,\beta,\theta}(u) := \int_\Omega |u|^{\theta(x)}\,dx + \sum_{i=1}^n \int_\Omega |u|^{\gamma(x)}\,|D_i u|^{\beta(x)}\,dx$, and the pseudo-norm on this space is

$$[u]_{S_{\gamma,\beta,\theta}} \equiv \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{u}{\lambda}\right|^{\theta(x)} dx + \sum_{i=1}^n \left(\int_\Omega \left|\frac{|u|^{\frac{\gamma(x)}{\beta(x)}}\,D_i u}{\lambda^{\frac{\gamma(x)}{\beta(x)}+1}}\right|^{\beta(x)}\right) dx \leq 1\right\}.$$

Lemma 5. *Assume that conditions of Proposition 2 are fulfilled. Let* $u \in S_{1,\gamma(x),\beta(x),\theta(x)}(\Omega)$ *and* $\lambda_u := [u]_{S_{\gamma,\beta,\theta}}$*, then the following inequality*

$$\max\left\{\lambda_u^{\gamma^-+\beta^-}, \lambda_u^{\theta^+}\right\} \geq R^{\gamma,\beta,\theta}(u) \geq \min\left\{\lambda_u^{\gamma^-+\beta^-}, \lambda_u^{\theta^+}\right\}$$

holds.

Theorem 2. *Suppose that conditions of Proposition 2 are satisfied and let* $p \in M_0(\Omega)$, $p(x) \geq \theta(x)$ *a.e.* $x \in \Omega$*. Then, the embedding*

$$W^{1,\,p(x)}(\Omega) \subset S_{1,\gamma(x),\beta(x),\theta(x)}(\Omega)$$

holds.

Definition 5. *Let* $\eta \in M_0(\Omega)$, *we introduce* $L^{1,\,\eta(x)}(\Omega)$ *the class*[2] *of functions* $u : \Omega \to \mathbb{R}$

$$L^{1,\,\eta(x)}(\Omega) \equiv \left\{u \in L^1(\Omega) :\ D_i u \in L^{\eta(x)}(\Omega),\ i = \overline{1,n}\right\}.$$

Theorem 3. *Let* γ, $\beta \in M_0(\Omega) \cap C^1(\bar{\Omega})$ *and* $L^{1,\,\beta(x)}(\Omega)$ *be the space given in Definition 5. Then the function* $\varphi : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$, $\varphi(x,t) := |t|^{\frac{\gamma(x)}{\beta(x)}} t$ *is a bijective mapping between* $S_{1,\gamma(x),\beta(x),\theta(x)}(\Omega)$ *and* $L^{1,\,\beta(x)}(\Omega) \cap L^{\psi(x)}(\Omega)$*, where* $\psi(x) := \frac{\theta(x)\beta(x)}{\gamma(x)+\beta(x)}$.

Theorem 4. *Suppose that conditions of Theorem 3 are satisfied. Let* $p \in M_0(\Omega)$, *additionally* $1 \leq \beta^- \leq \beta(x) < n$, $x \in \Omega$ *holds and for* $\varepsilon > 0$*, the inequality*

$$p(x) + \varepsilon < \frac{n(\gamma(x)+\beta(x))}{n-\beta(x)},\ x \in \Omega,$$

is satisfied. Then the following compact embedding

$$S_{1,\gamma(x),\beta(x),\theta(x)}(\Omega) \hookrightarrow L^{p(x)}(\Omega)$$

exists.

2 Some relations between constant exponent pn-spaces and Sobolev spaces

In this section, we give some embedding results for constant exponent pn-spaces with proofs.

Theorem 5. *Let* $\alpha \geq 0$, $\beta \geq 1$*. Then for all* p *satisfying the following conditions*

(i) if $\beta = n$*, then* $p > \beta$,

(ii) if $\beta > n$*, then* $p \geq \beta$,

(iii) if $\beta < n$*, then* $p \geq \frac{n(\alpha+\beta)}{\alpha+n}$,

the embedding

$$W_0^{1,p}(\Omega) \subset \mathring{S}_{1,\alpha,\beta}(\Omega) \tag{2}$$

holds.

[2] This space is not Banach one unlike to the space $W^{1,\,\eta(x)}(\Omega)$ [6].

Proof. The cases (i) and (ii) are evident as by virtue of the Sobolev imbedding theorems occurs the inclusion

$$W_0^{1,p}(\Omega) \subset C(\bar{\Omega}).$$

For the last case (iii), if $\beta < n$ and $p > n$, then the proof is same with the proofs of the cases (i) and (ii).

On the other side, let $\beta < n$ and $p \in \left[\frac{n(\alpha+\beta)}{\alpha+n}, n\right)$, by Sobolev imbedding theorems we have

$$W_0^{1,p}(\Omega) \subset L^{\tilde{q}}(\Omega) \tag{3}$$

for all $\tilde{q} \in \left[1, \frac{np}{n-p}\right]$. Hence, for $u \in W_0^{1,p}(\Omega)$ we have the following estimate by Young's inequality

$$\int\limits_{\Omega} |u|^{\alpha} |D_i u|^{\beta} \, dx \le \left(\frac{p-\beta}{p}\right) \int\limits_{\Omega} |u|^{\frac{\alpha p}{p-\beta}} \, dx + \left(\frac{p}{\beta}\right) \int\limits_{\Omega} |D_i u|^{p} \, dx. \tag{4}$$

We deduce from the equation $\frac{\alpha p}{p-\beta} - \frac{np}{n-p} = \frac{p[n(\alpha+\beta)-p(\alpha+n)]}{(p-\beta)(n-p)}$ and $p \in \left[\frac{n(\alpha+\beta)}{\alpha+n}, n\right)$ that

$$\frac{\alpha p}{p-\beta} \le \frac{np}{n-p}.$$

Thus, by (3) and (4) we arrive at

$$[u]_{\mathring{S}_{1,\alpha,\beta}}^{\alpha+\beta} = \int\limits_{\Omega} |u|^{\alpha} |D_i u|^{\beta} \, dx \le \tilde{C} \, \|u\|_{W_0^{1,p}(\Omega)}^{\frac{\alpha p}{p-\beta}} + \tilde{C}_1 \, \|u\|_{W_0^{1,p}(\Omega)}^{p},$$

which implies $[u]_{\mathring{S}_{1,\alpha,\beta}}^{\alpha+\beta} \le \tilde{C}_2 \, \|u\|_{W_0^{1,p}(\Omega)}^{p} + C_3$.

To complete the proof if $p = n > \beta$, by employing the embedding $W_0^{1,p}(\Omega) \subset L^r(\Omega)$, $1 \le r < \infty$, one can obtain the desired result by the help of above approach. ☐

Remark 1. *Under the conditions of Theorem 5, if $p \ge \alpha + \beta$ is satisfied, then we have the imbedding* (2) *independently from dimension of Ω.*

Actually for $u \in W_0^{1,p}(\Omega)$, we deduce from Lemma 2 that

$$\int\limits_{\Omega} |u|^{\alpha} |D_i u|^{\beta} \, dx \le C \int\limits_{\Omega} |D_i u|^{p} \, dx + C_1,$$

which yields $[u]_{\mathring{S}_{1,\alpha,\beta}}^{\alpha+\beta} \le C \, \|u\|_{W_0^{1,p}(\Omega)}^{p} + C_1$.

Theorem 6. *Suppose that $\beta > \alpha \ge 0$, $\beta \ge 2$. Then for all p satisfying the following conditions*

(i) if $\alpha + \beta = n$, then $1 \le p < 2\beta$,

(ii) if $\alpha + \beta > n$, then $1 \le p \le 2\beta$,

(iii) if $\alpha + \beta < n$, then $1 \le p \le \frac{2n\beta(\alpha+\beta)}{2n\beta-(\alpha+\beta)(\beta-\alpha)}$,

the embedding

$$\mathring{S}_{2,\alpha,\beta}(\Omega) \subset W_0^{1,p}(\Omega) \tag{5}$$

holds.

Proof. Considering these conditions, by Lemma 3 when $1 \le p \le \alpha+\beta$ following inequality

$$\int_{\Omega} |D_i u|^p \, dx \le C \int_{\Omega} |u|^{\alpha} \left|D_i^2 u\right|^{\beta} dx + C_1$$

holds independently of the dimension n, that yields the imbedding (5). So, if $1 \le p \le 2$, then $1 \le p \le \alpha+\beta$, which concludes the proof.

First, we prove (5) in line with conditions of (i). Let $\alpha+\beta = n$ and $p > 2$ (from now on we assume $p > 2$).

For $u \in \mathring{S}_{2,\alpha,\beta}(\Omega)$, by Lemma 3 we have the following estimate

$$\int_{\Omega} |D_i u|^{\alpha+\beta} \, dx \le C \int_{\Omega} |u|^{\alpha} \left|D_i^2 u\right|^{\beta} dx. \tag{6}$$

On the other hand, from Sobolev imbedding theorems

$$W_0^{1,\alpha+\beta}(\Omega) \subset L^q(\Omega) \;\; \forall q,\; q \in [1,\infty). \tag{7}$$

Hence, from (6) and (7) for all q satisfying $1 \le q < \infty$ we get

$$\|u\|_q \le \tilde{C} \left(\sum_{i=1}^{n} \|D_i u\|_{\alpha+\beta}^{\alpha+\beta} \right)^{\frac{1}{\alpha+\beta}} \le \tilde{C}_0 \left(\sum_{i=1}^{n} \left[\int_{\Omega} |u|^{\alpha} \left|D_i^2 u\right|^{\beta} dx \right] \right)^{\frac{1}{\alpha+\beta}} = \tilde{C}_0 \, [u]_{\mathring{S}_{2,\alpha,\beta}}. \tag{8}$$

Therefore, for all $u \in \mathring{S}_{2,\alpha,\beta}(\Omega)$ and $i = \overline{1,n}$

$$\begin{aligned} \int_{\Omega} |D_i u|^p \, dx &= \int_{\Omega} \left(D_i u \, |D_i u|^{p-2} \right) D_i u dx = (p-1) \int_{\Omega} u D_i^2 u \, |D_i u|^{p-2} \, dx \\ &\le (p-1) \int_{\Omega} |u|^{\frac{\beta-\alpha}{\beta}} \, |u|^{\frac{\alpha}{\beta}} \left|D_i^2 u\right| |D_i u|^{p-2} \, dx. \end{aligned} \tag{9}$$

Employing Hölder's inequality in (9) with exponents $\left(\frac{p\beta}{2\beta-p}, \beta, \frac{p}{p-2}\right)$, we obtain

$$\begin{aligned} \int_{\Omega} |D_i u|^p \, dx &\le C \left(\int_{\Omega} |u|^{\frac{p(\beta-\alpha)}{2\beta-p}} \, dx \right)^{\frac{2\beta-p}{p\beta}} \left(\int_{\Omega} |u|^{\alpha} \left|D_i^2 u\right|^{\beta} dx \right)^{\frac{1}{\beta}} \left(\int_{\Omega} |D_i u|^p \, dx \right)^{\frac{p-2}{p}} \\ &= C \, \|u\|_{\frac{p(\beta-\alpha)}{2\beta-p}}^{\frac{\beta-\alpha}{\beta}} \, [u]_{\mathring{S}_{2,\alpha,\beta}}^{\frac{\alpha+\beta}{\beta}} \, \|D_i u\|_p^{p-2}. \end{aligned} \tag{10}$$

Estimating (10) by using (8) we get

$$\int_{\Omega} |D_i u|^p \, dx \le \tilde{C} \, [u]_{\mathring{S}_{2,\alpha,\beta}}^{\frac{\beta-\alpha}{\beta}} \, [u]_{\mathring{S}_{2,\alpha,\beta}}^{\frac{\alpha+\beta}{\beta}} \, \|D_i u\|_p^{p-2} = \tilde{C} \, [u]_{\mathring{S}_{2,\alpha,\beta}}^{2} \, \|D_i u\|_p^{p-2}. \tag{11}$$

By using Young's inequality in (11), we arrive at

$$\|D_i u\|_p^p \le \tilde{C}(\varepsilon) \, [u]_{\mathring{S}_{2,\alpha,\beta}}^{p} + \tilde{C}\varepsilon \, \|D_i u\|_p^p,$$

choosing ε such that $\tilde{C}\varepsilon < 1$, then we acquire

$$\|D_i u\|_p \le \tilde{C}\,[u]_{\mathring{S}_{2,\alpha,\beta}} < \infty,$$

which completes the proof for the case (i).

Assume that (ii) holds, i.e. $\alpha+\beta > n$ and $2 < p \le 2\beta$. Then

$$W^{1,\alpha+\beta}(\Omega) \subset C(\bar{\Omega}),$$

by (6) and (8), we obtain

$$\|u\|_{C(\bar{\Omega})} \le \tilde{C}\,[u]_{\mathring{S}_{2,\alpha,\beta}}. \tag{12}$$

For all $u \in \mathring{S}_{2,\alpha,\beta}(\Omega)$ from (9) one concludes

$$\begin{aligned}
\|D_i u\|_p^p &\le (p-1)\int_\Omega |u|^{\frac{\beta-\alpha}{\beta}}\,|u|^{\frac{\alpha}{\beta}}\left|D_i^2 u\right| |D_i u|^{p-2}\,dx \\
&\le (p-1)C(\varepsilon)\int_\Omega |u|^{\beta-\alpha}\,|u|^{\alpha}\left|D_i^2 u\right|^{\beta} dx + (p-1)\varepsilon\int_\Omega |D_i u|^{\frac{\beta(p-2)}{\beta-1}}\,dx \\
&\le (p-1)C(\varepsilon)\,\|u\|_{C(\bar{\Omega})}^{\beta-\alpha}\int_\Omega |u|^{\alpha}\left|D_i^2 u\right|^{\beta} dx + (p-1)\varepsilon\,\|D_i u\|_{\frac{\beta(p-2)}{\beta-1}}^{\frac{\beta(p-2)}{\beta-1}}.
\end{aligned}$$

By using (12) and $\frac{\beta(p-2)}{\beta-1} - p = \frac{p-2\beta}{\beta-1}$ with $p \le 2\beta$ to estimate $\|u\|_{C(\bar{\Omega})}^{\beta-\alpha}$ and $\|D_i u\|_{\frac{\beta(p-2)}{\beta-1}}^{\frac{\beta(p-2)}{\beta-1}}$ respectively, we arrive at

$$\begin{aligned}
\|D_i u\|_p^p &\le C(\varepsilon)\,(p-1)\,[u]_{\mathring{S}_{2,\alpha,\beta}}^{\beta-\alpha}\,[u]_{\mathring{S}_{2,\alpha,\beta}}^{\alpha+\beta} + (p-1)\varepsilon\tilde{C}C\,\|D_i u\|_p^p + (p-1)\varepsilon C_1 \\
&= C(\varepsilon)\,[u]_{\mathring{S}_{2,\alpha,\beta}}^{2\beta} + \varepsilon\tilde{C}C\,\|D_i u\|_p^p + \varepsilon C_1,
\end{aligned}$$

which implies

$$\|D_i u\|_p^p \le \tilde{C}\,[u]_{\mathring{S}_{2,\alpha,\beta}}^{2\beta} + C_1,$$

that ends the proof.

For the last case (iii), let $\alpha+\beta < n$ and $1 \le p \le \frac{2n\beta(\alpha+\beta)}{2n\beta-(\alpha+\beta)(\beta-\alpha)}$. From Sobolev imbedding theorems

$$W^{1,\alpha+\beta}(\Omega) \subset L^{\tilde{q}}(\Omega) \quad \forall\tilde{q},\ \tilde{q} \in \left[1, \frac{n(\alpha+\beta)}{n-(\alpha+\beta)}\right]. \tag{13}$$

By (6) and (13), we attain

$$\|u\|_{\tilde{q}} \le C\,[u]_{\mathring{S}_{2,\alpha,\beta}}. \tag{14}$$

For all $u \in \mathring{S}_{2,\alpha,\beta}(\Omega)$, we deduce from the inequality $p \le \frac{2n\beta(\alpha+\beta)}{2n\beta-(\alpha+\beta)(\beta-\alpha)} < 2\beta$ that

$$\|D_i u\|_p^p \le C\,\|u\|_{\frac{p(\beta-\alpha)}{2\beta-p}}^{\frac{\beta-\alpha}{\beta}}\,[u]_{\mathring{S}_{2,\alpha,\beta}}^{\frac{\alpha+\beta}{\beta}}\,\|D_i u\|_p^{p-2}. \tag{15}$$

If we take the inequality $\frac{p(\beta-\alpha)}{2\beta-p} \le \frac{n(\alpha+\beta)}{n-(\alpha+\beta)}$ into account and estimate $\|u\|_{\frac{p(\beta-\alpha)}{2\beta-p}}$ in (15) by (14) we obtain

$$\|D_i u\|_p^p \le \tilde{C}\,[u]_{\mathring{S}_{2,\alpha,\beta}}^{\frac{\beta-\alpha}{\beta}}\,[u]_{\mathring{S}_{2,\alpha,\beta}}^{\frac{\alpha+\beta}{\beta}}\,\|D_i u\|_p^{p-2} = \tilde{C}\,[u]_{\mathring{S}_{2,\alpha,\beta}}^{2}\,\|D_i u\|_p^{p-2}. \tag{16}$$

Applying Young's inequality in (16) we attain

$$\|D_i u\|_p^p \leq \tilde{C}(\varepsilon)\,[u]^p_{\mathring{S}_{2,\alpha,\beta}} + \tilde{C}\varepsilon\,\|D_i u\|_p^p,$$

that yields

$$\|D_i u\|_p \leq \tilde{C}\,[u]_{\mathring{S}_{2,\alpha,\beta}},$$

so, the proof is complete. ☐

We now turn our attention to some examples and results for one dimensional case.

Definition 6. *Let $\alpha > \beta - 1 \geq 0$ we define the following function space*

$$\tilde{S}_{2,\alpha,\beta}(a,b) := \Big\{ u \in L^1(a,b) : [u]^{\alpha+\beta}_{\tilde{S}_{1,\alpha,\beta}(a,b)} = \int_a^b |u|^{\alpha+\beta}\,dx + \int_a^b |u|^{\alpha-\beta}\,|Du|^{2\beta}\,dx + \int_a^b |u|^{\alpha}\left|D^2 u\right|^{\beta} dx < \infty \Big\}.$$

The proofs of the following lemmas can be attained readily, thus we skip the proofs for the sake of brevity.

Lemma 6. *Let $\tilde{S}_{2,\alpha,\beta}(a,b)$ be the space given in Definition 6, then the imbedding*

$$\tilde{S}_{2,\alpha,\beta}(a,b) \subset S_{1,\alpha,\beta}(a,b)$$

holds.

Lemma 7. *Let $\alpha > \beta - 1 > 0$ and $g(t) \equiv |t|^{\frac{\alpha}{\beta}}\, t$ for any $t \in \mathbb{R}$. Then following assertions are true*

1) *if $u \in \tilde{S}_{2,\alpha,\beta}(a,b)$, then $g(u) \in W^{2,\beta}(a,b)$;*
2) *for a function $u \in L^1(a,b)$, if $g(u) \equiv v \in W^{2,\beta}(a,b)$, then $u \in \tilde{S}_{2,\alpha,\beta}(a,b)$.*

Consequently, we can define the space $\tilde{S}_{2,\alpha,\beta}(a,b)$ in the following way by virtue of the general definition of the nonlinear spaces.

Definition 7. *Let $g : \mathbb{R} \to \mathbb{R}$, $g(t) = |t|^{\frac{\alpha}{\beta}}\, t$ and $\alpha > \beta - 1 > 0$, then $\tilde{S}_{2,\alpha,\beta}(a,b)$ has the following representation*

$$\tilde{S}_{2,\alpha,\beta}(a,b) = \left\{ u \in L^1(a,b) : [u]^{\alpha+\beta}_{S_{gW^{2,\beta}}} \equiv \sum_{0\leq s\leq 2} \|D^s g(u)\|_\beta^\beta < \infty \right\} \equiv S_{gW^{2,\beta}}(a,b).$$

Remark 2. *The following equivalences are true*

$$\tilde{S}_{2,\alpha,\beta}(a,b) \cap \{u : u \mid_{\partial\Omega} = 0\} \equiv \mathring{S}_{2,\alpha,\beta}(a,b)$$

and

$$\sum_{0\leq s\leq k} \|D^s g(u)\|_\beta^\beta \equiv \sum_{0\leq s\leq k} \left\|g^{-1}\left(D^s g(u)\right)\right\|_{\alpha+\beta}^{\alpha+\beta}$$

for $k = 0, 1$, but for $k = 2$

$$\left\|g'(u) D^2 u\right\|_\beta^\beta \equiv \left\|g^{-1}\left(g'(u) D^2 u\right)\right\|_{\alpha+\beta}^{\alpha+\beta}$$

and

$$\left\|g''(u)\,(Du)^2\right\|_\beta^\beta \equiv \left\|g^{-1}\left(g''(u)\,(Du)^2\right)\right\|_{\alpha+\beta}^{\alpha+\beta}.$$

The following example shows the nonlinear structure of the pn-spaces.

Example 1. *Suppose that* $\beta > 1$. *Then* $S_{1,1,\beta}(0,1)$ *is a nonlinear space.*

Let $\tau \in \left(\frac{\beta-1}{\beta+1}, \frac{\beta-1}{\beta}\right]$ and define the functions

$$u_0(x) := x^\tau \text{ and } u_1(x) := \theta,\, x \in (0,1), \ \left(\theta \in \mathbb{R}^+ \text{ is a constant}\right).$$

It is easy to show that $u_0,\ u_1 \in S_{1,1,\beta}(0,1)$ by the definition of $S_{1,1,\beta}(0,1)$. Besides $u(x) := u_0(x) + u_1(x) = x^\tau + \theta \notin S_{1,1,\beta}(0,1)$.

$$[u]^{\beta+1}_{S_{1,1,\beta}(0,1)} = \int_0^1 |u|^{\beta+1}\,dx + \int_0^1 |u|\,|Du|^\beta dx = \int_0^1 (x^\tau+\theta)^{\beta+1}\,dx + \tau^\beta \int_0^1 (x^\tau+\theta)\,x^{\beta(\tau-1)}dx$$
$$= \int_0^1 (x^\tau+\theta)^{\beta+1}\,dx + \tau^\beta \int_0^1 \left(x^{\tau(\beta+1)-\beta} + \theta x^{\beta(\tau-1)}\right) dx.$$

Since $\beta(\tau-1) \le -1$ so, the right and side of the above equation is divergent which implies $u \notin S_{1,1,\beta}(0,1)$.

3 VARIABLE EXPONENT NONLINEAR SPACES AND EMBEDDING THEOREMS

In this section, we present certain new results with detailed proofs for variable exponent pn-spaces mentioned in Section 1. First, we derive integral inequalities (see, also [20]) to understand the structure of these spaces. Afterwards, we prove some lemmas and theorems on continuous embeddings of these spaces and on topology of them. Throughout this section, we assume that $\Omega \subset \mathbb{R}^n$ $(n \ge 2)$ is a bounded domain with Lipschitz boundary.

Lemma 8. *Let* $\alpha, \beta \in M_0(\Omega)$ *and* $\alpha(x) \ge \beta(x)$ *a.e.* $x \in \Omega$. *Then the inequality*

$$\int_\Omega |u|^{\beta(x)}\,dx \le \int_\Omega |u|^{\alpha(x)}\,dx + |\Omega|, \quad \forall u \in L^{\alpha(x)}(\Omega) \tag{17}$$

holds.

Proof. Let $\Omega_1 := \{x \in \Omega : \alpha(x) = \beta(x)\}$ and $\Omega_2 := \Omega \setminus \Omega_1$. Hence

$$\int_\Omega |u|^{\beta(x)}\,dx = \int_{\Omega_1} |u|^{\alpha(x)}\,dx + \int_{\Omega_2} |u|^{\beta(x)}\,dx.$$

Estimating the second integral on the right member of the above equation by utilizing Young inequality ($\alpha(x) > \beta(x)$ on Ω_2), we achieve that

$$\int_\Omega |u|^{\beta(x)}\,dx \le \int_{\Omega_1} |u|^{\alpha(x)}\,dx + \int_{\Omega_2} \left(\frac{\beta(x)}{\alpha(x)}\right) |u|^{\alpha(x)}\,dx + \int_{\Omega_2} \left(\frac{\alpha(x)-\beta(x)}{\alpha(x)}\right) dx,$$

since $\frac{\beta(x)}{\alpha(x)} < 1$ and $\frac{\alpha(x)-\beta(x)}{\alpha(x)} < 1$, for $x \in \Omega_2$ we deduce from the last inequality that

$$\int_\Omega |u|^{\beta(x)}\,dx \le \int_{\Omega_1} |u|^{\alpha(x)}\,dx + \int_{\Omega_2} |u|^{\alpha(x)}\,dx + |\Omega| = \int_\Omega |u|^{\alpha(x)}\,dx + |\Omega|.$$

On the other side if $\alpha(x) = \beta(x)$ a.e. $x \in \Omega$, then (17) is clear. □

Lemma 9. *Assume that $\zeta \in M_0(\Omega)$ and $\beta \geq 1, \varepsilon > 0$. Then for every $u \in L^{\zeta(x)+\varepsilon}(\Omega)$*

$$\int_{\Omega} |u|^{\zeta(x)} |\ln |u||^{\beta} dx \leq N_1 \int_{\Omega} |u|^{\zeta(x)+\varepsilon} dx + N_2$$

is satisfied. Here $N_1 \equiv N_1(\varepsilon, \beta) > 0$ and $N_2 \equiv N_2(\varepsilon, \beta, |\Omega|) > 0$ are constants.

Proof. Let us consider the function $f(t) = |t|^{\varepsilon} - \ln |t|$ for $t \in \mathbb{R} - \{0\}$. Since f is an even function it is sufficient to investigate only $f(t) = t^{\varepsilon} - \ln t$, $t > 0$. It can be readily shown that this function is decreasing on $\left(0, \frac{1}{\sqrt[\varepsilon]{\varepsilon}}\right]$ and increasing on the interval $\left[\frac{1}{\sqrt[\varepsilon]{\varepsilon}}, \infty\right)$. Also $f \nearrow \infty$ when $x \searrow 0$ and $x \nearrow \infty$ and $f\left(\frac{1}{\sqrt[\varepsilon]{\varepsilon}}\right) = \frac{1}{\varepsilon}(1 + \ln \varepsilon)$. Here we have two situations: (i) if $\varepsilon \in \left(\frac{1}{e}, \infty\right)$, then $f\left(\frac{1}{\sqrt[\varepsilon]{\varepsilon}}\right) > 0$; (ii) if $\varepsilon \in \left(0, \frac{1}{e}\right]$, then $f\left(\frac{1}{\sqrt[\varepsilon]{\varepsilon}}\right) \leq 0$. For the first case (i) $\forall t \in (0, \infty)$, $f(t) > 0$ or equivalently $\ln t < t^{\varepsilon}$. For the case (ii), the function f has two zeros, say $m_1 > 0$ and $m_2 > 0$, and for $t \in \mathbb{R}^+ - (m_1, m_2)$ it is obvious that $\ln t < t^{\varepsilon}$. For $t \in [m_1, m_2]$, $\exists N_0 > 1$ $\left(N_0 \equiv N_0\left(\frac{1}{\sqrt[\varepsilon]{\varepsilon}}\right)\right)$ such that $\ln t < N_0 t^{\varepsilon}$. Hence, the inequality $\ln t \leq N_0 t^{\varepsilon}$ will be satisfied on $(0, \infty)$. As a result, from the cases (i) and (ii) for arbitrary $\varepsilon > 0$ and $t \in \mathbb{R} - \{0\}$, we have the inequality

$$\ln |t| \leq N_0(\varepsilon) |t|^{\varepsilon},$$

that implies on the set $\{x \in \Omega : |u(x)| \geq 1\}$ the inequality $|u|^{\zeta(x)} |\ln |u||^{\beta} \leq N_0(\varepsilon, \beta) |u|^{\zeta(x)+\varepsilon}$ be fulfilled. Moreover, from $\lim\limits_{t \to 0^+} t^{\varepsilon} |\ln t|^{\beta} = 0$ and for every fixed $x_0 \in \Omega$, $\lim\limits_{t \to 0^+} \frac{|t|^{\zeta(x_0)} |\ln |t||^{\beta}}{t^{\zeta(x_0)+\varepsilon}+1} = 0$, we arrive at the inequality $|u|^{\zeta(x)-1} |u| |\ln |u||^{\beta} \leq \tilde{N}_0 \left(|u|^{\zeta(x)+\varepsilon} + 1\right)$ is fulfilled on the set $\{x \in \Omega : |u(x)| < 1\}$ for some $\tilde{N}_0 = \tilde{N}_0(\varepsilon, \beta) > 0$. So, the proof is complete by the combination of these inequalities. □

Lemma 10. *Let $\tilde{\varepsilon} > 0$ and $\beta_1 : \Omega \to [\tilde{\varepsilon}, \infty)$ be a measurable function, which satisfies $\tilde{\varepsilon} \leq \beta_1^- \leq \beta_1(x) \leq \beta_1^+ < \infty$ and $\xi, \beta \in M_0(\Omega)$, then the inequality*

$$\int_{\Omega} |u|^{\xi(x)} |\ln |u||^{\beta(x)} dx \leq C_1 \int_{\Omega} |u|^{\xi(x)+\beta_1(x)} dx + C_2, \ \forall u \in L^{\xi(x)+\beta_1(x)}(\Omega) \tag{18}$$

holds. Here $C_1 \equiv C_1(\tilde{\varepsilon}, \beta^+) > 0$ and $C_2 \equiv C_2(\tilde{\varepsilon}, \beta^+, |\Omega|) > 0$ are constants.

Proof. For arbitrary $\gamma \in (0,1)$, $\frac{\beta^+ + \gamma}{\beta(x)} > 1$, by utilizing the Young's inequality with this exponent to $|\ln |u||^{\beta(x)}$ we obtain the following inequality$|\ln |u||^{\beta(x)} \leq |\ln |u||^{\beta^+ + \gamma} + 1$, by multiplying each side of this inequality with $|u|^{\xi(x)}$, we get

$$|u|^{\xi(x)} |\ln |u||^{\beta(x)} \leq |u|^{\xi(x)} |\ln |u||^{\beta^+ + \gamma} + |u|^{\xi(x)}, \ \ x \in \Omega.$$

Thus, integrating both sides over Ω,

$$\int_{\Omega} |u|^{\xi(x)} |\ln |u||^{\beta(x)} dx \leq \int_{\Omega} |u|^{\xi(x)} |\ln |u||^{\beta^+ + \gamma} dx + \int_{\Omega} |u|^{\xi(x)} dx$$

is established. For $\varepsilon < \tilde{\varepsilon}$, estimating the first integral on the right side of the last inequality by Lemma 9, we acquire

$$\int_{\Omega} |u|^{\xi(x)} |\ln |u||^{\beta(x)} dx \leq C_3 \int_{\Omega} |u|^{\xi(x)+\varepsilon} dx + C_4 + \int_{\Omega} |u|^{\xi(x)} dx.$$

As $\frac{\xi(x)+\varepsilon}{\xi(x)} > 1$, applying Lemma 8 to estimate the second integral on the right member of the last inequality, we gain

$$\int_{\Omega} |u|^{\xi(x)} |\ln |u||^{\beta(x)} \, dx \leq C_1 \int_{\Omega} |u|^{\xi(x)+\varepsilon} \, dx + C_2 \,,$$

here $C_1 \equiv C_1 (\varepsilon, \beta^+) > 0$ and $C_2 \equiv C_2 (\varepsilon, \beta^+, |\Omega|) > 0$ are constants.

Since $\xi (x) + \varepsilon < \xi (x) + \beta_1 (x)$, a.e. $x \in \Omega$, estimating the integral on the right side of the above equation by using Lemma 8, we attain (18). □

In the following discussions, we examine elaborate properties of the pn-spaces $S_{1,\gamma(x),\beta(x),\theta(x)} (\Omega)$, presented in Section 1 (for other results, see [19,20]).

Lemma 11. *Let $S_{1,\gamma(x),\beta(x),\theta(x)} (\Omega)$ and $S_{1,\xi(x),\alpha(x),\theta_1(x)} (\Omega)$ be the spaces given in Definition 4. Assume that one of the conditions given below are satisfied*

(i) $\theta_1 (x) \leq \theta (x) , \beta (x) \geq \alpha (x)$ *and* $\xi (x) \beta (x) = \gamma (x) \alpha (x)$, *a.e.* $x \in \Omega$,

(ii) $\theta_1 (x) \leq \theta (x) , \xi (x) \beta (x) > \gamma (x) \alpha (x) , \gamma (x) + \beta (x) \geq \xi (x) + \alpha (x)$ *and* $\beta (x) \geq \alpha (x) + \varepsilon$ *for some* $\varepsilon > 0$.

Under these conditions the embedding

$$S_{1,\gamma(x),\beta(x),\theta(x)} (\Omega) \subset S_{1,\xi(x),\alpha(x),\theta_1(x)} (\Omega) \tag{19}$$

holds.

Proof. First, suppose that (i) holds. Let $u \in S_{1,\gamma(x),\beta(x),\theta(x)} (\Omega)$, to show the embedding (19) it is sufficient to verify the finiteness of

$$R^{\xi,\alpha,\theta_1} (u) = \int_{\Omega} |u|^{\theta_1(x)} \, dx + \sum_{i=1}^{n} \int_{\Omega} |u|^{\xi(x)} |D_i u|^{\alpha(x)} \, dx,$$

estimating the first integral on the right member of the above equation with the help of Lemma 8 and second one by employing Young's inequality, we acquire

$$R^{\xi,\alpha,\theta_1} (u) \leq (n+1) |\Omega| + \int_{\Omega} |u|^{\theta(x)} \, dx + \sum_{i=1}^{n} \int_{\Omega} |u|^{\frac{\xi(x)\beta(x)}{\alpha(x)}} |D_i u|^{\beta(x)} \, dx.$$

From the conditions, $\frac{\xi(x)\beta(x)}{\alpha(x)} = \gamma (x)$ that yields

$$R^{\xi,\alpha,\theta_1} (u) \leq R^{\gamma,\beta,\theta} (u) + (n+1) |\Omega| \,,$$

so (19) is gained. We note that when the case $\beta (x) = \alpha (x)$ a.e. $x \in \Omega$, then $\xi (x) = \gamma (x)$, hence (19) can be obtained by similar operations as above.

Now, assume that (ii) fulfills. We need to show that $R^{\xi,\alpha,\theta_1} (u)$ is finite. We have

$$\begin{aligned} R^{\xi,\alpha,\theta_1} (u) &= \int_{\Omega} |u|^{\theta_1(x)} \, dx + \sum_{i=1}^{n} \int_{\Omega} |u|^{\xi(x)} |D_i u|^{\alpha(x)} \, dx \\ &= \int_{\Omega} |u|^{\theta_1(x)} \, dx + \sum_{i=1}^{n} \int_{\Omega} |u|^{\xi(x) - \frac{\gamma(x)\alpha(x)}{\beta(x)}} |u|^{\frac{\gamma(x)\alpha(x)}{\beta(x)}} |D_i u|^{\alpha(x)} \, dx. \end{aligned}$$

If we estimate the first integral on the right member of the above equation with the help of Lemma 8 and second one by employing Young's inequality with the exponent $\frac{\beta(x)}{\alpha(x)}$ at every point, one can acquire that

$$R^{\xi,\alpha,\theta_1}(u) \le \int_{\Omega} |u|^{\theta(x)}\,dx + |\Omega| + \sum_{i=1}^{n} \int_{\Omega} |u|^{\gamma(x)} |D_i u|^{\beta(x)}\,dx + n \int_{\Omega} |u|^{\frac{\xi(x)\beta(x)-\gamma(x)\alpha(x)}{\beta(x)-\alpha(x)}}\,dx.$$

In the light of the condition (ii), the inequality $\frac{\xi(x)\beta(x)-\gamma(x)\alpha(x)}{\beta(x)-\alpha(x)} < \gamma(x) + \beta(x)$ holds, so estimating the third integral in the right side of the last inequality by Lemma 8, we arrive at

$$\begin{aligned} R^{\xi,\alpha,\theta_1}(u) &\le (n+1) \int_{\Omega} |u|^{\theta(x)}\,dx + (n+1)\,|\Omega| + \sum_{i=1}^{n} \int_{\Omega} |u|^{\gamma(x)} |D_i u|^{\beta(x)}\,dx \\ &\le (n+1) \left(R^{\gamma,\beta,\theta}(u) + |\Omega| \right), \end{aligned}$$

hence from here desired inequality is achieved. Also if $\theta_1(x) = \theta(x)$ a.e. $x \in \Omega$, by employing the same operations one can show (19). ☐

Lemma 12. *Let β, γ and ψ satisfy the conditions of Theorem 3, then $S_{1,\gamma(x),\beta(x),\theta(x)}(\Omega)$ is a metric space with the metric which is defined below*

$$d_{S_1}(u,v) := \|\varphi(u) - \varphi(v)\|_{L^{\psi(x)}(\Omega)} + \sum_{i=1}^{n} \left\| \varphi_t'(u)\, D_i u - \varphi_t'(v)\, D_i u \right\|_{L^{\beta(x)}(\Omega)},$$

$\forall u, v \in S_{1,\gamma(x),\beta(x),\theta(x)}(\Omega)$, here $\varphi(x,t) = |t|^{\frac{\gamma(x)}{\beta(x)}}\, t$ and for every fixed $x \in \Omega$

$$\varphi_t'(t) = \left(\frac{\gamma(x)}{\beta(x)} + 1 \right) |t|^{\frac{\gamma(x)}{\beta(x)}}.$$

Proof. It has been shown in Theorem 3 that[3] $\varphi(u) \in L^{\psi(x)}(\Omega)$ and $\varphi_t'(u)\, D_i u \in L^{\beta(x)}(\Omega)$ whenever $u \in S_{1,\gamma(x),\beta(x),\theta(x)}(\Omega)$, thus one can verify that $d_{S_1}(\cdot,\cdot) : S_{1,\gamma(x),\beta(x),\theta(x)}(\Omega) \to \mathbb{R}$ satisfy the metric axioms, i.e.

(i) $d_{S_1}(u,v) \ge 0$,

(ii) $d_{S_1}(u,v) = d_{S_1}(v,u)$,

(iii) $u = v \Rightarrow d_{S_1}(u,v) = 0$,

(iv) $d_{S_1}(u,v) = 0 \Rightarrow \|\varphi(u) - \varphi(v)\|_{L^{\psi(x)}(\Omega)} = 0 \Rightarrow \varphi(u) = \varphi(v)$ since φ is 1-1, then $u = v$,

(v) from the subadditivity of norm, $d_{S_1}(u,v) \le d_{S_1}(u,w) + d_{S_1}(w,v)$.

☐

[3] From now on, we denote $\varphi(x,u) := \varphi(u) = |u|^{\frac{\gamma(x)}{\beta(x)}}\, u$ for simplicity.

Theorem 7. *Under the conditions of Theorem 3, φ is a homeomorphism between the spaces $S_{1,\gamma(x),\beta(x),\theta(x)}(\Omega)$ and $L^{1,\beta(x)}(\Omega)\cap L^{\psi(x)}(\Omega)$.*

Proof. The function φ is a bijection between $S_{1,\gamma(x),\beta(x),\theta(x)}(\Omega)$ and $L^{1,\beta(x)}(\Omega)\cap L^{\psi(x)}(\Omega)$ by Theorem 3. Thus it is ample to prove the continuity of φ as well as φ^{-1} in the sense of topology induced by the metric $d_{S_1}(\cdot,\cdot)$. For this, we need to show that

(i) $d_{S_1}(u_m,u_0) \underset{m\nearrow\infty}{\longrightarrow} 0 \Rightarrow \varphi(u_m) \overset{L^{1,\beta(x)}(\Omega)\cap L^{\psi(x)}(\Omega)}{\underset{m\nearrow\infty}{\longrightarrow}} \varphi(u_0)$ for every $\{u_m\}\in S_{1,\gamma(x),\beta(x),\theta(x)}(\Omega)$ which converges to u_0 and

(ii) $v_m \overset{L^{1,\beta(x)}(\Omega)\cap L^{\psi(x)}(\Omega)}{\underset{m\nearrow\infty}{\longrightarrow}} v_0 \Rightarrow d_{S_1}(\varphi^{-1}(v_m),\varphi^{-1}(v_0)) \underset{m\nearrow\infty}{\longrightarrow} 0$ for every $v_m\in L^{1,\beta(x)}(\Omega)\cap L^{\psi(x)}(\Omega)$ which converges to v_0.

Since for every v_m and v_0 there exist a unique u_m and $u_0\in S_{1,\gamma(x),\beta(x),\theta(x)}(\Omega)$ such that $\varphi(u_m)=v_m$ and $\varphi(u_0)=v_0$, the implication ***(ii)*** can be written equivalently as follows $\varphi(u_m) \overset{L^{1,\beta(x)}(\Omega)\cap L^{\psi(x)}(\Omega)}{\underset{m\nearrow\infty}{\longrightarrow}} \varphi(u_0) \Rightarrow d_{S_1}(u_m,u_0) \underset{m\nearrow\infty}{\longrightarrow} 0$ for every $\{u_m\}\in S_{1,\gamma(x),\beta(x),\theta(x)}(\Omega)$ which converges to u_0.

Since the proofs of ***(i)*** and ***(ii)*** are similar, we only prove ***(ii)***. Let $v_0, v_m\in L^{1,\beta(x)}(\Omega)\cap L^{\psi(x)}(\Omega)$ and $v_m \overset{L^{1,\beta(x)}(\Omega)\cap L^{\psi(x)}(\Omega)}{\longrightarrow} v_0 \Leftrightarrow \varphi(u_m) \overset{L^{1,\beta(x)}(\Omega)\cap L^{\psi(x)}(\Omega)}{\longrightarrow} \varphi(u_0)$.

To verify $d_{S_1}(u_m,u_0)\to 0$, by definition of metric d_{S_1} it is ample to demonstrate that

$$\left\|\varphi_t'(u_m)D_iu_m-\varphi_t'(u_0)D_iu_0\right\|_{L^{\beta(x)}(\Omega)}\to 0 \quad\text{and}\quad \left\|\varphi(u_m)-\varphi(u_0)\right\|_{L^{\psi(x)}(\Omega)}\to 0$$

as $m\nearrow\infty$.

From $\varphi(u_m) \overset{L^{1,\beta(x)}(\Omega)\cap L^{\psi(x)}(\Omega)}{\longrightarrow} \varphi(u_0)$, we have

$$\left\|\varphi(u_m)-\varphi(u_0)\right\|_{L^{\psi(x)}(\Omega)}\to 0 \quad\text{and}\quad \left\|D_i(\varphi(u_m)-\varphi(u_0))\right\|_{L^{\beta(x)}(\Omega)}\to 0.$$

Hence, we only need to show that

$$\left\|\varphi_t'(u_m)D_iu_m-\varphi_t'(u_0)D_iu_0\right\|_{L^{\beta(x)}(\Omega)}\longrightarrow 0 \text{ as } m\nearrow\infty.$$

From Lemma 4, we have

$$\left\|\varphi_t'(u_m)D_iu_m-\varphi_t'(u_0)D_iu_0\right\|_{L^{\beta(x)}(\Omega)}\to 0 \Leftrightarrow \sigma_\beta\left(\varphi_t'(u_m)D_iu_m-\varphi_t'(u_0)D_iu_0\right)\to 0. \quad (20)$$

Based on (20), for $i=\overline{1,n}$

$$\sigma_\beta\left(\varphi_t'(u_m)D_iu_m-\varphi_t'(u_0)D_iu_0\right)=\int\limits_\Omega\left|\varphi_t'(u_m)D_iu_m-\varphi_t'(u_0)D_iu_0\right|^{\beta(x)}dx, \quad (21)$$

one can show that the following equality holds

$$\begin{aligned}\varphi_t'(u_m)D_iu_m-\varphi_t'(u_0)D_iu_0 &= \left(\tfrac{\beta(x)}{\beta(x)+\gamma(x)}\right)D_i(\varphi(u_m)-\varphi(u_0))\\ &\quad-\left(\tfrac{D_i\gamma.\beta-\gamma.D_i\beta}{\beta(\gamma+\beta)}\right)\left(|u_m|^{\frac{\gamma(x)}{\beta(x)}}u_m\ln|u_m|-|u_0|^{\frac{\gamma(x)}{\beta(x)}}u_0\ln|u_0|\right).\end{aligned} \quad (22)$$

Substituting (22) into (21), we acquire

$$\begin{aligned}
&\sigma_\beta\left(\varphi_t'(u_m)\,D_iu_m-\varphi_t'(u_0)\,D_iu_0\right)\\
&\qquad=\int_\Omega\Bigg|\left(\tfrac{\beta(x)}{\gamma(x)+\beta(x)}\right)D_i\left(\varphi(u_m)-\varphi(u_0)\right)\\
&\qquad\qquad-\left(\tfrac{D_i\gamma.\beta-\gamma.D_i\beta}{\beta(\gamma+\beta)}\right)\left(|u_m|^{\frac{\gamma(x)}{\beta(x)}}u_m\ln|u_m|-|u_0|^{\frac{\gamma(x)}{\beta(x)}}u_0\ln|u_0|\right)\Bigg|^{\beta(x)}
\end{aligned}$$

taking $\beta(x)$ into the absolute value and applying known inequality, we gain

$$\begin{aligned}
\sigma_\beta\left(\varphi_t'(u_m)\,D_iu_m-\varphi_t'(u_0)\,D_iu_0\right)&\le 2^{\beta^+-1}\int_\Omega\left|D_i\left(\varphi(u_m)\right)-D_i\left(\varphi(u_0)\right)\right|^{\beta(x)}dx\\
&\quad+C_3\int_\Omega\left||u_m|^{\frac{\gamma(x)}{\beta(x)}}u_m\ln|u_m|-|u_0|^{\frac{\gamma(x)}{\beta(x)}}u_0\ln|u_0|\right|^{\beta(x)}dx,
\end{aligned}\tag{23}$$

here $C_3=C_3\left(\beta^+,\|\gamma\|_{C^1(\bar\Omega)},\|\beta\|_{C^1(\bar\Omega)}\right)>0$ is constant.

Since $\|D_i(\varphi(u_m)-\varphi(u_0))\|_{L^{\beta(x)}(\Omega)}\longrightarrow 0$ as $m\nearrow\infty$, the first integral in the right member of (23) converges to zero when m tends to infinity (Lemma 4).

From Theorem 3, function φ is bijective between the spaces $L^{\theta(x)}(\Omega)$ and $L^{\psi(x)}(\Omega)$. Also since $\|\varphi(u_m)-\varphi(u_0)\|_{L^{\psi(x)}(\Omega)}\longrightarrow 0$, we arrive at

$$\varphi(u_m)\xrightarrow[\Omega]{a.e}\varphi(u_0)\Rightarrow u_m\xrightarrow[\Omega]{a.e}u_0\tag{24}$$

and

$$\sigma_\theta(u_m)=\int_\Omega|u_m|^{\theta(x)}dx=\int_\Omega\left||u_m|^{\frac{\gamma(x)}{\beta(x)}}u_m\right|^{\psi(x)}dx=\int_\Omega|\varphi(u_m)|^{\psi(x)}dx\le M\tag{25}$$

for some $M>0$.

Employing (24), (25) and Vitali's Theorem[4], we attain

$$\int_\Omega|u_m|^{\theta(x)}dx\longrightarrow\int_\Omega|u_0|^{\theta(x)}dx,\quad m\nearrow\infty.\tag{26}$$

Since u_m converges to u_0 in measure on Ω, using this and (26), we deduce from Lemma 4 that

$$\sigma_\theta(u_m-u_0)\longrightarrow 0\Rightarrow\|u_m-u_0\|_{L^{\theta(x)}(\Omega)}\longrightarrow 0.\tag{27}$$

[4] **Theorem** (Vitali, [16]). *Let (Ω,Σ,μ) be a finite measure space, and $f_n:\Omega\to\mathbb{R}$ be a sequence of measurable functions converging a.e. to a measurable f. Then $\|f_n-f\|_{L^1(\Omega)}\to 0$ as $n\to\infty$ iff $\{f_n:\ n\ge 1\}$ is uniformly integrable. When the condition is satisfied, we have*

$$\lim_{n\to\infty}\int_\Omega f_n d\mu=\int_\Omega f d\mu.$$

Denote $w_m := |u_m|^{\frac{\gamma(x)}{\beta(x)}} u_m \ln |u_m|$ and $w_0 := |u_0|^{\frac{\gamma(x)}{\beta(x)}} u_0 \ln |u_0|$, then

$$\sigma_\beta (w_m) = \int_\Omega |u_m|^{\gamma(x)+\beta(x)} |\ln |u_m||^{\beta(x)} dx.$$

Estimating the above integral by using Lemma 10, one can obtain

$$\sigma_\beta (w_m) \leq C_4 \int_\Omega |u_m|^{\theta(x)} dx + C_5 = C_4 \sigma_\theta (u_m) + C_5.$$

From (27), $\sigma_\beta (w_m) \leq \tilde{M}$ for all $m \geq 1$, for some $\tilde{M} > 0$. Thus as shown above for u_m similarly we conclude that as $m \nearrow \infty$

$$\sigma_\beta (w_m - w_0) \longrightarrow 0 \Rightarrow \int_\Omega \left| |u_m|^{\frac{\gamma(x)}{\beta(x)}} u_m \ln |u_m| - |u_0|^{\frac{\gamma(x)}{\beta(x)}} u_0 \ln |u_0| \right|^{\beta(x)} \longrightarrow 0,$$

hence from (23) we attain,

$$\left\| \varphi_t' (u_m) D_i u_m - \varphi_t' (u_0) D_i u_0 \right\|_{L^{\beta(x)}(\Omega)} \longrightarrow 0, \quad m \nearrow \infty.$$

So, the proof is complete. □

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У даній роботі ми вивчаємо властивості класу функціональних просторів, так званих pn-просторів, які з'являються при дослідженні нелінійних диференціальних рівнянь. Ми встановили деякі інтегральні нерівності для аналізу структури pn-просторів зі сталими та змінними показниками. Ми довели теореми про вкладення, які встановлюють співвідношення цих просторів з добре відомими класичними просторами Лебега і Соболєва зі сталими та змінними показниками.

Ключові слова і фрази: pn-простір, змінний показник, інтегральна нерівність, нелінійне диференціальне рівняння, теорема про вкладення.

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ON TWO LONG STANDING OPEN PROBLEMS ON L_p-SPACES

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The present note was written during the preparation of the talk at the International Conference dedicated to 70-th anniversary of Professor O. Lopushansky, September 16-19, 2019, Ivano-Frankivsk, Ukraine. We focus on two long standing open problems. The first one, due to Lindenstrauss and Rosenthal (1969), asks of whether every complemented infinite dimensional subspace of L_1 is isomorphic to either L_1 or ℓ_1. The second problem was posed by Enflo and Rosenthal in 1973: does there exist a nonseparable space $L_p(\mu)$ with finite atomless μ and $1 < p < \infty$, $p \neq 2$, having an unconditional basis? We analyze partial results and discuss on some natural ideas to solve these problems.

Key words and phrases: L_p-spaces, complemented subspace, unconditional basis.

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1 INTRODUCTION

Investigation of the geometry of Lebesgue spaces $L_p := L_p[0,1]$ has long and rich history (see [3]) due to famous mathematicians: D.E. Alspach, S. Banach, J. Bourgain, D.L. Burkholder, L.E. Dor, P. Enflo, W.B. Johnson, M.I. Kadets, N. Kalton, J. Lindenstrauss, B. Maurey, E. Odell, R.E.A.C. Paley, A. Pełczyński, H.P. Rosenthal, G. Schechtman, T.W. Starbird, S. Szarek, M. Talagrand, L. Tzafriri and others. More is known on the isomorphic structure of these classical spaces. Isomorphic embeddability of $L_r(\nu)$ into L_p is completely known. We use the notation $X \hookrightarrow Y$ to express that X embeds isomorphically into Y, and $X \simeq Y$ means that the Banach spaces X and Y are isomorphic. The relation $\ell_p \hookrightarrow L_p$, which is easily seen, was first noted by S. Banach [4, p. 175]. The embedding $\ell_2 \hookrightarrow L_p$ follows from Khintchin's inequality [30, p. 66]. It is not hard to see that $\ell_p \not\hookrightarrow L_2$ for $p \neq 2$ (for the proof, see [4, p. 175]). The relation $\ell_r \not\hookrightarrow L_p$ for $2 < p < r$ and $1 \leq r < p < 2$ was proved by S. Banach [4, p. 175]. Paley's results [37] imply $\ell_r \not\hookrightarrow L_p$ for $1 \leq r < 2 < p$, $2 < r < p$ and $1 \leq p < 2 < r$.

A special case is $1 \leq p < r < 2$, where isometric embeddings of L_r into L_p are possible. First it was proved by P. Levy [25] that ℓ_r is *finitely representable*[1] in L_p if $1 \leq p < r < 2$. Later M.I. Kadets proved that $\ell_r \hookrightarrow L_p$ for $1 \leq p < r < 2$ [20]. Then the latter result was strengthen to the embedding $L_r \hookrightarrow L_p$ by J. Bretagnolle, D. Dacunha-Castelle and J. L. Krivine [9] and independently by J. Lindenstrauss J. and A. Pełczyński [27], who proved more: if a Banach space X is finitely representable in L_p then $X \hookrightarrow L_p$.

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[1] A Banach space X is said to be finitely representable in a Banach space Y if for every $\varepsilon > 0$ and every finite dimensional subspace F of X there exists a subspace G of Y of the same dimension such that $d(F,G) < 1+\varepsilon$, where $d(F,G)$ denotes the Banach-Mazur distance between F and G.

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As we see, the properties of the spaces L_p are different for the cases $p < 2$ and $p > 2$. Moreover, if $2 < p < \infty$ then every subspace of L_p possesses the following properties:

– either is isomorphic to a Hilbert space or contains a complemented subspace isomorphic to ℓ_p [21];

– either contains a subspace isomorphic to ℓ_2 or embeds isomorphically into ℓ_p [19].

On the other hand, if $1 \le p < 2$ then every subspace of L_p either contains a complemented subspace isomorphic to ℓ_p or embeds isomorphically into L_r for some $p < r \le 2$ [44].

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2 Complemented subspaces of L_p

2.1 As p goes to 1, the complementability properties of subspaces of L_p, $p \neq 2$, get worse

By Khintchin's inequality, the closed linear span R of the Rademacher system in L_p is isomorphic to ℓ_2 and is actually independent on p, as a set. Remark that R is complemented in L_p for $1 < p < \infty$ [30, p. 66] and is uncomplemented in L_1, as well as any other subspace of L_1 isomorphic to ℓ_2 [38]. So, it became interesting, whether there exists an uncomplemented subspace of L_p isomorphic to ℓ_2 for $p > 1$. If $2 < p < \infty$ then every subspace X of L_p isomorphic to ℓ_2 is complemented, and, moreover, the L_p- and L_2-norms[2] on X are equivalent [21]. To the contrast, if $1 < p < 2$ then there exists an uncomplemented subspace of L_p isomorphic to ℓ_2 (first it was proved for $1 < p < 4/3$ in [42] and then for the rest of values in [5]).

It is clear that L_p contains a complemented subspace isomorphic to ℓ_p. If $1 \le p < \infty$, $p \neq 2$, then there is an uncomplemented subspace of L_p isomorphic to ℓ_p, and hence, it is not difficult to show that there is an uncomplemented subspace of L_p isomorphic to L_p itself (first it was proved for $2 < p < \infty$ and $1 < p < 4/3$ in [43], then in a different way for all $1 < p < 2$ in [5], and finally for $p = 1$ in [6]).

2.2 Primarity of L_p and Enflo operators

By the famous Enflo theorem, if $L_p = X \oplus Y$, $1 \le p < \infty$, is a decomposition into mutually complemented subspaces, then at least one of the subspaces X, Y is isomorphic to L_p (first it was announced by P. Enflo; then B. Maurey [34] published a proof, see also [2] for all p, [14] for $p = 1$ and [31, p. 179] for a generalization to rearrangement invariant spaces). This nice property of the spaces L_p is called the *primarity.*

Let X, Y be Banach spaces. Denote by $\mathcal{L}(X, Y)$ the Banach space of all continuous linear operators from X to Y, and write $\mathcal{L}(X)$ instead of $\mathcal{L}(X, X)$. Recall that an operator $T \in \mathcal{L}(X, Y)$ is said to *fix a copy of a Banach space Z*, if there exists a subspace X_1 of X isomorphic to Z such that the restriction $T|_{X_1}$ of T to X_1 is an into isomorphism. An operator $T \in \mathcal{L}(L_p, Y)$, $1 \le p < \infty$, is called an *Enflo operator* provided T fixes a copy of L_p. Note that every Enflo operator $T \in \mathcal{L}(L_p)$ fixes a complemented copy of L_p, that is, there is a complemented subspace X_1 of X isomorphic to L_p such that the restriction $T|_{X_1}$ is an into isomorphism, because every subspace X of L_p, which is isomorphic to L_p, contains a further subspace $Y \subseteq X$ isomorphic to

[2] which are well defined for these values of p

L_p and complemented in L_p (see [18, p. 239] for $p > 1$ and [14] for $p = 1$). The Enflo theorem implies that, if the identity operator Id on L_p is a sum of two projections $Id = P + Q$, then at least one of the projections P, Q is an Enflo operator. Moreover, the range of a projection P on L_p is isomorphic to L_p if and only if P is an Enflo operator (to prove, use the mentioned above result from [18] and Pełczyński's decomposition method [31, p. 54]).

2.3 Isomorphic types of complemented subspaces of L_p

How many do there exist pairwise non-isomorphic complemented subspaces of L_p for $1 \le p < \infty$, $p \ne 2$? If $p > 1$ then there are obviously the following pairwise non-isomorphic Banach spaces isomorphic to complemented subspaces of L_p:

$$L_p,\ \ell_p,\ \ell_2,\ \ell_p \oplus \ell_2,\ \Big(\bigoplus_{n=1}^{\infty} \ell_2\Big)_p.$$

Further finitely many examples, different from the above obvious ones, was obtained by H.P. Rosenthal in [43]. Later G. Schechtman provided infinitely many pairwise non-isomorphic examples in [48], and then J. Bourgain, H.P. Rosenthal and G. Schechtman constructed uncountably many pairwise non-isomorphic complemented subspaces of L_p for $1 < p < \infty$, $p \ne 2$ in [8] (it is unknown, whether there exists continuum such subspaces).

The exceptional case is $p = 1$: there are only two known obvious examples of pairwise non-isomorphic infinite dimensional subspaces of L_1, they are L_1 itself and ℓ_1.

Problem 1 (Lindenstrauss and Rosenthal, 1969, [29]). *Is every complemented infinite dimensional subspace of L_1 isomorphic to either L_1 or ℓ_1?*

2.4 Progress in the solution of Problem 1

The following assertions have been established for an arbitrary complemented subspace E of L_1.

Theorem 1 (Pełczyński, 1960, [38]). *E contains a subspace isomorphic to ℓ_1 and complemented in L_1.*

Theorem 2 (Lindenstrauss, Pełczyński, 1968, [27]). *If E has an unconditional basis then E is isomorphic to ℓ_1.*

Recall that *the Radon-Nikodým property* (*RNP*) for a Banach space X means that for every finite measure space (Ω, Σ, μ) and every μ-continuous X-valued measure $G : \Sigma \to X$ of bounded variation there exists $g \in L_1(\mu, X)$ such that $G(A) = \int_A g \, d\mu$ for all $A \in \Sigma$. One can show that the characteristic function $G(A) = \mathbf{1}_A$ is an example of L_1-valued such measure for which the function g does not exist [12, p. 61]; thus, L_1 does not have the RNP. However, ℓ_1 has the RNP (this can be proved directly, using the Radon-Nikodým theorem for separate coordinates [12, p. 64]).

Theorem 3 (Lewis, Stegall, 1973, [26]). *If E has the RNP then E is isomorphic to ℓ_1.*

A Banach space X is said to have the *Schur property* if the weak convergence of a sequence in X implies its norm convergence. It is well known that ℓ_1 has the Schur property.

Theorem 4 (Rosenthal, 1975, [45]). *If E does not have the Schur property then ℓ_2 embeds into E.*

Theorem 5 (Enflo, Starbird, 1979, [14]). *If E contains a subspace isomorphic to L_1 then E is itself isomorphic to L_1.*

Simultaneously, W.B. Johnson, B. Maurey, G. Schechtman and L. Tzafriri [18] obtained the same result as Theorem 5 asserts for L_p with $1 < p < \infty$.

The next result strengthens Theorem 4.

Theorem 6 (Bourgain, 1980, [7]). *If E does not have the Schur property then $(\bigoplus_{n=1}^{\infty} \ell_2)_1$ embeds into E.*

There is a natural idea to solve Problem 1. Obviously, **the hypothesis that every complemented infinite dimensional subspace of L_1 is isomorphic to either L_1 or ℓ_1, is equivalent to the hypothesis that the following two climes hold true**.

Let E be an infinite dimensional complemented subspace of L_1.

Claim 1. *If E has the Schur property then E is isomorphic to ℓ_1.*

Claim 2. *If E does not have the Schur property then E is isomorphic to L_1.*

As to the best of our knowledge, there is no information about Claim 1 in the literature. Remark that there is no direct way to prove Claim 1 without taking into account peculiarity of L_1, because there exists a Banach space with the Schur property but without the RNP, and so, not isomorphic to ℓ_1 (see J. Hagler [17]).

However, Claim 2 has been considered by different mathematicians as a weak version of Problem 1 in the sense that a positive solution to Problem 1 implies a positive answer to Problem 2.

Problem 2 ([45], [14] and [7]). *Must a non-Dunford-Pettis projection $P \in \mathcal{L}(L_1)$ be an Enflo operator? Equivalently, whether each non-Schur complemented subspace of L_1 is isomorphic to L_1?*

The most unclear thing concerning Problem 2 is how to use the information that P is a projection, not just a continuous linear operator. H.P. Rosenthal constructed an example of a non-Dunford-Pettis operator $T \in \mathcal{L}(L_1)$ failing to be an Enflo operator [45]. This is the so-called biased coin convolution operator. To explain the details, recall that the *Rademacher system* is defined by $r_n(t) = \text{sign}\,\sin(2^{n+1}\pi t)$ for each $n \in \mathbb{N}$ and $t \in [0,1]$. Denote by $\mathbb{N}^{<\omega}$ the set of all finite subsets of $\mathbb{N}$. The *Walsh system* $(w_I)_{I \in \mathbb{N}^{<\omega}}$ is defined by setting $w_I = \prod_{i \in I} r_i$, where $(r_n)_{n=1}^{\infty}$ is the Rademacher system (in particular, $w_\varnothing = \mathbf{1}$, by convention). The Walsh system with respect to the lexicographical order $w_\varnothing, w_{\{1\}}, w_{\{2\}}, w_{\{1,2\}}, w_{\{3\}}, w_{\{1,3\}}, w_{\{2,3\}}, w_{\{1,2,3\}}, \ldots$ is a Schauder basis of L_p for $1 < p < \infty$, an orthonormal basis of L_2, a conditional basis of L_p for $p \neq 2$, and a Markushevich basis of L_1.

Theorem 7 (H.P. Rosenthal, [45]). *There is $\varepsilon_0 \in (0,1)$ such that for every $\varepsilon \in (0, \varepsilon_0)$ there is an operator $R_\varepsilon \in \mathcal{L}(L_1)$ possessing the equality $R_\varepsilon w_I = \varepsilon^{|I|} w_I$ for all $I \in \mathbb{N}^{<\omega}$, where $|I|$ is the cardinality of I.*

The operator R_ε is called the ε-biased coin convolution operator. Since $R_\varepsilon r_n = \varepsilon r_n$ for all $n \in \mathbb{N}$, the operator R_ε is not Dunford-Pettis. H.P. Rosenthal proved in [45] that R_ε is not an Enflo operator.

2.5 All operators on L_1 are regular

Recall some information. Let E, F be vector lattices. An operator $T : E \to F$ is called *positive* if $T(E^+) \subseteq F^+$, and $T : E \to F$ is called *regular* if T equals a difference of two positive operators. Obviously, every positive (and hence, every regular) operator $T : E \to F$ is *order bounded*, that is, T sends order bounded subsets of E to order bounded subsets of F. Two elements $x, y \in E$ are said to be disjoint (write $x \perp y$) if $|x| \wedge |y| = 0$. The notation $x = \bigsqcup_{k=1}^n x_k$ means that $x = \sum_{k=1}^n x_k$ and $x_i \perp x_j$ for $i \neq j$.

It is an amazing and seldom used fact on operators on L_1 that all of them are regular [47, p. 232]. More precisely, every operator $T \in \mathcal{L}(L_1)$ admits the representation $T = T^+ - T^-$, where for every $x \in L_1^+$ one has

$$T^+ x = \sup\Big\{ \sum_{k=1}^m T x_k : \ x = \bigsqcup_{k=1}^n x_k, \ n \in \mathbb{N} \Big\}.$$

As a consequence, we obtain that for any operator $T \in \mathcal{L}(L_1)$ the modulus $|T| = T^+ + T^- \in \mathcal{L}(L_1)$ exists and could be defined by setting for every $x \in E^+$

$$|T|\, x = \sup\Big\{ \sum_{k=1}^n |Tx_i| : \ x = \sum_{k=1}^n x_k, \ x_k \in E^+, \ n \in \mathbb{N} \Big\}.$$

Moreover, $\||T|\| = \|T\|$ for every $T \in \mathcal{L}(L_1)$ [47, p. 232].

As was noted by H.P. Rosenthal [46], the regularity of operators $T \in \mathcal{L}\big(L_1(\mu), L_1(\nu)\big)$ is a consequence of the following Grothendieck's inequality [16, Corollaire, p. 67]: given any $f_1, \ldots, f_1 \in L_1(\mu)$, one has

$$\int_{\Omega_\nu} \max_i |Tf_i|\, d\nu \le \|T\| \int_{\Omega_\mu} \max_i |Tf_i|\, d\mu.$$

A very useful development of Grothendieck's inequality is M. Lévy's extension theorem (see [24]) asserting that, for every subspace X of $L_1(\mu)$ every order bounded operator $T \in \mathcal{L}(X, L_1(\nu))$ has an extension to some operator $\widehat{T} \in \mathcal{L}(L_1(\mu), L_1(\nu))$, which is therefore order bounded as well. The latter fact was then generalized to regular operators from $L_p(\mu)$ to $L_p(\nu)$ for $1 \le p \le \infty$ by G. Pisier in [40].

The regularity of all operators on L_1 in fact means that there are few operators on L_1, only regular ones. This explains why common subspaces of all L_p (like the closed linear span of the Rademacher system), which are complemented in L_p for $p > 1$ becomes uncomplemented in L_1: they are complemented in L_p by means of non-regular projections. The same reason makes the Haar system a conditional basis in L_1. This argument made the authors of [33] and [41, Problem 10.45] to generalize Problem1 as follows. We say that a subspace X of a Banach lattice is regularly complemented if there is a regular projection of E onto X.

Problem 3. *Let $1 \le p < \infty$, $p \neq 2$. Is every regularly complemented subspace of L_p isomorphic to either ℓ_p or L_p?*

2.6 Complemented subspaces of L_p for $0 < p < 1$

Consider now the quasi-Banach spaces L_p for $0 < p < 1$. The list of known isomorphic types of complemented subspaces of these spaces becomes smaller by one space, namely by ℓ_p, because L_p has trivial dual and hence cannot have a complemented subspace with nontrivial dual, like those that are isomorphic to ℓ_p. So, the problem is as follows.

Problem 4. *Let $0 \le p < 1$. Is every complemented subspace of L_p isomorphic to L_p?*

This problem has been systematically studied by N.J. Kalton in a number of papers. The best progress is Kalton's theorem, which asserts that, if there exists a complemented subspace of L_p not isomorphic to L_p, then at most one, up to an isomorphism [22].

3 Unconditional bases in $L_p(\mu)$

3.1 Preliminary information

For convenience of the reader, we recall some necessary information on bases [1, 30]. A sequence $(x_n)_{n=1}^\infty$ of elements of a Banach space X is called a *Schauder basis* (or just a *basis*) of X if for every $x \in X$ there is a unique sequence of scalars $(a_n)_{n=1}^\infty$ such that

$$x = \sum_{k=1}^{\infty} a_k x_k. \tag{1}$$

A sequence in X, which is a basis in its closed linear span, is called a *basic sequence*. The partial sums $P_n x = \sum_{k=1}^n a_k x_k$ of the expansion (1) are linear bounded projections on X with $K := \sup_n \|P_n\| < \infty$, and the number K is called the *basis constant* of $(x_n)_{n=1}^\infty$. In particular, the coefficients $x_k^*(x) := a_k$ of the expansion (1) are elements of X^* with $\sup_n \|x_n\| \|x_n^*\| \le 2K$ and are called the *biorthogonal functionals* to $(x_n)_{n=1}^\infty$. The best possible basis constant is 1; a basis with basis constant 1 is said to be *monotone*. The biorthogonal functionals $(x_n^*)_{n=1}^\infty$ form a basic sequence in X^* with the same basis constant K. A basis $(x_n)_{n=1}^\infty$ of X is called *unconditional* if for every $x \in X$ the series $x = \sum_{k=1}^\infty x_k^*(x) x_k$ converges unconditionally; otherwise the basis is said to be conditional. If $(x_n)_{n=1}^\infty$ is unconditional then for every sequence of signs $\Theta = (\theta_n)_{n=1}^\infty$, $\theta_n \in \{-1, 1\}$, and every $x \in X$ the series $T_\Theta x := \sum_{n=1}^\infty \theta_n x_n^*(x) x_n$ converges and T_Θ is a linear bounded operator. Moreover, $M := \sup_\Theta \|T_\Theta\| < \infty$. The number M is called the *unconditional constant* of the unconditional basis $(x_n)_{n=1}^\infty$.

Let $(x_n)_{n=1}^\infty$ be a basic sequence in X, $(a_n)_{n=1}^\infty$ be a sequence of scalars and $0 \le k_1 < k_2 < \dots$ be integers. A sequence $(u_n)_{n=1}^\infty$ of nonzero elements of X of the form

$$u_n = \sum_{i=k_n+1}^{k_{n+1}} a_i x_i$$

is called a *block basis* of $(x_n)_{n=1}^\infty$. It is not hard to see that $(u_n)_{n=1}^\infty$ is a basic sequence itself, the basis constant of which does not exceed that of $(x_n)_{n=1}^\infty$. Two basic sequences $(x_n)_{n=1}^\infty$ in X and $(y_n)_{n=1}^\infty$ in Y are called *λ-equivalent* if there exists an isomorphism $T : [x_n] \to [y_n]$ between the closed linear spans of these systems with $Tx_n = y_n$ for all n such that $\|T\| \|T^{-1}\| \le \lambda$. Basic sequences are said to be *equivalent* if they are λ-equivalent for some $\lambda \in [1, +\infty)$. Using the Closed Graph theorem, one can easily show that basic sequences $(x_n)_{n=1}^\infty$ and $(y_n)_{n=1}^\infty$ are equivalent if and only if for every sequence of scalars $(a_n)_{n=1}^\infty$ the convergence of the series $\sum_{n=1}^\infty a_n x_n$ and $\sum_{n=1}^\infty a_n x_n$ are equivalent. It is clear that if one of two λ-equivalent basic sequences is unconditional then the other one is unconditional as well, and the basic (unconditional) constants K_1, K_2 are estimated as follows: $\lambda^{-1} K_1 \le K_2 \le \lambda K_1$.

3.2 The Haar system in L_p

Define the *dyadic intervals* by setting $I_n^k = [\frac{k-1}{2^n}, \frac{k}{2^n})$ for $n = 0, 1, \dots$ and $k = 1, \dots, 2^n$. The L_∞-normalized *Haar system* is the following sequence in L_∞: $\overline{h}_1 = \mathbf{1}$ and

$$\overline{h}_{2^n+k} = \mathbf{1}_{I_{n+1}^{2k-1}} - \mathbf{1}_{I_{n+1}^{2k}} \tag{2}$$

for $n = 0, 1, 2, \dots$ and $k = 1, 2, \dots, 2^n$ (by $\mathbf{1}_A$ we denote the characteristic function of a set A). The Haar system is a monotone basis of every space L_p with $1 \le p < \infty$ [30, p. 3], and an unconditional basis of L_p for any $1 < p < \infty$ [31, p. 155] (the first fact one can obtain using a criterium of bases, and the second fact is a deep result of Paley [36] (1932), the proof of which was then simplified by Burkholder [11] (1985)). The unconditional constant of the Haar system in L_p equals $K_p = \max\{p, q\} - 1$, where $1/p + 1/q = 1$ [10].

The Haar system possesses the following useful property, called the *precise reproducibility* [28], [31, p. 158]: for every isomorphic embedding $T : L_p \to X$, $1 \le p < \infty$, where X is a Banach space with a basis $(x_n)_{n=1}^\infty$, and every $\varepsilon > 0$ there is a block basis $(u_n)_{n=1}^\infty$ of $(x_n)_{n=1}^\infty$, which is $(\|T\|\|T^{-1}\| + \varepsilon)$-equivalent to the Haar system in L_p. This gives that the Haar system is the "best" basis: once we have an unconditional basis in L_p, the Haar system is unconditional as well, and its unconditional constant is the minimal possible one. Since the Haar system is a conditional basis in L_1 [31, p. 156], we obtain that L_1 cannot be isomorphically embedded in a Banach space with an unconditional basis (initially this was proved by A. Pełczyński [39]).

3.3 Nonseparable $L_p(\mu)$-spaces

There is a nice complete isomorphic classification of the spaces $L_p(\mu)$ over finite atomless measure spaces (Ω, Σ, μ). A canonical representative of measure spaces (Ω, Σ, μ) with[3] $\dim L_p(\mu) = \aleph_\alpha$ for $0 < p < \infty$ is $\big(\{-1,1\}^{\omega_\alpha}, \Sigma_{\omega_\alpha}, \mu_{\omega_\alpha}\big)$, where ω_α is the cardinal of cardinality $\aleph_\alpha$, Σ_{ω_α} is the Borel σ-algebra of subsets of $\{-1,1\}^{\omega_\alpha}$ endowed with the Tykhonov topology on the power of the discrete two-point space $\{-1,1\}$, and μ_{ω_α} is the corresponding power of the measure μ_0 on the subsets of $\{-1,1\}$ defined by $\mu_0\{-1\} = \mu_0\{1\} = 1/2$. In other words, μ_{ω_α} is the Haar measure on the compact Abelian group $\{-1,1\}^{\omega_\alpha}$ with the point-wise product. By the famous Maharam theorem (see [32] for the original paper, and [15,23] for different proofs), every finite atomless measure space (Ω, Σ, μ) is isomorphic (in the sense of measure spaces) to a unique (up to a permutation of summands) direct sum of the measure spaces $\bigoplus_{\alpha \in \mathcal{A}} \big(\{-1,1\}^{\omega_\alpha}, \Sigma_{\omega_\alpha}, \varepsilon_\alpha \mu_{\omega_\alpha}\big)$, where $\mathcal{A}$ is an at most countable set of ordinals, called the *Maharam invariants* of (Ω, Σ, μ), and $\varepsilon_\alpha > 0$ are weights with $\sum_{\alpha \in \mathcal{A}} \varepsilon_\alpha = \mu(\Omega)$. The Lebesgue measure space $\big([0,1], \Sigma, \lambda\big)$, where λ is the Lebesgue measure on the Borel σ-algebra Σ of subsets of $[0,1]$, is isomorphic to $\big(\{-1,1\}^{\omega_0}, \Sigma_{\omega_0}, \mu_{\omega_0}\big)$. As a consequence, we obtain that every $L_p(\mu)$-space over a finite atomless measure μ with $0 < p \le \infty$ is isometrically isomorphic to the ℓ_p-sum $\big(\sum_{\alpha \in \mathcal{A}} L_p\{-1,1\}^{\omega_\alpha}\big)_p$.

A (not ordered) family $(x_i)_{i \in I}$ of elements of a (non-separable) Banach space X is called an *unconditional basis* of X if every $x \in X$ admits a unique representation $x = \sum_{i \in I} a_i x_i$, where the set of all indices $i \in I$ with $a_i \neq 0$ is at most countable, and the series converges unconditionally. One can show directly, that a family $(x_i)_{i \in I}$ with dense linear span is an unconditional basis of X if and only if every its countable subfamily is an unconditional basic sequence. If this is the case then the unconditional constants of countable subfamilies are bounded from

[3] By $\dim X$ we mean the smallest cardinality of subsets of X with dense linear span.

above, and their supremum equals the unconditional constant of the entire family, which is defined similarly.

P. Enflo and H.P. Rosenthal (1973) [13] proved that, if $\dim L_p(\mu) \geq \aleph_{\omega_0}$, where μ is finite atomless and $1 < p < \infty$, $p \neq 2$, then $L_p(\mu)$ does not embed isomorphically into a Banach space with an unconditional basis. They proved preliminarily that, for any $n \in \mathbb{N}$, assuming the isomorphic embedding $T : L_p\{-1,1\}^{\omega_n} \to X$ into a Banach space X with an unconditional basis $(x_i)_{i\in I}$, the finite Walsh system $(w_J)_{|J|\leq n}$ is $\|T\|\|T^{-1}\|$-reproducible in $(x_i)_{i\in I}$, even more, $\|T\|\|T^{-1}\|$-equivalent to a suitable block basis of $(x_i)_{i\in I}$. As a consequence, the unconditional constant M_n of $(w_J)_{|J|\leq n}$ does not exceed $M\|T\|\|T^{-1}\|$, where M is the unconditional constant of $(x_i)_{i\in I}$. Since for every $n \in \mathbb{N}$ the space $L_p\{-1,1\}^{\omega_n}$ isometrically embeds into $L_p(\mu)$, it then remained to show that $M_n \to \infty$ as $n \to \infty$, which is true. Unfortunately, their method could not give more, remaining the following problem to be open.

Problem 5 (P. Enflo and H.P. Rosenthal, 1973, [13])**.** *Let $1 \leq p < \infty$, $p \neq 2$, and let (Ω, Σ, μ) be a finite atomless measure space with $\aleph_0 < \dim L_p(\mu) < \aleph_{\omega_0}$. Is there an unconditional basis of $L_p(\mu)$?*

Below we describe two different possible ideas to solve this problem.

3.4 The Olevskii system

In 1966 A.M. Olevskii constructed a system of functions on $[0,1]$, which is a basis of L_1 containing the Rademacher system as a part [35]. This system, called in the literature the Olevskii system, is a conditional basis in L_p for $p \neq 2$, a result of E.M. Semenov [49]. If one tries to prove that $L_p\{-1,1\}^{\omega_1}$ (and therefore, $L_p\{-1,1\}^{\omega_n}$ for each $n \geq 1$) has no unconditional basis, then it would be enough to prove that the Olevskii system is reproducible in any unconditional basis of $L_p\{-1,1\}^{\omega_1}$. Let us present an author's description of the Olevskii system, which may be convenient for this purpose.

First, we represent the Haar system (2), collected by bunches, via the Rademacher system $(r_n)_{n=1}^{\infty}$ as follows:

$$\begin{aligned}
&\text{bunch } 1: \ \mathbf{1},\\
&\text{bunch } 2: \ r_1,\\
&\text{bunch } 3: \ \frac{r_1+1}{2}\cdot r_2,\ \ \frac{r_1-1}{2}\cdot r_2,\\
&\text{bunch } 4: \ \frac{r_1+1}{2}\cdot\frac{r_2+1}{2}\cdot r_3,\ \ \frac{r_1+1}{2}\cdot\frac{r_2-1}{2}\cdot r_3,\ \ \frac{r_1-1}{2}\cdot\frac{r_2+1}{2}\cdot r_3,\ \ \frac{r_1-1}{2}\cdot\frac{r_2-1}{2}\cdot r_3,\\
&\ldots
\end{aligned}$$

The Olevskii system can be constructed using the following scheme. First, we take the function $\mathbf{1}$. Then, to obtain the $(n+1)$-th Olevskii bunch, we multiply the beginning of the Haar system including its n-th bunch by r_n.

$$\begin{aligned}
&\text{bunch } 1: \ \mathbf{1},\\
&\text{bunch } 2: \ r_1,\\
&\text{bunch } 3: \ r_2, \quad r_1\cdot r_2,
\end{aligned}$$

$$\text{bunch } 4: \quad r_3, \quad r_1 \cdot r_3, \quad \frac{r_1+1}{2} \cdot r_2 \cdot r_3, \quad \frac{r_1-1}{2} \cdot r_2 \cdot r_3,$$

$$\text{bunch } 5: \quad r_4, \quad r_1 \cdot r_4, \quad \frac{r_1+1}{2} \cdot r_2 \cdot r_4, \quad \frac{r_1-1}{2} \cdot r_2 \cdot r_4, \quad \frac{r_1+1}{2} \cdot \frac{r_2+1}{2} \cdot r_3 \cdot r_4,$$
$$\frac{r_1+1}{2} \cdot \frac{r_2-1}{2} \cdot r_3 \cdot r_4, \quad \frac{r_1-1}{2} \cdot \frac{r_2+1}{2} \cdot r_3 \cdot r_4, \quad \frac{r_1-1}{2} \cdot \frac{r_2-1}{2} \cdot r_3 \cdot r_4,$$

...

A partial question to this concern: *is the beginning* $\mathbf{1}$, r_1, r_2, $r_1 \cdot r_2$ *of the Olevskii system isometrically reproducible in any unconditional basis of* $L_p\{-1,1\}^{\omega_1}$*?* Remark that it is isometrically reproducible in any unconditional basis of $L_p\{-1,1\}^{\omega_2}$, by the Enflo-Rosenthal results, because it coincides with the Walsh system of order two.

3.5 A close separable problem

Consider the following important partial case of Problem 5.

Problem 6. *Let* $1 \le p < \infty$, $p \ne 2$. *Does there exist an unconditional basis in* $L_p\{-1,1\}^{\omega_1}$*?*

We now pose a separable problem and then provide arguments to show that it is close to Problem 6. Let $E_p = L_p[0,1]^2$ be the L_p-space over the Lebesgue measure space of Borel subsets of the square $[0,1]^2$, and let F_p be the subspace of E_p consisting of all functions depending only on the first variable.

Problem 7. *Let* $1 \le p < \infty$, $p \ne 2$. *Does there exist an unconditional basis* $(f_n) \cup (g_n)$ *in* E_p *consisting of two parts such that* $[f_n] = F_p$ *and the unconditional constant of* (f_n) *equals the unconditional constant of the entire basis* $(f_n) \cup (g_n)$*?*

Theorem 8. *An affirmative answer to Problem 6 implies an affirmative answer to Problem 7.*

Before the proof, we provide with some necessary information. Given an infinite set I, $i \in I$, $x \in \{-1,1\}^{I \setminus \{i\}}$, and $\theta \in \{-1,1\}$, we denote by $\theta \times x$ the element $y \in \{-1,1\}^I$ such that $y(i) = \theta$ and $y(j) = x(j)$ for all $j \in I \setminus \{i\}$. Following [13], a μ_I-measurable function $f : \{-1,1\}^I \to \mathbb{R}$ is said to be *independent* of $i \in I$, if $f(1 \times x) = f(-1 \times x)$ for $\mu_{I \setminus \{i\}}$-almost all values of $x \in \{-1,1\}^{I \setminus \{i\}}$. In the opposite case we say that f *depends* on i. For any measurable function $f : \{-1,1\}^I \to \mathbb{R}$, the set $\{i \in I : f \text{ depends on } i\}$ is at most countable. By the obvious reason, the same terminology we apply to equivalence classes of measurable functions.

Proof. Let $(f_\alpha)_{\alpha<\omega_1}$ be an unconditional basis of $L_p\{-1,1\}^{\omega_1}$ with unconditional constant M. For any $\alpha < \omega_1$ we denote by X_α the subspace of all $f \in L_p\{-1,1\}^{\omega_1}$ depending on coordinates $< \alpha$ only. Obviously, X_α is isometrically isomorphic to $L_p\{-1,1\}^{\alpha}$, which is separable and atomless, and hence, isometrically isomorphic to L_p.

Lemma 1. *There exists a strictly increasing* ω_1*-sequence of limited ordinals* $(\xi_\gamma)_{\gamma<\omega_1}$, $\xi_\gamma < \omega_1$, *such that* $[f_\alpha]_{\alpha<\xi_\gamma} = X_{\xi_\gamma}$ *for all* $\gamma < \omega_1$.

Proof of Lemma 1. Since every function $f \in L_p\{-1,1\}^{\omega_1}$ depends on at most countable set of ordinals $\alpha < \omega_1$, for every separable subspace Z of $L_p\{-1,1\}^{\omega_1}$ the value $\varphi(Z) = \min\{\alpha < \omega_1 : Z \subseteq X_\alpha\}$ is well defined. Then

$$Z \subseteq X_{\varphi(Z)} \quad \text{for every separable subspace } Z. \tag{3}$$

Since every function $f \in L_p\{-1,1\}^{\omega_1}$ has an expansion $f = \sum_{\alpha<\omega_1} a_\alpha f_\alpha$, where the set $\{\alpha < \omega_1 : a_\alpha \neq 0\}$ is at most countable, for every separable subspace Z of $L_p\{-1,1\}^{\omega_1}$ the value $\psi(Z) = \min\{\beta < \omega_1 : Z \subseteq [f_\alpha]_{\alpha<\beta}\}$ is well defined as well. Then

$$Z \subseteq [f_\alpha]_{\alpha<\psi(Z)} \text{ for every separable subspace } Z. \tag{4}$$

Now define recursively ω_1-sequences $(\alpha_\eta)_{\eta<\omega_1}$ and $(\beta_\eta)_{\eta<\omega_1}$ possessing the following properties for every $\eta < \zeta < \omega_1$:

1. $\alpha_\eta \leq \beta_\eta < \alpha_\zeta$;
2. $[f_\alpha]_{\alpha<\alpha_\eta} \subseteq X_{\beta_\eta} \subseteq [f_\alpha]_{\alpha<\alpha_\zeta}$.

Set $\alpha_0 = \omega_0$ and $\beta_0 = \max\{\varphi([f_\alpha]_{\alpha<\alpha_0}), \omega_0\}$. Then $\alpha_0 \leq \beta_0$ and $[f_\alpha]_{\alpha<\alpha_0} \subseteq X_{\beta_0}$. Given any $\delta < \omega_1$, we assume that δ-sequences $(\alpha_\eta)_{\eta<\delta}$ and $(\beta_\eta)_{\eta<\delta}$ possessing 1 and 2 for every $\eta < \zeta < \delta$ have been already constructed. To define α_δ and β_δ, we consider cases.

(i) δ is an isolated ordinal, that is, $\delta = \delta' + 1$. In this case we set

$$\alpha_\delta = \max\{\psi(X_{\beta_{\delta'}}), \beta_{\delta'} + 1\} \text{ and } \beta_\delta = \max\{\varphi([f_\alpha]_{\alpha<\alpha_\delta}), \alpha_\delta\}.$$

(ii) δ is a limited ordinal. In this case we set

$$\alpha_\delta = \beta_\delta = \bigcup_{\eta<\delta} \alpha_\eta = \bigcup_{\eta<\delta} \beta_\eta$$

(the latter equality is guaranteed by property 1 for every $\eta < \zeta < \delta$).

Property 1 for $\eta < \zeta \leq \delta$ follows directly from the construction. To prove 2, observe that in case (i) by (3), (4), $X_{\beta_{\delta'}} \subseteq [f_\alpha]_{\alpha<\psi(X_{\beta_{\delta'}})} \subseteq [f_\alpha]_{\alpha<\alpha_\delta}$ and $[f_\alpha]_{\alpha<\alpha_\delta} \subseteq X_{[f_\alpha]_{\alpha<\alpha_\delta}} \subseteq X_{\beta_\delta}$. In case (ii) inclusions 2 are obvious. Thus, the desired ω_1-sequences $(\alpha_\eta)_{\eta<\omega_1}$ and $(\beta_\eta)_{\eta<\omega_1}$ are constructed.

By (ii) and 2, for every limited ordinal $\delta < \omega_1$ one has $[f_\alpha]_{\alpha<\alpha_\delta} = X_{\alpha_\delta}$. By (ii), for every limited ordinal $\delta < \omega_1$, the ordinal α_δ is limited as well. Since there are uncountably many such ordinals, we can renumber them to obtain the desired ω_1-sequence. □

Lemma 2. *Let $I \subset J$ be countable subsets with $J \setminus I$ infinite. Then there is an isometric isomorphism $T : E_p \to L_p\{-1,1\}^J$ such that $T(F_p)$ equals the subspace of $L_p\{-1,1\}^J$ consisting of all functions which depend on coordinates $i \in I$ only.*

Proof of Lemma 2. It is straightforward that the linear span of the Walsh system $(w_A)_{A\in\mathbb{N}^{<\omega}}$ coincides with that of the Haar system, hence it is dense in L_p. So, to define an isometrical isomorphism on the entire $L_p(\mu)$, it is enough to define it on the Walsh system and prove that it is an isometry on the linear span. Observe that the Walsh system in $E_p = L_p[0,1]^2$ is given by $w_A(x)w_B(y)$, where A, B are finite subsets of $\mathbb{N}$.

Let $I = \{i_1, i_2, \ldots\}$ and $J \setminus I = \{j_1, j_2, \ldots\}$ be any numerations. Given any $A, B \in \mathbb{N}^{<\omega}$, we define functions $\widehat{w}'_A, \widehat{w}''_B : \{-1,1\}^J \to \mathbb{R}$ by setting $\widehat{w}'_A(x) = \prod_{n\in A} x(i_n)$ and $\widehat{w}''_B(x) = \prod_{n\in B} x(j_n)$. Likewise, the Walsh system in $L_p\{-1,1\}^J$ can be represented as follows: $\widehat{w}'_A \cdot \widehat{w}''_B$, $A, B \in \mathbb{N}^{<\omega}$. Now we define $T : E_p \to L_p\{-1,1\}^J$, first on the Walsh system by $Tw_A(x)w_B(y) = \widehat{w}'_A \cdot \widehat{w}''_B$ for all $A, B \in \mathbb{N}^{<\omega}$, and then extend to the linear span of the Walsh system W by linearity. We omit a routine proof that the obtained mapping is an isometry on W. It remains to observe that $T(F_p) = L_p\{-1,1\}^I$. □

We continue the proof of the theorem. Take a sequence $(\xi_\gamma)_{\gamma<\omega_1}$ satisfying the claims of Lemma 1. Denote by M_γ the unconditional constant of the system $(f_\alpha)_{\alpha<\xi_\gamma}$. Then $M_\gamma \uparrow M$. Since there is no strictly increasing ω_1-sequence of reals, we obtain that there is $\gamma_0 < \omega_1$ such that $M_\gamma = M$ for all $\gamma_0 \le \gamma < \omega_1$. Choose by Lemma 2 an isometric isomorphism $T : E_p \to X_{\xi_{\gamma_0+1}}$ with $T(F_p) = X_{\xi_{\gamma_0}}$. Since $(f_\alpha)_{\alpha<\xi_{\gamma_0+1}} = (f_\alpha)_{\alpha<\xi_{\gamma_0}} \cup (f_\alpha)_{\xi_{\gamma_0}\le\alpha<\xi_{\gamma_0+1}}$ is an unconditional basis of $X_{\xi_{\gamma_0+1}}$ with unconditional constant M and $(f_\alpha)_{\alpha<\xi_{\gamma_0}}$ is an unconditional basis of $X_{\xi_{\gamma_0}}$ with with the same unconditional constant M, we obtain that $(T^{-1}f_\alpha)_{\alpha<\xi_{\gamma_0+1}} = (T^{-1}f_\alpha)_{\alpha<\xi_{\gamma_0}} \cup (T^{-1}f_\alpha)_{\xi_{\gamma_0}\le\alpha<\xi_{\gamma_0+1}}$ is an unconditional basis of $T^{-1}(X_{\xi_{\gamma_0+1}}) = E_p$ with unconditional constant M and $(T^{-1}f_\alpha)_{\alpha<\xi_{\gamma_0}}$ is an unconditional basis of $T^{-1}(X_{\xi_{\gamma_0}}) = F_p$ with with the same unconditional constant M. $\square$

Remarks.

1. In Problem 7, one can equivalently replace the unconditional constants of unconditional bases with the supremum of norms of projections with respect to the bases.

2. We do not know of whether an affirmative solution to Problem 7 formally implies an affirmative solution to Problem 6, however, an affirmative solution to Problem 7 would give a possible way to construct an unconditional basis of $L_p\{-1,1\}^{\omega_1}$ by a recursive procedure.

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Дану замітку написано при підготовці доповіді на міжнародній конференції, присвяченій 70-річчю професора О. Лопушанського, 16-19 вересня 2019 р. Ми зосереджуємося на двох давніх відкритих проблемах. Перша, що належить Лінденштраусу і Розенталю (1969 р.), формулюється так: чи кожний доповнювальний нескінченновимірний підпростір простору L_1 ізоморфний до L_1 чи до ℓ_1? Друга проблема була поставлена Енфло і Розенталем у 1973 р.: чи існує несепарабельний простір $L_p(\mu)$ зі скінченною безатомною мірою μ та $1 < p < \infty$, $p \neq 2$, з безумовним базисом? У замітці наведено аналіз часткових результатів та природних ідей розв'язання даних проблем.

Ключові слова і фрази: простори L_p, доповнювальний підпростір, безумовний базис.

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THE GENERALIZED CENTRALLY EXTENDED LIE ALGEBRAIC STRUCTURES AND RELATED INTEGRABLE HEAVENLY TYPE EQUATIONS

There are studied Lie-algebraic structures of a wide class of heavenly type non-linear integrable equations, related with coadjoint flows on the adjoint space to a loop vector field Lie algebra on the torus. These flows are generated by the loop Lie algebras of vector fields on a torus and their coadjoint orbits and give rise to the compatible Lax-Sato type vector field relationships. The related infinite hierarchy of conservations laws is analysed and its analytical structure, connected with the Casimir invariants, is discussed. We present the typical examples of such equations and demonstrate in details their integrability within the scheme developed. As examples, we found and described new multidimensional generalizations of the Mikhalev-Pavlov and Alonso-Shabat type integrable dispersionless equation, whose seed elements possess a special factorized structure, allowing to extend them to the multidimensional case of arbitrary dimension.

Key words and phrases: heavenly type equations, Lax integrability, Hamiltonian system, torus diffeomorphisms, loop Lie algebra, central extension, Lie-algebraic scheme, Casimir invariants, Lie-Poisson structure, R-structure, Mikhalev-Pavlov equations.

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1 INTRODUCTION

The main object of our study are integrable multidimensional dispersionless differential equations, which possess modified Lax-Sato type representations, related with their hidden Hamiltonian structures. Equations of this type arise and widely applied in mechanics, general relativity, differential geometry and the theory of integrable systems. Among the most one can mention the Boyer-Finley equation, heavenly type Plebański equations, which are descriptive of a class of self-dual four-manifolds, as well as the dispersionless Kadomtsev-Petviashvili (dKP) equation, also known as the Khokhlov-Zabolotskaya equation, which arises in non-linear acoustics and the theory of Einstein-Weyl structures. Their integrability have been investigated by a whole variety of modern techniques including symmetry analysis, differential-geometric and algebro-geometric methods, dispersionless $\bar{\partial}$-dressing, factorization techniques, Virasoro constraints, hydrodynamic reductions, etc. The first examples and the importance of the related Hamiltonian structures were before demonstrated in [29, 36, 38] and later were developed in [25, 43], where there were analyzed in detail many examples of dispersionless differential equations as flows on orbits of the coadjoint action of loop vector

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field algebras $\widetilde{diff}(\mathbb{T}^n)$, generated by specially chosen seed elements $\tilde{l} \in \widetilde{diff}(\mathbb{T}^n)^*$. In these works there was observed that many integrable multidimensional dispersionless differential equations are generated by seed elements of a very special structure, namely for them there exist such analytical functional elements $\tilde{\eta}, \tilde{\rho} \in \Lambda^0(C^\infty(\mathbb{T}^n; \mathbb{R})) \otimes \mathbb{C}$ that $\tilde{l} = \tilde{\eta} d\tilde{\rho}$. As the latter naturally generates the symplectic structure $\tilde{\omega}^{(2)} := \int_{\mathbb{T}^n} d\tilde{\eta} \wedge d\tilde{\rho} \in \Lambda^2(\mathbb{T}^n) \otimes \mathbb{C}$ on the moduli space [2, 42] of flat connections on $\mathbb{T}^n$, related to coadjoint actions of the corresponding Casimir functionals, the geometric nature of many integrable multidimensional dispersionless differential equations can be also studied using cohomological techniques, devised in [2, 10] in the case of Riemannian surfaces. It is worth also to mention a revealed in [25] deep connection of the related Hamiltonian flows on $\widetilde{diff}(\mathbb{T}^n)^*$ with the well known in classical mechanics Lagrange–d'Alembert principle.

In this article, in part developing the approach, devised in [29, 38], we describe a Lie algebraic structure and integrability properties of a generalized hierarchy of the Lax-Sato type compatible systems of Hamiltonian flows and related integrable multidimensional dispersionless differential equations. Such systems are called the heavenly type equations and were first introduced by Plebański in [41]. The heavenly type equations were analyzed in many articles (see, e.g., [16, 19–22, 32, 38, 39] and [40, 46, 47, 52, 53]) using several different approaches. In [7–9, 50] the heavenly type equations were analyzed by using nonassociative and noncommutative current algebras on the torus $\mathbb{T}^m, m \in \mathbb{N}$. Mention also that [49, 51] B. Szablikowski and A. Sergyeyev developed some generalizations of the classical AKS-algebraic and related R-structures [11, 13, 15, 45, 54]. In [38, 39] and recently in [25] these ideas were applied to a semi-direct Lie algebra $\mathcal{T}^n)^*$ of the loop Lie algebra $\widetilde{diff}(\mathbb{T}^n) := \widetilde{Vect}(\mathbb{T}^n)$ of vector fields on the torus $\mathbb{T}^n, n \in \mathbb{Z}_+$, and its dual space $\widetilde{diff}(\mathbb{T}^n)^*$. Several interesting and deep results about orbits of the corresponding coadjoint actions on the space $\tilde{\mathcal{G}}^* \simeq \tilde{\mathcal{G}}$ and the classical Lie-Poisson type structures on them were presented. It is worth to specially remark here that the AKS-algebraic scheme is naturally imbedded into the classical R-structure approach via the following construction.

Let $(\tilde{\mathcal{G}}; [\cdot, \cdot])$ denote a Lie algebra over $\mathbb{C}$ and $\tilde{\mathcal{G}}^*$ be its natural adjoint space. Take some tensor element $r \in \tilde{\mathcal{G}} \otimes \tilde{\mathcal{G}} \simeq Hom(\tilde{\mathcal{G}}^*; \tilde{\mathcal{G}})$ and consider its splitting into symmetric and anti-symmetric parts

$$r = k \oplus \sigma,$$

respectively, and assume that the symmetric tensor $k \in \tilde{\mathcal{G}} \otimes \tilde{\mathcal{G}}$ is not degenerate. That allows to define on the Lie algebra $\tilde{\mathcal{G}}$ a symmetric nondegenerate bi-linear product $(\cdot|\cdot) : \tilde{\mathcal{G}} \otimes \tilde{\mathcal{G}} \to \mathbb{C}$ via the expression

$$(a|b) := k^{-1}a(b) \tag{1}$$

for any $a, b \in \tilde{\mathcal{G}}$. The composed mapping $R := \sigma \circ k^{-1} : \tilde{\mathcal{G}} \to \tilde{\mathcal{G}}$, following the scheme $\tilde{\mathcal{G}} \xrightarrow{k^{-1}} \tilde{\mathcal{G}}^* \xrightarrow{\sigma} \tilde{\mathcal{G}}$, defines the following R-structure on the Lie algebra $\tilde{\mathcal{G}}$:

$$[a, b]_R := [Ra, b] + [a, Rb]$$

for all elements $a, b \in \tilde{\mathcal{G}}$. The following theorem, defining the related Poisson structure [10, 12, 45, 48] on the adjoint space $\tilde{\mathcal{G}}$ holds.

Theorem 1. *Let $\alpha, \beta \in \tilde{\mathcal{G}}^*$ be arbitrary and define the bracket*

$$\{\alpha, \beta\} := ad^*_{r\alpha}\beta - ad^*_{r\beta}\alpha. \tag{2}$$

Then the bracket (2) *is Poisson if and only if the R-structure on the Lie algebra $\tilde{\mathcal{G}}$ defines the Lie structure on $\tilde{\mathcal{G}}$, that is there holds the Yang-Baxter equation*

$$[Ra, Rb] - R[a, b]_R = -[a, b]$$

for any $a, b \in \tilde{\mathcal{G}}$.

The above theorem makes it possible to consider the Hamiltonian flows on the coadjoint space $\tilde{\mathcal{G}}^*$ as those determined on the Lie algebra $\tilde{\mathcal{G}}$. The latter is exceptionally useful if for the scalar product (1) there exists such a trace-type $Tr(\cdot)$ symmetric and ad-invariant functional (of Killing type) that

$$Tr(ab) := (a|b), \quad (a|[b,c]) = (([a,b]|, c)$$

for any a, b and $c \in \tilde{\mathcal{G}}$. Then any Hamiltonian flow of an element $a \in \tilde{\mathcal{G}}$ is representable in the standard Lax type form

$$da/dt = [\nabla(h), a],$$

where $\nabla(h) \in \tilde{\mathcal{G}}$ is generated by the corresponding Gateaux derivative of the corresponding smooth Hamiltonian function $h \in \mathcal{D}(\tilde{\mathcal{G}})$.

Concerning the loop Lie algebra $\tilde{\mathcal{G}} := \widetilde{diff}(\mathbb{T}^n)$ on the torus $\mathbb{T}^n$, it is well known that such a trace-type functional on $\tilde{\mathcal{G}}$ does not exist, thus we need to study the Hamiltonian flows on the adjoint loop space $\tilde{\mathcal{G}}^* \simeq \tilde{\Lambda}^1(\mathbb{T}^n)$ of meromorphic differential forms on the torus $\mathbb{T}^n$ and obtain, as a result, integrable dispersionless differential equations as compatibility conditions for the related loop vector fields, generated by Casimir functionals on $\tilde{\mathcal{G}}^*$. This procedure is much more complicated for analysis than the standard one and employs more geometrical tools and considerations about the orbit space structure of the seed elements $\tilde{l} \in \tilde{\mathcal{G}}^*$, generating a hierarchy of integrable Hamiltonian flows. The latter, in part, is deeply related to its reduction properties, guaranteeing the existence of nontrivial Casimir invariants on its coadjoint orbits.

By applying and extending these ideas to central extensions of Lie algebras, we construct new classes of commuting Hamiltonian flows on an extended adjoint space $\bar{\mathcal{G}} := \tilde{\mathcal{G}}^* \oplus \mathbb{C}$. These Hamiltonian flows are generated by seed elements $(\tilde{a} \ltimes \tilde{l}; \alpha) \in \bar{\mathcal{G}}^*$ and specially constructed Casimir invariants on the corresponding orbits of $\tilde{\mathcal{G}}^*$. In most cases these seed elements appeared to be represented as specially factorized differential objects, whose real geometric nature is still much hidden and not clear. Moreover, we found that the corresponding compatibility condition of constructed Hamiltonian flows coincides exactly with the compatibility condition for a system of related three Lax-Sato type linear vector field equations. As examples, we found and described new multidimensional generalizations of the Mikhalev-Pavlov and Alonso-Shabat type integrable dispersionless equation, whose seed elements possess a special factorized structure, allowing to extend them to the multidimensional case of arbitrary dimension.

2 DIFFEOMORPHISMS GROUP $Diff(\mathbb{T}^n)$ AND ITS DESCRIPTION

Consider the n-dimensional torus $\mathbb{T}^n$ and call points $X \in \mathbb{T}^n$ as the Lagrangian variables of a configuration $\eta \in Diff(\mathbb{T}^n)$. The manifold $\mathbb{T}^n$, thought of as the target space of a configuration $\eta \in Diff(\mathbb{T}^n)$, is called the spatial or Eulerian configuration, whose points, called spatial or Eulerian points, will be denoted by small letters $x \in \mathbb{T}^n$. Then any one-parametric configuration of $Diff(\mathbb{T}^n)$ is a time $t \in \mathbb{R}$ dependent family [1,4,6,28,34] of diffeomorphisms written as

$$\mathbb{T}^n \ni x = \eta(X,t) := \eta_t(X) \in \mathbb{T}^n$$

for any initial configuration $X \in \mathbb{T}^n$ and some mappings $\eta_t \in Diff(\mathbb{T}^n), t \in \mathbb{R}$.

Being interested in studying flows on the space of Lagrangian configurations $\eta \in Diff(\mathbb{T}^n)$ with respect to the temporal variable $t \in \mathbb{R}$, which are generated by group diffeomorphisms $\eta_t \in Diff(\mathbb{T}^n)$, $t \in \mathbb{R}$, let us proceed to describing the structure of tangent $T_{\eta_t}(Diff(\mathbb{T}^n))$ and cotangent $T^*_{\eta_t}(Diff(\mathbb{T}^n))$ spaces to the diffeomorphism group $Diff(\mathbb{T}^n)$ at the points $\eta_t \in Diff(\mathbb{T}^n)$ for any $t \in \mathbb{R}$. Determine first the tangent space $T_{\eta_t}(Diff(\mathbb{T}^n))$ to the diffeomorphism group manifold $Diff(\mathbb{T}^n)$ at point $\eta \in Diff(\mathbb{T}^n)$ for which we will make use of the construction, devised before in [1,4,27]. Namely, let $\eta \in Diff(\mathbb{T}^n)$ be a Lagrangian configuration and try to determine the tangent space $T_\eta(Diff(\mathbb{T}^n))$ at $\eta \in Diff(\mathbb{T}^n)$ as the collection of vectors $\xi_\eta := d\eta_\tau/d\tau|_{\tau=0}$, where $\mathbb{R} \ni \tau \to \eta_\tau \in Diff(\mathbb{T}^n)$, $\eta_\tau|_{\tau=0} = \eta$, is a smooth curve on $Diff(\mathbb{T}^n)$, and for arbitrary reference point $X \in \mathbb{T}^n$ there holds $\xi_\eta(X) = d\eta_\tau(X)/d\tau|_{\tau=0}$. The latter equivalently means that the vectors $\xi_\eta(X) \in T_{\eta(X)}(\mathbb{T}^n)$, $X \in \mathbb{T}^n$, represent a vector field $\xi : \mathbb{T}^n \to T(\mathbb{T}^n)$ on the manifold $\mathbb{T}^n$ for any $\eta \in Diff(\mathbb{T}^n)$. Thus, the tangent space $T_\eta(Diff(\mathbb{T}^n))$ coincides with the set of vector fields on $\mathbb{T}^n$:

$$T_\eta(Diff(\mathbb{T}^n)) \simeq \{\xi_\eta \in \Gamma(T(\mathbb{T}^n)) : \xi_\eta(X) \in T_{\xi(X)}(\mathbb{T}^n)\}$$

and similarly, the cotangent space $T^*_\eta(Diff(\mathbb{T}^n))$ consists of all one-form densities on $\mathbb{T}^n$ over $\eta \in Diff(\mathbb{T}^n)$:

$$T^*_\eta(Diff(\mathbb{T}^n)) = \{\alpha_\eta \in \Lambda^1(\mathbb{T}^n) \otimes \Lambda^3(\mathbb{T}^n) : \alpha_\eta(X) \in T^*_{\eta(X)}(\mathbb{T}^n) \otimes |\Lambda^3(\mathbb{T}^n)|\}$$

subject to the canonical nondegenerate pairing $(\cdot|\cdot)_c$ on $T^*_\eta(Diff(\mathbb{T}^n)) \times T_\eta(Diff(\mathbb{T}^n))$: if $\alpha_\eta \in T^*_\eta(Diff(\mathbb{T}^n))$, $\xi_\eta \in T_\eta(Diff(\mathbb{T}^n))$, where

$$\alpha_\eta|_X = \langle \alpha_\eta(X)|dx\rangle \otimes d^3X, \quad \xi_\eta|_X = \langle \xi_\eta(X)|\partial/\partial x\rangle,$$

then

$$(\alpha_\eta|\xi_\eta)_c := \int_{\mathbb{T}^n} \langle \alpha_\eta(X)|\xi_\eta(X)\rangle d^3X.$$

The construction above makes it possible to identify the cotangent bundle $T^*_\eta(Diff(\mathbb{T}^n))$ at the fixed Lagrangian configuration $\eta \in Diff(\mathbb{T}^n)$ to the tangent space $T_\eta(Diff(\mathbb{T}^n))$, as the tangent space $T(\mathbb{T}^n)$ is endowed with the natural internal tangent bundle metric $\langle\cdot|\cdot\rangle$ at any point $\eta(X) \in \mathbb{T}^n$, identifying $T(\mathbb{T}^n)$ with $T^*(\mathbb{T}^n)$ via the related metric isomorphism $\sharp : T^*(\mathbb{T}^n) \to T(\mathbb{T}^n)$. The latter can be also naturally lifted to $T^*_\eta(Diff(\mathbb{T}^n))$ at $\eta \in Diff(\mathbb{T}^n)$, namely: for any elements $\alpha_\eta, \beta_\eta \in T^*_\eta(Diff(\mathbb{T}^n)), \alpha_\eta|_X = \langle \alpha_\eta(X)|dx\rangle \otimes d^3X$ and $\beta_\eta|_X = \langle \beta_\eta(X)|dx\rangle \otimes d^3X \in T^*_\eta(Diff(\mathbb{T}^n))$ we can define the metric

$$(\alpha_\eta|\beta_\eta) := \int_{\mathbb{T}^n} \langle \alpha^\sharp_\eta(X)|\beta^\sharp_\eta(X)\rangle d^3X,$$

where, by definition, $\alpha^\sharp_\eta(X) := \sharp\langle \alpha_\eta(X)|dx\rangle)$, $\beta^\sharp_\eta(X) := \sharp\langle \beta_\eta(X)|dx\rangle \in T_{\eta(X)}(\mathbb{T}^n)$ for any $X \in \mathbb{T}^n$. Based on the notions above one can proceed to constructing smooth invariant functionals on the cotangent bundle $T^*(Diff(\mathbb{T}^n))$ subject to the corresponding co-adjoint actions of the diffeomorphism group $Diff(\mathbb{T}^n)$. Moreover, as the cotangent bundle $T^*(Diff(\mathbb{T}^n))$ is *a priori* endowed with the canonical symplectic structure, equivalent [1,4,5,11,13,26,30,31,34,45] to the corresponding Poisson bracket on the space of smooth functionals on $T^*(Diff(\mathbb{T}^n))$, one can study both the related Hamiltonian flows on it and their adjoint symmetries and complete integrability.

Consider now the cotangent bundle $T^*(Diff(\mathbb{T}^n))$ as a smooth manifold endowed with the canonical symplectic structure [1,5] on it, equivalent to the corresponding canonical Poisson bracket on the space of smooth functionals on it. Taking into account that the cotangent space $T^*_\eta(Diff(\mathbb{T}^n))$ at $\eta \in Diff(\mathbb{T}^n)$, shifted by the right $R_{\eta^{-1}}$- action to the space $T^*_{Id}(Diff(\mathbb{T}^n))$, $Id \in Diff(\mathbb{T}^n)$, becomes diffeomorphic to the adjoint space $diff^*(\mathbb{T}^n)$ to the Lie algebra $diff(\mathbb{T}^n) \simeq \Gamma(T(\mathbb{T}^n))$ of vector fields on $\mathbb{T}^n$, as there was stated [34,35,56,57] still by S. Lie in 1887, this canonical Poisson bracket on $T^*_\eta(Diff(\mathbb{T}^n))$ transforms [4,5,24,31,33,34,55–57] into the classical Lie-Poisson bracket on the adjoint space $\mathcal{G}^*$. Moreover, the orbits of the diffeomorphism group $Diff(\mathbb{T}^n)$ on $T^*(Diff(\mathbb{T}^n))$ respectively transform into the coadjoint orbits on the adjoint space $\mathcal{G}^*$, generated by suitable elements of the Lie algebra $\mathcal{G}$. To construct in detail this Lie-Poisson bracket, we formulate preliminary the following simple lemma.

Lemma 1. *The Lie algebra $diff(\mathbb{T}^n) \simeq \Gamma(T(\mathbb{T}^n))$ is determined by the following Lie commutator relationships:*

$$[a_1, a_2] = \langle a_1|\nabla\rangle a_2 - \langle a_2|\nabla\rangle a_1 \tag{3}$$

for any vector fields $a_1, a_2 \in \Gamma(T(\mathbb{T}^n))$ on the manifold $\mathbb{T}^n$.

Proof. Proof of the commutation relationships (3) easily follows from the group multiplication

$$(\varphi_{1,t} \circ \varphi_{2,t})(X) = \varphi_{2,t}(\varphi_{1,t}(X))$$

for any local group diffeomorphisms $\varphi_{1,t}, \varphi_{2,t} \in Diff(\mathbb{T}^n), t \in \mathbb{R}$, and $X \in \mathbb{T}^n$ under condition that $a_j(X) := d\varphi_{j,t}(X)/dt|_{t=0}$ and $\varphi_{j,t}|_{t=0} = Id \in Diff(\mathbb{T}^n), j = \overline{1,2}$. □

To calculate the Poisson bracket on the cotangent space $T^*_\eta(Diff(\mathbb{T}^n))$ at any $\eta \in Diff(\mathbb{T}^n)$, let us consider the cotangent space $T^*_\eta(Diff(\mathbb{T}^n)) \simeq diff^*(\mathbb{T}^n)$, the adjoint space to the tangent space $T_\eta(Diff(\mathbb{T}^n))$ of left invariant vector fields on $Diff(\mathbb{T}^n)$ at any $\eta \in Diff(\mathbb{T}^n)$, and take the canonical symplectic structure on $T^*_\eta(Diff(\mathbb{T}^n))$ in the form $\omega^{(2)}(\mu,\eta) := \delta\alpha(\mu,\eta)$, where the canonical Liouville form $\alpha(\mu,\eta) := (\mu|\delta\eta)_c \in \Lambda^1_{(\mu,\eta)}(T^*_\eta(Diff(\mathbb{T}^n)))$ at a point $(\mu,\eta) \in T^*_\eta(Diff(\mathbb{T}^n))$ is defined *a priori* on the tangent space $T_\eta(Diff(\mathbb{T}^n)) \simeq \Gamma(T(M))$ of right-invariant vector fields on the torus manifold $\mathbb{T}^n$. Having calculated the corresponding Poisson bracket of smooth functions $(\mu|a)_c, (\mu|b)_c \in C^\infty(T^*_\eta(Diff(\mathbb{T}^n)); \mathbb{R})$ on $T^*_\eta(Diff(\mathbb{T}^n)) \simeq diff^*(\mathbb{T}^n), \eta \in Diff(\mathbb{T}^n),$ one can formulate the following proposition.

Proposition 1. *The Lie-Poisson bracket on the coadjoint space $T^*_\eta(Diff(\mathbb{T}^n)), \eta \in M$, is equal to the expression*

$$\{f,g\}(\mu) = (\mu|[\delta g(\mu)/\delta\mu, \delta f(\mu)/\delta\mu])_c \tag{4}$$

for any smooth right-invariant functionals $f,g \in C^\infty(\mathcal{G}^*; \mathbb{R})$.

Proof. By definition (see [1,5]) of the Poisson bracket of smooth functions $(\mu|a)_c, (\mu|b)_c \in C^\infty(T^*_\eta(Diff(\mathbb{T}^n)); \mathbb{R})$ on the symplectic space $T^*_\eta(Diff(\mathbb{T}^n))$, it is easy to calculate that

$$\{\mu(a), \mu(b)\} := \delta\alpha(X_a, X_b) = X_a(\alpha|X_b)_c - X_b(\alpha|X_a)_c - (\alpha|[X_a, X_b])_c, \tag{5}$$

where $X_a := \delta(\mu|a)_c/\delta\mu = a \in diff(\mathbb{T}^n), X_b := \delta(\mu|b)_c/\delta\mu = b \in diff(\mathbb{T}^n)$. Since the expressions $X_a(\alpha|X_b)_c = 0$ and $X_b(\alpha|X_a)_c = 0$ owing the right-invariance of the vector fields $X_a, X_b \in T_\eta(Diff(\mathbb{T}^n))$, the Poisson bracket (5) transforms into

$$\{(\mu|a)_c, (\mu|b)_c\} = -(\alpha|[X_a, X_b])_c = (\mu|[b,a])_c = (\mu|[\delta(\mu|b)_c/\delta\mu, \delta(\mu|a)_c/\delta\mu])_c$$

for all $(\mu,\eta) \in T^*_\eta(Diff(\mathbb{T}^n)) \simeq diff^*(\mathbb{T}^n), \eta \in Diff(\mathbb{T}^n)$ and any $a,b \in diff(\mathbb{T}^n)$. The Poisson bracket (5) is easily generalized to

$$\{f,g\}(\mu) = (\mu|[\delta g(\mu)/\delta\mu, \delta f(\mu)/\delta\mu])_c$$

for any smooth functionals $f,g \in C^\infty(\mathcal{G}^*;\mathbb{R})$, finishing the proof. □

Based on the Lie-Poisson bracket (4), one can naturally construct Hamiltonian flows on the adjoint space $diff^*(\mathbb{T}^n)$ via the expressions

$$\partial l/\partial t = -ad^*_{\nabla h(l)} l$$

for any element $l \in diff^*(\mathbb{T}^n), t \in \mathbb{R}$, where, by definition, $\frac{d}{d\varepsilon} h(l+\varepsilon m)|_{\varepsilon=0} := (m|\nabla h(l))_c$, for some smooth Hamiltonian function $h \in C^\infty(diff^*(\mathbb{T}^n);\mathbb{R})$. If the system possesses enough additional invariants except the Hamiltonian function, one can expect its simplification often reducing to its complete integrability. Below we proceed to developing an effective enough analytical scheme, before suggested in [25, 37] for suitably constructed holomorphic loop diffeomorphism groups on tori, allowing to generate infinite hierarchies of such completely integrable Hamiltonian systems on related functional phase spaces.

3 HEAVENLY TYPE SYSTEMS: THE MODIFIED LIE-ALGEBRAIC INTEGRABILITY SCHEME

Let $\widetilde{Diff}_\pm(\mathbb{T}^n)$, $n \in \mathbb{Z}_+$, be subgroups of the loop diffeomorphisms group $\widetilde{Diff}(\mathbb{T}^n) := \{\mathbb{C} \supset \mathbb{S}^1 \to Diff(\mathbb{T}^n)\}$, holomorphically extended, respectively, on the interior $\mathbb{D}^1_+ \subset \mathbb{C}$ and on the exterior $\mathbb{D}^1_- \subset \mathbb{C}$ regions of the unit centrally located disk $\mathbb{D}^1 \subset \mathbb{C}^1$ and such that for any $\tilde{g}(\lambda) \in \widetilde{Diff}_-(\mathbb{T}^n)$, $\lambda \in \mathbb{D}^1_-$, $\tilde{g}(\infty) = 1 \in Diff(\mathbb{T}^n)$. The corresponding Lie subalgebras $\widetilde{diff}_\pm(\mathbb{T}^n) \simeq \widetilde{Vect}_\pm(\mathbb{T}^n)$ of the loop subgroups $\widetilde{Diff}_\pm(\mathbb{T}^n)$ are vector fields on $\mathbb{S}^1 \times \mathbb{T}^n$, extended holomorphically, respectively, on regions $\mathbb{D}^1_\pm \subset \mathbb{C}^1$, where for any $\tilde{a}(\lambda) \in \widetilde{diff}_-(\mathbb{T}^n)$ the value $\tilde{a}(\infty) = 0$. The loop Lie algebra splitting $\widetilde{diff}(\mathbb{T}^n) = \widetilde{diff}_+(\mathbb{T}^n) \oplus \widetilde{diff}_-(\mathbb{T}^n)$ can be naturally identified with a dense subspace of the dual space $\widetilde{diff}(\mathbb{T}^n)^*$ through the pairing

$$(\tilde{l}|\tilde{a}) := \underset{\lambda\in\mathbb{C}}{res}(l(x;\lambda)|a(x;\lambda))_{H^0} \tag{6}$$

with respect to the scalar product

$$(l(x;\lambda)|a(x;\lambda))_{H^0} := \int_{\mathbb{T}^n} dx \langle l(x;\lambda), a(x;\lambda)\rangle$$

on the usual Hilbert space $H^0 := L_2(\mathbb{T}^n;\mathbb{C}^n)$ for any elements $\tilde{l} \in \widetilde{diff}(\mathbb{T}^n)^*$ and $\tilde{a} \in \widetilde{diff}(\mathbb{T}^n)$, naturally represented in their reduced canonical form

$$\tilde{a} = \sum_{j=1}^n a^{(j)}(x;\lambda)\frac{\partial}{\partial x_j} := \left\langle a(x;\lambda), \frac{\partial}{\partial x}\right\rangle,$$

$$\tilde{l} = \sum_{j=1}^n l_j(x;\lambda)dx_j := \langle l(x;\lambda), dx\rangle,$$

where we have introduced for brevity the gradient operator $\frac{\partial}{\partial x} := \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, ..., \frac{\partial}{\partial x_n}\right)^{\intercal}$ in the Euclidean space $(E^n; \langle\cdot,\cdot\rangle)$. The corresponding Lie commutator $[\tilde{a}, \tilde{b}] \in \widetilde{diff}(\mathbb{T}^n)$ of any vector fields $\tilde{a}, \tilde{b} \in \widetilde{diff}(\mathbb{T}^n)$ is calculated the standard way and equals

$$[\tilde{a}, \tilde{b}] = \tilde{a}\tilde{b} - \tilde{b}\tilde{a} = \left\langle \left\langle a(x;\lambda), \frac{\partial}{\partial x}\right\rangle b(x;\lambda), \frac{\partial}{\partial x}\right\rangle - \left\langle \left\langle b(x;\lambda), \frac{\partial}{\partial x}\right\rangle a(x;\lambda), \frac{\partial}{\partial x}\right\rangle.$$

The Lie algebra $\tilde{\mathcal{G}}$ is naturally split into the direct sum of two Lie subalgebras

$$\widetilde{diff}(\mathbb{T}^n) = \widetilde{diff}_+(\mathbb{T}^n)_+ \oplus \widetilde{diff}_-(\mathbb{T}^n),$$

for which one can identify the following dual spaces:

$$\widetilde{diff}_+(\mathbb{T}^n)^* \simeq \widetilde{diff}_-(\mathbb{T}^n), \qquad \widetilde{diff}_-(\mathbb{T}^n)^* \simeq \widetilde{diff}_+(\mathbb{T}^n),$$

where for any $\tilde{l}(\lambda) \in \widetilde{diff}_-(\mathbb{T}^n)^*$ there holds the constraint $\tilde{l}(0) = 0$.

Construct now the Lie algebra $\tilde{\mathcal{G}} := \widetilde{diff}(\mathbb{T}^n) \ltimes \widetilde{diff}(\mathbb{T}^n)^*$ as the semi-direct sum of the Lie algebra $\widetilde{diff}(\mathbb{T}^n)$ and its dual space $\widetilde{diff}(\mathbb{T}^n)^*$, whose Lie structure is given by the following expression

$$[\tilde{a}_1 \ltimes \tilde{l}_1, \tilde{a}_2 \ltimes \tilde{l}_2] := [\tilde{a}_1, \tilde{a}_2] \ltimes (ad^*_{\tilde{a}_2}\tilde{l}_1 - ad^*_{\tilde{a}_1}\tilde{l}_2) \tag{7}$$

for any pair of elements $(\tilde{a}_1 \ltimes \tilde{l}_1), (\tilde{a}_2 \ltimes \tilde{l}_2) \in \tilde{\mathcal{G}}$, where $ad^*_{\widetilde{diff}(\mathbb{T}^n)} : \widetilde{diff}(\mathbb{T}^n)^* \to \widetilde{diff}(\mathbb{T}^n)^*$, $(ad^*_{\tilde{a}}\tilde{l}|\tilde{b}) := (\tilde{l}|[\tilde{a}, \tilde{b}])$ for $\tilde{l} \in \widetilde{diff}(\mathbb{T}^n)^*$ and any $\tilde{a}, \tilde{b} \in \widetilde{diff}(\mathbb{T}^n)$, is the standard coadjoint mapping of the Lie algebra $\widetilde{diff}(\mathbb{T}^n)$ on its adjoint space $\widetilde{diff}(\mathbb{T}^n)^*$ with respect to the pairing (6). The Lie algebra $\tilde{\mathcal{G}}$ can be metricized, as it can be endowed with the nondegenerate symmetric product

$$(\tilde{a}_1 \ltimes \tilde{l}_1 | \tilde{a}_2 \ltimes \tilde{l}_2) := (\tilde{l}_2|\tilde{a}_1) + (\tilde{l}_1|\tilde{a}_2), \tag{8}$$

where $\tilde{a}_1 \ltimes \tilde{l}_1, \tilde{a}_2 \ltimes \tilde{l}_2 \in \tilde{\mathcal{G}}$ are arbitrary elements. Owing to the holomorphic structure of the Lie algebra $\widetilde{diff}(\mathbb{T}^n)$, the ad-invariant product (8) makes it possible to identify the Lie algebra $\tilde{\mathcal{G}}$ with its dual $\tilde{\mathcal{G}}^*$, that is $\tilde{\mathcal{G}}^* \simeq \tilde{\mathcal{G}}$. Moreover, the Lie algebra $\tilde{\mathcal{G}}$ can be naturally split [38,39,49] with respect to the pairing (6) and the Lie bracket (7) into two subalgebras $\tilde{\mathcal{G}} = \tilde{\mathcal{G}}_+ \oplus \tilde{\mathcal{G}}_-$, where, by definition,

$$\tilde{\mathcal{G}}_+ := \widetilde{diff}(\mathbb{T}^n)_+ \ltimes \widetilde{diff}(\mathbb{T}^n)^*_-, \quad \tilde{\mathcal{G}}_- := \widetilde{diff}(\mathbb{T}^n)_- \ltimes \widetilde{diff}(\mathbb{T}^n)^*_+.$$

The latter allows to define on the Lie algebra $\tilde{\mathcal{G}}$ a new Lie bracket

$$[\tilde{w}_1, \tilde{w}_2]_{\mathcal{R}} := [\mathcal{R}\tilde{w}_1, \tilde{w}_2] + [\tilde{w}_1, \mathcal{R}\tilde{w}_2]$$

for any elements $\tilde{w}_1, \tilde{w}_2 \in \tilde{\mathcal{G}}$, where $R := (P_+ - P_-)/2$ is the standard R-matrix homomorphism [11,14,44,54] on $\tilde{\mathcal{G}}$ and, by definition, $P_\pm : \tilde{\mathcal{G}} \to \tilde{\mathcal{G}}_\pm \subset \tilde{\mathcal{G}}$ are projectors. The construction above makes it possible to apply to the Lie algebra $\tilde{\mathcal{G}}$ the classical AKS-scheme and, respectively, to generate a wide class of completely integrable Hamiltonian systems as the commuting flows on the adjoint space $\tilde{\mathcal{G}}^* \simeq \tilde{\mathcal{G}}$, generated by the corresponding hierarchies of the Casimir invariants subject to the basic Lie bracket (7).

To describe this scheme in more details, we need to find the corresponding Casimir functionals $h \in I(\tilde{\mathcal{G}}^*)$, satisfying, by definition, the following relationship:

$$ad^*_{\nabla h(\tilde{l};\tilde{a})}(\tilde{l};\tilde{a}) = 0 \tag{9}$$

at $(\tilde{l};\tilde{a}) \in \tilde{\mathcal{G}}^* \simeq \tilde{\mathcal{G}}$, where, by definition, the gradient $\nabla h(\tilde{l};\tilde{a}) := \nabla h_{\tilde{l}} \ltimes \nabla h_{\tilde{a}} \in \widetilde{diff}(\mathbb{T}^n) \ltimes \widetilde{diff}(\mathbb{T}^n)^* = \tilde{\mathcal{G}}$ satisfies the following from (9) differential-algebraic equations:

$$[\nabla h_{\tilde{l}}, \tilde{a}] = 0, \quad ad^*_{\nabla h_{\tilde{l}}} \tilde{l} - ad^*_{\tilde{a}} \nabla h_{\tilde{a}} = 0 \tag{10}$$

for arbitrarily chosen element $\tilde{a} \ltimes \tilde{l} \in \tilde{\mathcal{G}}$. The equations (10) can be rewritten [25] in details as

$$\begin{gathered} \langle \nabla h_l, \partial/\partial x\rangle \, a - \langle a, \partial/\partial x\rangle \, \nabla h_l = 0, \\ \langle \partial/\partial x, \nabla h_l\rangle \, l + \langle l, (\partial/\partial x \nabla h_l)\rangle - \langle \partial/\partial x, a\rangle \, \nabla h_a - \langle \nabla h_a, (\partial/\partial x a)\rangle = 0, \end{gathered} \tag{11}$$

where we put, by definition, that

$$\begin{gathered} \nabla h_{\tilde{l}} := \langle \nabla h_l, \partial/\partial x\rangle, \qquad \tilde{a} := \langle a, \partial/\partial x\rangle, \\ \tilde{l} := \langle l, dx\rangle, \qquad \nabla h_{\tilde{a}} := \langle \nabla h_a, dx\rangle. \end{gathered} \tag{12}$$

The system of linear equation (11) for a given element $\tilde{a} \ltimes \tilde{l} \in \tilde{\mathcal{G}}$, singular as $\lambda \to \infty$, can be, in general, resolved by means of the asymptotical expressions

$$\nabla h_l \sim \sum_{j \in \mathbb{Z}_+} \nabla h_l^{(j)} \lambda^{-j}, \qquad \nabla h_a \sim \sum_{j \in \mathbb{Z}_+} \nabla h_a^{(j)} \lambda^{-j}, \tag{13}$$

giving rise to an infinite hierarchy of gradients $\nabla h^{(p)}(\tilde{a}, \tilde{l}) = \lambda^p \nabla h(\tilde{a}, \tilde{l}) \in \tilde{\mathcal{G}}$, $p \in \mathbb{Z}_+$, for the corresponding Casimir functionals $h^{(p)} \in I(\tilde{\mathcal{G}}^*), p \in \mathbb{Z}_+$. Similarly, if a given element $\tilde{a} \ltimes \tilde{l} \in \tilde{\mathcal{G}}$ is chosen to be singular as $\lambda \to 0$, the system of linear equations (11) can be resolved by means of the asymptotical expressions

$$\nabla h_l \sim \sum_{j \in \mathbb{Z}_+} \nabla {h_l}^{(j)} \lambda^{j}, \qquad \nabla h_a \sim \sum_{j \in \mathbb{Z}_+} \nabla h_a^{(j)} \lambda^{-j}, \tag{14}$$

also generating an infinite hierarchy of gradients $\nabla h^{(p)}(\tilde{l}, \tilde{a}) = \lambda^{-p} \nabla h(\tilde{a}, \tilde{l}) \in \tilde{\mathcal{G}}$, $p \in \mathbb{Z}_+$, for the corresponding Casimir functionals $h^{(p)} \in I(\tilde{\mathcal{G}}^*), p \in \mathbb{Z}_+$.

Let us now assume that we have already found the gradients $\nabla h^{(y)}(\tilde{a}, \tilde{l}) := \lambda^{p_y} \nabla h^{(1)}(\tilde{a}, \tilde{l})$, $\nabla h^{(t)}(\tilde{a}, \tilde{l}) := \lambda^{p_y} \nabla h^{(2)}(\tilde{a}, \tilde{l}) \in \tilde{\mathcal{G}}$, related with two Casimir invariants $h^{(1)}, h^{(2)} \in I(\tilde{\mathcal{G}}^*)$ (not necessary different) for some integers $p_y, p_t \in \mathbb{Z}$, satisfying the determining equations (11). Then, owing to the classical AKS-scheme [11, 14, 48, 54], one can construct two commuting to each other flows with respect to the evolution parameters $y, t \in R$ on the adjoint space $\tilde{\mathcal{G}}^* \simeq \tilde{\mathcal{G}}$

$$\frac{\partial}{\partial y} \tilde{a} = -[\nabla h^{(y)}_{\tilde{l},+}, \tilde{a}], \qquad \frac{\partial}{\partial t} \tilde{a} = -[\nabla h^{(t)}_{\tilde{l},+}, \tilde{a}], \tag{15}$$

and

$$\frac{\partial}{\partial y} \tilde{l} = -ad^*_{\nabla h^{(y)}_{\tilde{l},+}} \tilde{l} + ad^*_{\tilde{a}} (\nabla h^{(y)}_{\tilde{a},+}), \qquad \frac{\partial}{\partial t} \tilde{l} = -ad^*_{\nabla h^{(t)}_{\tilde{l},+}} \tilde{l} + ad^*_{\tilde{a}} (\nabla h^{(t)}_{\tilde{a},+}), \tag{16}$$

where, we have denoted by $(\nabla h^{(y)}_{\tilde{l},+} \ltimes \nabla h^{(y)}_{\tilde{a},+}) := P_+ \nabla h^{(y)}(\tilde{a}, \tilde{l}) \in \tilde{\mathcal{G}}_+$ and $(\nabla h^{(t)}_{\tilde{l},+} \ltimes \nabla h^{(t)}_{\tilde{a},+}) := P_+ \nabla h^{(t)}(\tilde{a}, \tilde{l}) \in \tilde{\mathcal{G}}_+$ the corresponding projections on positive degree parts of the corresponding asymptotic expansions (12)–(14). The flows (15) and (16) are, by construction, Hamiltonian, as they are a result of the expressions

$$\frac{\partial}{\partial y} (\tilde{a} \ltimes \tilde{l}) = \{\tilde{a} \ltimes \tilde{l}, h^{(y)}\}_{\mathcal{R}}, \frac{\partial}{\partial t} (\tilde{a} \ltimes \tilde{l}) = \{\tilde{a} \ltimes \tilde{l}, h^{(t)}\}_{\mathcal{R}} \tag{17}$$

for a chosen element $\tilde{a} \ltimes \tilde{l} \in \tilde{\mathcal{G}}^* \simeq \tilde{\mathcal{G}}$, stemming from the R-deformed Lie-Poisson [11,14,48,54] bracket

$$\{h, f\}_{\mathcal{R}} := (\tilde{a} \ltimes \tilde{l}, [\nabla h(\tilde{l}, \tilde{a}), \nabla f(\tilde{l}, \tilde{a})]_{\mathcal{R}}) \tag{18}$$

on the adjoint space $\tilde{\mathcal{G}}^* \simeq \tilde{\mathcal{G}}$, defined for any smooth functionals $h, f \in D(\tilde{\mathcal{G}}^*)$. Their commutativity condition is equivalent to two equations such as

$$[\nabla h^{(y)}_{\tilde{l},+}, \nabla h^{(t)}_{\tilde{l},+}] - \frac{\partial}{\partial t}\nabla h^{(y)}_{\tilde{l},+} + \frac{\partial}{\partial y}\nabla h^{(t)}_{\tilde{l},+} = 0, \tag{19}$$

and

$$ad^*_{\tilde{a}}\tilde{P} = 0,$$
$$\tilde{P} = ad^*_{\nabla h^{(y)}_{\tilde{l},+}}(\nabla h^{(t)}_{\tilde{a},+}) - ad^*_{\nabla h^{(t)}_{\tilde{l},+}}(\nabla h^{(y)}_{\tilde{a},+}) - \frac{\partial}{\partial t}\nabla h^{(y)}_{\tilde{a},+} + \frac{\partial}{\partial y}\nabla h^{(t)}_{\tilde{a},+}$$

for any $\tilde{a} \ltimes \tilde{l} \in \tilde{\mathcal{G}}$. Thus, the following important proposition holds.

Proposition 2. *The Hamiltonian flows* (17) *generate the separately commuting evolution equations* (15) *and* (16). *The evolution equations* (15) *give rise to the Lax type compatibility condition* (19)*, being equivalent to some system of nonlinear heavenly type equations in partial derivatives.*

The presented above construction of Hamiltonian flows on the adjoint space $\tilde{\mathcal{G}}^*$ still allows the next important generalization. Namely, let us endow the point product $\tilde{\mathcal{G}}^{\mathbb{S}^1}: = \prod_{z\in\mathbb{S}^1}\tilde{\mathcal{G}}$ of the loop Lie algebra $\tilde{\mathcal{G}}$ with the central extension generated by a two-cocycle $\omega_2 : \tilde{\mathcal{G}} \times \tilde{\mathcal{G}} \to \mathbb{C}$, where

$$\omega_2(\tilde{a}_1 \ltimes \tilde{l}_1, \tilde{a}_2 \ltimes \tilde{l}_2) := \int_{\mathbb{S}^1}[(l_1, \partial\tilde{a}_2/\partial z) - (l_2, \partial\tilde{a}_1/\partial z)]$$

for any elements $\tilde{a}_1 \ltimes \tilde{l}_1, \tilde{a}_2 \ltimes \tilde{l}_2 \in \tilde{\mathcal{G}}$. The resulting centrally extended Lie-algebra $\tilde{\mathcal{G}} := \tilde{\mathcal{G}} \oplus \mathbb{C}$ is defined by the commutator

$$[(\tilde{a}_1 \ltimes \tilde{l}_1; \alpha_1), (\tilde{a}_2 \ltimes \tilde{l}_2; \alpha_1)] := ([\tilde{a}_1, a_2] \ltimes (ad^*_{\tilde{a}_1}\tilde{l}_2 - ad^*_{\tilde{a}_2}\tilde{l}_1); \omega_2(\tilde{a}_1 \ltimes \tilde{l}_1, \tilde{a}_2 \ltimes \tilde{l}_2)$$

for any pair of elements $(\tilde{a}_1 \ltimes \tilde{l}_1; \alpha_1), (\tilde{a}_2 \ltimes \tilde{l}_2; \alpha_1) \in \tilde{\mathcal{G}}$. The resulting R-deformed Lie-Poisson bracket (18) for any smooth functionals $h, f \in D(\mathcal{G}^*)$ on the adjoint space $\tilde{\mathcal{G}}^*$ becomes equal to

$$\begin{aligned}\{h, f\}_{\mathcal{R}} := {}& (\tilde{a} \ltimes \tilde{l}, [\nabla h(\tilde{l}, \tilde{a}), \nabla f(\tilde{l}, \tilde{a})]_{\mathcal{R}}) \\ &+ \omega_2(\mathcal{R}\nabla h(\tilde{l}, \tilde{a}), \nabla f(\tilde{l}, \tilde{a})) + \omega_2(\nabla h(\tilde{l}, \tilde{a}), \mathcal{R}\nabla f(\tilde{l}, \tilde{a})).\end{aligned} \tag{20}$$

The corresponding Casimir functionals $h^{(p)} \in I(\tilde{\mathcal{G}}^*), p \in \mathbb{Z}_+$, are defined with respect to the standard Lie-Poisson bracket as

$$\{h^{(p)}, f\} := (\tilde{a} \ltimes \tilde{l}, [\nabla h^{(p)}(\tilde{l}, \tilde{a}), \nabla f(\tilde{a}, \tilde{l})]) + \omega_2(\nabla h^{(p)}(\tilde{a}, \tilde{l}), \nabla f(\tilde{a}, \tilde{l})) = 0 \tag{21}$$

for all smooth functionals $f \in D(\tilde{\mathcal{G}}^*)$. Based on the equality (21) one easily finds that the gradients $\nabla h^{(p)} \in \tilde{\mathcal{G}}$ of the Casimir functionals $h^{(p)} \in I(\tilde{\mathcal{G}}^*), p \in \mathbb{Z}_+$, satisfy the following equations:

$$[\nabla h_{\tilde{l}}, \tilde{a}] - \frac{\partial}{\partial z}\nabla h_{\tilde{l}} = 0, \qquad ad^*_{\nabla h_{\tilde{l}}}\tilde{l} - ad^*_{\tilde{a}}\nabla h_{\tilde{a}} - \frac{\partial}{\partial z}\nabla h_{\tilde{a}} = 0$$

for any chosen element $\tilde{a} \ltimes \tilde{l} \in \bar{\mathcal{G}}^*$. Making use of suitably constructed Casimir functionals $h^{(y)}, h^{(t)} \in I(\tilde{\mathcal{G}})$, one can construct from (20) the following commuting Hamiltonian flows on the adjoint space $\tilde{\mathcal{G}}^*$:

$$\frac{\partial}{\partial y}(\tilde{a} \ltimes \tilde{l}) = \{\tilde{a} \ltimes \tilde{l}, h^{(y)}\}_{\mathcal{R}}, \qquad \frac{\partial}{\partial t}(\tilde{a} \ltimes \tilde{l}) = \{\tilde{a} \ltimes \tilde{l}, h^{(t)}\}_{\mathcal{R}}, \tag{22}$$

which are equivalent to the evolution equations

$$\frac{\partial}{\partial y}\tilde{a} = -[\nabla h^{(y)}_{\tilde{l},+}, \tilde{a}] + \frac{\partial}{\partial z}\nabla h^{(y)}_{\tilde{l},+}, \qquad \frac{\partial}{\partial t}\tilde{a} = -[\nabla h^{(t)}_{\tilde{l},+}, \tilde{a}] + \frac{\partial}{\partial z}\nabla h^{(t)}_{\tilde{l},+}, \tag{23}$$

and

$$\begin{aligned} \frac{\partial}{\partial y}\tilde{l} &= -ad^*_{\nabla h^{(y)}_{\tilde{l},+}}\tilde{l} + ad^*_{\tilde{a}}(\nabla h^{(y)}_{\tilde{a},+}) + \frac{\partial}{\partial z}\nabla h^{(y)}_{\tilde{a},+}, \\ \frac{\partial}{\partial t}\tilde{l} &= -ad^*_{\nabla h^{(t)}_{\tilde{l},+}}\tilde{l} + ad^*_{\tilde{a}}(\nabla h^{(t)}_{\tilde{a},+}) + \frac{\partial}{\partial z}\nabla h^{(t)}_{\tilde{a},+}. \end{aligned} \tag{24}$$

The commutativity condition for these flows is split into two equations such as

$$[\nabla h^{(y)}_{\tilde{l},+}, \nabla h^{(t)}_{\tilde{l},+}] - \frac{\partial}{\partial t}\nabla h^{(y)}_{\tilde{l},+} + \frac{\partial}{\partial y}\nabla h^{(t)}_{\tilde{l},+} = 0, \tag{25}$$

and

$$\frac{\partial \tilde{P}}{\partial z} + ad^*_{\tilde{a}}\tilde{P} = 0,$$

$$\tilde{P} = ad^*_{\nabla h^{(y)}_{\tilde{l},+}}(\nabla h^{(t)}_{\tilde{a},+}) - ad^*_{\nabla h^{(t)}_{\tilde{l},+}}(\nabla h^{(y)}_{\tilde{a},+}) - \frac{\partial}{\partial t}\nabla h^{(y)}_{\tilde{a},+} + \frac{\partial}{\partial y}\nabla h^{(t)}_{\tilde{a},+}$$

for any $\tilde{a} \ltimes \tilde{l} \in \tilde{\mathcal{G}}$. The first of them can be considered as the Lax type compatibility condition for the evolution equations (23). As a consequence of the obtained above results one can formulate the following proposition.

Proposition 3. *The Hamiltonian flows* (22) *on the adjoint space $\tilde{\mathcal{G}}^*$ generate the separately commuting evolution equations* (23) *and* (24). *The evolution equations* (23) *give rise to the Lax type compatibility condition* (25)*, being equivalent to some system of nonlinear heavenly type equations in partial derivatives. Moreover, the system of evolution equations* (23) *can be considered as the compatibility condition for the following set of linear vector equations*

$$\partial\psi/\partial y + \nabla h^{(y)}_{\tilde{l},+}\psi = 0, \qquad \partial\psi/\partial z + \tilde{a}\psi = 0, \qquad \partial\psi/\partial t + \nabla h^{(t)}_{\tilde{l},+}\psi = 0$$

for all $(y, t; \lambda, z, x) \in \mathbb{R}^2 \times (\mathbb{C} \times \mathbb{S}^1) \times \mathbb{T}^n)$ *and a function* $\psi \in C^2(\mathbb{R}^2 \times \mathbb{C} \times (\mathbb{S}^1 \times \mathbb{T}^n); \mathbb{C})$.

The following example demonstrates the analytical applicability of the devised above Lie-algebraic scheme for construction a wide class of nonlinear multidimensional heavenly type integrable Hamiltonian systems on functional spaces.

3.1 Example: the modified Mikhalev-Pavlov heavenly type system

Let a seed element $\tilde{a} \ltimes \tilde{l} \in \tilde{\mathcal{G}}^*$ be chosen in its reduced form as

$$\tilde{a} \ltimes \tilde{l} = ((u_x + v_x\lambda - \lambda^2)\partial/\partial x \ltimes (w_x + \zeta_x\lambda)dx, \tag{26}$$

where $u, v, w, \zeta \in C^2(\mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{T}^1; \mathbb{R})$. The asymptotic splits for the components of the gradient of the corresponding Casimir functional $h \in I(\bar{\mathcal{G}}^*)$, as $|\lambda| \to \infty$ have the following forms:

$$\begin{aligned}
\nabla h_{\tilde{l}} \simeq\, & 1 - v_x\lambda^{-1} - u_x\lambda^{-2} - v_z\lambda^{-3} - (u_z + v_xv_z - 2(\partial_x^{-1}v_{xx}v_z))\lambda^{-4} \\
& + v_y\lambda^{-5} - (-u_y - v_xv_y + 2(\partial_x^{-1}v_{xx}v_y))\lambda^{-6} + \ldots, \\
\nabla h_{\tilde{a}} \simeq\, & -\zeta_x\lambda^{-1} - w_x\lambda^{-2} - \zeta_z\lambda^{-3} - (w_z - \zeta_xv_z + 2v_x\zeta_z + (\partial_x^{-1}v_x\zeta_x)_z)\lambda^{-4} \\
& + \zeta_y\lambda^{-5} - (-w_y + \zeta_xv_y - 2v_x\zeta_y + (\partial_x^{-1}v_x\zeta_x)_y)\lambda^{-6} + \ldots.
\end{aligned}$$

In the case when

$$\begin{aligned}
\nabla h_{\tilde{l},+}^{(y)} &:= \lambda^4 - v_x\lambda^3 - u_x\lambda^2 - v_z\lambda - (u_z + v_xv_z - 2(\partial_x^{-1}v_{xx}v_z)), \\
\nabla h_{\tilde{a},+}^{(y)} &:= -\zeta_x\lambda^3 - w_x\lambda^2 - \zeta_z\lambda - (w_z - \zeta_xv_z + 2v_x\zeta_z - (\partial_x^{-1}v_x\zeta_x)_z),
\end{aligned}$$

and

$$\begin{aligned}
\nabla h_{\tilde{l},+}^{(t)} &:= \lambda^6 - v_x\lambda^5 - u_x\lambda^4 - v_z\lambda^3 - (u_z + v_xv_z - 2(\partial_x^{-1}v_{xx}v_z))\lambda^2 \\
&\quad + v_y\lambda - (-u_y - v_xv_y + 2(\partial_x^{-1}v_{xx}v_y)), \\
\nabla h_{\tilde{a},+}^{(t)} &:= -\zeta_x\lambda^5 - w_x\lambda^4 - \zeta_z\lambda^3 - (w_z - \zeta_xv_z + 2v_x\zeta_z - (\partial_x^{-1}v_x\zeta_x)_z)\lambda^2 \\
&\quad + \zeta_y\lambda - (-w_y + \zeta_xv_y - 2v_x\zeta_y + (\partial_x^{-1}v_x\zeta_x)_y),
\end{aligned}$$

the compatibility condition of the Hamiltonian vector flows (22) leads to the system of evolution equations:

$$\begin{aligned}
u_{zt} + u_{yy} &= -u_yu_{xz} + u_zu_{xy} - v_yv_{xy} + v_zv_{xt} - u_zv_yv_{xx} + u_yv_zv_{xx} \\
&\quad - v_x^2v_zv_{xy} + v_x^2v_yv_{xz} - 2eu_{xy} - 2su_{xz} + 2e_t - 2s_y + 2ev_yv_{xx} + 2sv_zv_{xx}, \\
v_{zt} + v_{yy} &= -u_yv_{xz} + u_zv_{xy} - v_yu_{xz} + v_zu_{xy} - 2ev_{xy} - 2sv_{xz} - 2v_xv_yv_{xz} + 2v_xv_zv_{xy}, \\
-u_{xy} - u_{zz} &= u_xu_{xz} - u_zu_{xx} - u_{xx}v_xv_z + u_xv_{xz}v_x - u_xv_{xx}v_z + (v_xv_z)_z + 2u_{xx}e - 2e_z, \\
-v_{xy} - v_{zz} &= u_{xz}v_x - u_zv_{xx} - u_{xx}v_z + u_xv_{xz} - 2v_{xx}v_xv_z + v_x^2v_{xz} + 2v_{xx}e, \\
-u_{xt} + u_{yz} &= -u_xu_{xy} + u_yu_{xx} + u_{xx}v_xv_y - u_xv_{xy}v_x + u_xv_{xx}v_y - (v_xv_y)_z + 2u_{xx}s - 2s_z, \\
-v_{xt} + v_{yz} &= -u_{xy}v_x + u_yv_{xx} + u_{xx}v_y - u_xv_{xy} + 2v_{xx}v_xv_y - v_x^2v_{xy} + 2v_{xx}s,
\end{aligned} \tag{27}$$

where

$$e_{xx} = v_{xx}v_z, \qquad s_{xx} = -v_{xx}v_y. \tag{28}$$

Under the constraint $v = 0$ one obtains a set of independent scalar differential equations before listed in [17, 18, 23]; two equations are spatially four-dimensional:

$$u_{zt} + u_{yy} = -u_yu_{xz} + u_zu_{xy} \tag{29}$$

and

$$-u_{xt} + u_{yz} = -u_xu_{xy} + u_yu_{xx}, \tag{30}$$

a one is spatially three-dimensional:

$$-u_{xy} - u_{zz} = u_x u_{xz} - u_z u_{xx}. \tag{31}$$

In particular, under the spatial variable reductions $x \to y \in \mathbb{R}$, $t \to z \in \mathbb{R}$, the second equation becomes trivial and the first (32) and third (31) equations bring about the reduced Mikhalev-Pavlov type equation

$$u_{zz} + u_{yy} = -u_y u_{yz} + u_z u_{yy}. \tag{32}$$

Proposition 4. *The constructed set of heavenly type equations* (27), (28) *has the Lax-Sato vector field representation* (19) *with the "spectral" parameter $\lambda \in \mathbb{C}$, which is related with the seed element $\tilde{a} \ltimes \tilde{l} \in \tilde{\mathcal{G}}^*$ in the form* (26).

Remark 1. *The following remark concerning the dimensionality of the differential systems obtained above proves to be essential. The generalized Mikhalev-Pavlov differential system* (29) *as the one considered on the related jet-manifold $J(\mathbb{R}^4; \mathbb{R}^2)$ for smooth mappings $(u, v) : \mathbb{R}^4 \to \mathbb{R}^2$ presents, in reality, a differential system with effective dimension equal* $2 = 4 - 2$. *This fact is important from the geometric point of view devised recently in E.V. Ferapontov and others [19,22] works, devoted to the Plücker manifold imbedding into the Grassmannians and a classification of related integrable differential systems. There was, in particular, stated that the corresponding integrable systems associated with fourfolds in $Gr(3, 5)$ also appeared to be effectively two-dimensional, ensuing at the present time in some sense a challenging problem. As it was also mentioned above concerning a generalization of spatially multidimensional Mikhalev-Pavlov type equations by means of the seed element* (33)*, there is a possibility to check directly the existence of effectively three and more dimensional integrable differential systems and then, eventually, to construct them.*

We can here observe that the seed element (26) can be presented in the following special compact form:

$$\tilde{a} \ltimes \tilde{l} := \frac{d\tilde{\eta}}{dx}\partial/\partial x \ltimes d\tilde{\rho}, \tilde{\eta} = u + v\lambda - \lambda^2 x, \tilde{\rho} = w + \zeta\lambda,$$

deeply connected with geometry of the related moduli space of flat connections, related to coadjoint actions of the corresponding Casimir functionals. Its possible generalization to spatially multidimensional Mikhalev-Pavlov type equations can be done by the seed element

$$\tilde{a} \ltimes \tilde{l} := \langle \nabla\tilde{\eta}, \nabla \rangle \ltimes d\tilde{\rho} \tag{33}$$

for some elements $\tilde{\eta}, \tilde{\rho} \in \Omega^0(\mathbb{T}^n) \otimes \mathbb{C}, n \in \mathbb{N}$. An analysis of the case (33) and corresponding systems of spatially multidimensional Mikhalev-Pavlov type equations is planned to be done in a separate study.

3.2 The modified Martinez Alonso-Shabat heavenly type system

If the seed element $\tilde{a} \ltimes \tilde{l} \in \tilde{\mathcal{G}}^*$ is chosen in its reduced form as

$$\begin{aligned} \tilde{a} \ltimes \tilde{l} = {} & (((u_{x_1} + cu_{x_2}) + \lambda)\partial/\partial x_1 + ((v_{x_1} + cv_{x_2}) + c\lambda)\partial/\partial x_2) \\ & \ltimes ((w_{x_1} + cw_{x_2})dx_1 + (\zeta_{x_1} + c\zeta_{x_2})dx_2), \end{aligned} \tag{34}$$

where $u, v, w, \zeta \in C^2(\mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{T}^2; \mathbb{R})$, $c \in \mathbb{R}\backslash\{0\}$, one has the following asymptotic splits for the components of the gradients of the corresponding Casimir functionals $h^{(1)}, h^{(2)} \in I(\bar{\mathcal{G}}^*)$ as $|\lambda| \to \infty$:

$$\nabla h^{(1)}_{\tilde{l}} \simeq \begin{pmatrix} 1 + (u_{x_1} + cu_{x_2})\lambda^{-1} - u_z\lambda^{-2} + \ldots \\ c + (v_{x_1} + cv_{x_2})\lambda^{-1} - v_z\lambda^{-2} + \ldots \end{pmatrix},$$
$$\nabla h^{(1)}_{\tilde{a}} \simeq \begin{pmatrix} (w_{x_1} + cw_{x_2})\lambda^{-1} - w_z\lambda^{-2} + \ldots \\ (\zeta_{x_1} + c\zeta_{x_2})\lambda^{-1} - \zeta_z\lambda^{-2} + \ldots \end{pmatrix},$$

and

$$\nabla h^{(2)}_{\tilde{l}} \simeq \begin{pmatrix} 1 + (u_{x_1} - cu_{x_2})\lambda^{-1} + \varkappa\lambda^{-2} + \ldots \\ -c + (v_{x_1} - cv_{x_2})\lambda^{-1} + \omega\lambda^{-2} + \ldots \end{pmatrix},$$
$$\nabla h^{(2)}_{\tilde{a}} \simeq \begin{pmatrix} (w_{x_1} - cw_{x_2})\lambda^{-1} + \varrho\lambda^{-2} + \ldots \\ (\zeta_{x_1} - c\zeta_{x_2})\lambda^{-1} + \chi\lambda^{-2} + \ldots \end{pmatrix},$$

where

$$\begin{aligned} \varkappa_{x_1} + c\varkappa_{x_2} &= -(u_{zx_1} - cu_{zx_2}) + 2c(u_{x_1}u_{x_1x_2} - u_{x_2}u_{x_1x_1} + v_{x_1}u_{x_2x_2} - v_{x_2}u_{x_1x_2}), \\ \omega_{x_1} + c\omega_{x_2} &= -(v_{zx_1} - cv_{zx_2}) + 2c(u_{x_1}v_{x_1x_2} - u_{x_2}v_{x_1x_1} + v_{x_1}v_{x_2x_2} - v_{x_2}v_{x_1x_2}), \end{aligned} \tag{35}$$

and

$$\begin{aligned} \varrho_{x_1} + c\varrho_{x_2} = &- (w_{zx_1} - cw_{zx_2}) + 2c(u_{x_1}w_{x_1x_2} - u_{x_2}w_{x_1x_1} + 2w_{x_2}u_{x_1x_1} \\ &- 2w_{x_1}u_{x_1x_2} + v_{x_1}w_{x_2x_2} - v_{x_2}w_{x_1x_2} + w_{x_2}v_{x_1x_2} - w_{x_2}v_{x_2x_2} + \zeta_{x_2}v_{x_1x_1} - \zeta_{x_1}v_{x_1x_2}), \\ \chi_{x_1} + c\chi_{x_2} = &- (\zeta_{zx_1} - c\zeta_{zx_2}) + 2c(v_{x_1}\zeta_{x_2x_2} - v_{x_2}\zeta_{x_1x_2} + 2\zeta_{x_2}v_{x_1x_2} \\ &- 2\zeta_{x_1}v_{x_2x_2} + u_{x_1}\zeta_{x_1x_2} - u_{x_2}\zeta_{x_1x_1} + \zeta_{x_2}u_{x_1x_1} - \zeta_{x_1}u_{x_1x_2} + w_{x_2}u_{x_1x_2} - w_{x_1}u_{x_2x_2}). \end{aligned}$$

In the case when

$$\nabla h^{(y)}_{\tilde{l},+} := \begin{pmatrix} \lambda^2 + (u_{x_1} + cu_{x_2})\lambda - u_z \\ c\lambda^2 + (v_{x_1} + cv_{x_2})\lambda - v_z \end{pmatrix},$$

$$\nabla h^{(y)}_{\tilde{a},+} := \begin{pmatrix} (w_{x_1} + cw_{x_2})\lambda - w_z \\ (\zeta_{x_1} + c\zeta_{x_2})\lambda - \zeta_z \end{pmatrix},$$

and

$$\nabla h^{(t)}_{\tilde{l},+} := \begin{pmatrix} \lambda^2 + (u_{x_1} - cu_{x_2})\lambda + \varkappa \\ -c\lambda^2 + (v_{x_1} - cv_{x_2})\lambda + \omega \end{pmatrix},$$

$$\nabla h^{(t)}_{\tilde{a},+} := \begin{pmatrix} (w_{x_1} - cw_{x_2})\lambda + \varrho \\ (\zeta_{x_1} - c\zeta_{x_2})\lambda + \chi \end{pmatrix},$$

the compatibility condition of the Hamiltonian vector flows (22) leads to the system of evolution equations:

$$\begin{aligned}
u_{zt} + \varkappa_y &= -u_{zx_1}\varkappa - u_{zx_2}\omega + u_z\varkappa_{x_1} + v_z\varkappa_{x_2},\\
v_{zt} + \omega_y &= -v_{zx_1}\varkappa - v_{zx_2}\omega + u_z\omega_{x_1} + v_z\omega_{x_2},\\
u_{yx_1} + cu_{yx_2} &= -(u_{x_1} + cu_{x_2})u_{zx_1} - (v_{x_1} + cv_{x_2})u_{zx_2} + (u_{x_1x_1} + cu_{x_1x_2})u_z\\
&\quad + (u_{x_1x_2} + cu_{x_2x_2})v_z - u_{zz},\\
v_{yx_1} + cv_{yx_2} &= -(u_{x_1} + cu_{x_2})v_{zx_1} - (v_{x_1} + cv_{x_2})v_{zx_2} + (v_{x_1x_1} + cv_{x_1x_2})u_z\\
&\quad + (v_{x_1x_2} + cv_{x_2x_2})v_z - v_{zz},\\
u_{tx_1} + cu_{tx_2} &= (u_{x_1} + cu_{x_2})\varkappa_{x_1} + (v_{x_1} + cv_{x_2})\varkappa_{x_2} - (u_{x_1x_1} + cu_{x_1x_2})\varkappa\\
&\quad - (u_{x_1x_2} + cu_{x_2x_2})\omega + \varkappa_z,\\
v_{tx_1} + cv_{tx_2} &= (u_{x_1} + cu_{x_2})\omega_{x_1} + (v_{x_1} + cv_{x_2})\omega_{x_2} - (v_{x_1x_1} + cv_{x_1x_2})\varkappa\\
&\quad - (v_{x_1x_2} + cv_{x_2x_2})\omega + \omega_z.
\end{aligned} \tag{36}$$

Thus, the following proposition holds.

Proposition 5. *The constructed system of heavenly type equations* (36) *and* (35) *has the Lax-Sato vector field representation* (19) *with the "spectral" parameter $\lambda \in \mathbb{C}$, which is related with the element $\tilde{a} \ltimes \tilde{l} \in \tilde{\mathcal{G}}^*$ in the form* (34).

The system of equations (36) and (35) admits the reduction when $v = u$ and $\omega = \varkappa$. In this case, under $c = 1$ one obtains

$$\begin{aligned}
u_{zt} + \varkappa_y &= -(u_{zx_1} + u_{zx_2})\varkappa + u_z(\varkappa_{x_1} + \varkappa_{x_2}),\\
\varkappa_{x_1} + \varkappa_{x_2} &= -(u_{zx_1} - u_{zx_2}) - 2((u_{x_1}u_{x_2})_{x_1} - (u_{x_1}u_{x_2})_{x_2}).
\end{aligned} \tag{37}$$

The change $u_z = u_{x_1} + u_{x_2}$ in (37) leads to the system:

$$\begin{aligned}
(u_{\tilde{t}x_1} + u_{\tilde{t}x_2}) - (u_{\tilde{y}x_1} - u_{\tilde{y}x_2}) &= u_{x_1x_2}(u_{x_1} - u_{x_2}) - u_{x_1x_1}u_{x_2} + u_{x_2x_2}u_{x_1}\\
&\quad - u_{x_1x_2}(u_{x_1}^2 - u_{x_2}^2) - u_{x_1x_1}u_{x_2}(u_{x_1} + u_{x_2}) + u_{x_2x_2}u_{x_1}(u_{x_1} + u_{x_2})\\
&\quad - 2\rho_{\tilde{y}} + (u_{x_1x_1} + 2u_{x_1x_2} + u_{x_2x_2})\rho,\\
\rho_{x_1} + \rho_{x_2} &= (u_{x_1}u_{x_2})_{x_1} - (u_{x_1}u_{x_2})_{x_2},
\end{aligned}$$

where $\tilde{t} = 2t$ and $\tilde{y} = 2y$. Thus, the system (37) can be considered as some modification of the Martinez Alonso-Shabat one [3].

4 Heavenly type systems: the generalized Lie-algebraic structures

Concerning a further generalization of the multi-dimensional case related with the loop group $\widetilde{Diff}(\mathbb{T}^n)$ on the torus $\mathbb{T}^n$, $n \in \mathbb{Z}_+$, one can proceed, as before, [25] the following natural way: as the Lie algebra $\widetilde{diff}(\mathbb{T}^n)$ consists of the loop group elements, holomorphically continued from the circle $S^1 := \partial\mathbb{D}^1$, being the boundary of the disk $\mathbb{D}^1 \subset \mathbb{C}$, by means of the complex "spectral" variable $\lambda \in \mathbb{C}$ both into the interior $\mathbb{D}^1_+ \subset \mathbb{C}$ and the exterior $\mathbb{D}^1_- \subset \mathbb{C}$ parts of the disk $\mathbb{D}^1 \subset \mathbb{C}$, one can take into account its analytical invariance subject to the circle $\mathbb{S}^1 := \partial\mathbb{D}^1$ diffeomorphism group $Diff(\mathbb{S}^1)$. The latter gives rise to the naturally extended holomorphic Lie algebra $\widetilde{diff}(\mathbb{T}^n) = \widetilde{diff}_+(\mathbb{T}^n)_+ \oplus \widetilde{diff_-}(\mathbb{T}^n)$ on the Cartesian product $\mathbb{C} \times \mathbb{T}^n$, whose elements are representable as

$$\bar{a} := \left\langle a(x;\lambda), \frac{\partial}{\partial \mathrm{x}} \right\rangle = a_0(x;\lambda)\frac{\partial}{\partial\lambda} + \sum_{j=1}^{n} a_j(x;\lambda)\frac{\partial}{\partial x_j}$$

for some holomorphic in $\lambda \in \mathbb{D}^1_\pm$ vectors $a(x;\lambda) \in E \times E^n$ for all $x \in \mathbb{T}^n$, and where we denoted by $\frac{\partial}{\partial \mathrm{x}} := (\frac{\partial}{\partial\lambda}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, ..., \frac{\partial}{\partial x_n})^\intercal$the generalized Euclidean vector gradient with respect to the vector variable $x := (\lambda, x) \in \mathbb{T}^n$.

Construct now the semi-direct sum $\bar{\mathcal{G}} := diff(\mathbb{T}^n) \ltimes diff(\mathbb{T}^n)^*$ of the loop Lie algebra $diff(\mathbb{T}^n)$ and its adjoint space $diff(\mathbb{T}^n)^*$, taking into account their natural pairing

$$(\bar{l}|\bar{a}) := \underset{\lambda\in\mathbb{C}}{res}(l(\mathrm{x})|a(\mathrm{x}))_{H^0}$$

for any $\bar{l} := \langle l(x;\lambda), dx\rangle = l_0(x;\lambda)d\lambda + \sum_{j=1}^{n} l_j(x;\lambda)dx_j \in diff(\mathbb{T}^n)^*$ and $\bar{a} \in diff(\mathbb{T}^n)$. The corresponding Lie commutator on the loop Lie algebra $\bar{\mathcal{G}}$ is naturally given by the expression

$$[\bar{a}_1 \ltimes \bar{l}_1, \bar{a}_2 \ltimes \bar{l}_2] = [\bar{a}_1, a_2] \ltimes ad^*_{a_2}\bar{l}_1 - ad^*_{a_1}\bar{l}_2$$

for any $\bar{a}_1 \ltimes \bar{l}_1, \bar{a}_2 \ltimes \bar{l}_2 \in \bar{\mathcal{G}}$. The Lie algebra $\bar{\mathcal{G}}$ also splits into the direct sum of two subalgebras

$$\bar{\mathcal{G}} = \bar{\mathcal{G}}_+ \oplus \bar{\mathcal{G}}_-,$$

allowing to introduce on it the classical R-structure

$$[\bar{a}_1 \ltimes \bar{l}_1, \bar{a}_2 \ltimes \bar{l}_2]_{\mathcal{R}} := [\mathcal{R}(\bar{a}_1 \ltimes \bar{l}_1), \bar{a}_2 \ltimes \bar{l}_2] + [\bar{a}_1 \ltimes \bar{l}_1, \mathcal{R}(\bar{a}_2 \ltimes \bar{l}_2)]$$

for any $\bar{a}_1 \ltimes \bar{l}_1, \bar{a}_2 \ltimes \bar{l}_2 \in \bar{\mathcal{G}}$, where, by definition,

$$\mathcal{R} := (P_+ - P_-)/2, \qquad \text{and} \qquad P_\pm\bar{\mathcal{G}} := \bar{\mathcal{G}}_\pm \subset \bar{\mathcal{G}}.$$

The space $\bar{\mathcal{G}}^*$ adjoint to the Lie algebra $\bar{\mathcal{G}}$ can be functionally identified with the space $\bar{\mathcal{G}}$ subject to the nondegenerate symmetric product

$$(\bar{a} \ltimes \bar{l}|\bar{r} \ltimes \bar{m}) := \underset{\lambda\in\mathbb{C}}{res}(\bar{a} \ltimes \bar{l}|\bar{r} \ltimes \bar{m})_{H^0},$$

where we put, by definition, that

$$(\bar{a} \ltimes \bar{l}|\bar{r} \ltimes \bar{m})_{H^0} = (\bar{m}|\bar{a})_{H^0} + (\bar{l}|\bar{r})_{H^0} \tag{38}$$

for any pair of elements $\bar{a} \ltimes \bar{l}, \bar{r} \ltimes \bar{m} \in \bar{\mathcal{G}}$.

Owing to the convolution (38), the Lie algebra $\bar{\mathcal{G}}$ becomes metricized. If now to take arbitrary smooth functions $f, g \in D(\bar{\mathcal{G}}^*)$, one can naturally determine two Lie-Poisson brackets

$$\{f, g\} := (\bar{a} \ltimes \bar{l}|[\nabla f(\bar{l}, \bar{a}), \nabla g(\bar{l}, \bar{a})])$$

and

$$\{f, g\}_{\mathcal{R}} := (\bar{a} \ltimes \bar{l}|[\nabla f(\bar{l}, \bar{a}), \nabla g(\bar{l}, \bar{a})]_{\mathcal{R}}), \tag{39}$$

where at any seed element $\bar{a} \ltimes \bar{l} \in \bar{\mathcal{G}}^* \simeq \bar{\mathcal{G}}$ the gradient element $\nabla f(\bar{l},\bar{a}) := \nabla f_{\bar{l}} \ltimes \nabla f_{\bar{a}} \simeq \langle \nabla f(l,a), (\partial/\partial x, dx)^\intercal \rangle \in \bar{\mathcal{G}}$ and $\nabla f_{\bar{l}} = \langle \nabla f_l, \partial/\partial x \rangle$, $\nabla f_{\bar{a}} = \langle \nabla f_a, dx \rangle$, and, similarly, the gradient element $\nabla g(\bar{l},\bar{a}) := \nabla g_{\bar{l}} \ltimes \nabla g_{\bar{a}} \simeq \langle \nabla g(l,a), (\partial/\partial x, dx)^\intercal \rangle \in \bar{\mathcal{G}}^*$ and $\nabla g_{\bar{l}} = \langle \nabla g_l, \partial/\partial x \rangle$, $\nabla g_{\bar{a}} = \langle \nabla g_a, dx \rangle$ are calculated with respect to the metric (38).

Let now assume that a smooth function $h \in I(\bar{\mathcal{G}}^*)$ is a Casimir invariant, that is

$$ad^*_{\nabla h(\bar{l},\bar{a})}(\bar{a} \ltimes \bar{l}) = 0 \tag{40}$$

for a chosen seed element $\bar{a} \ltimes \bar{l} \in \bar{\mathcal{G}}^* \simeq \bar{\mathcal{G}}$. Since for an element $\bar{a} \ltimes \bar{l} \in \bar{\mathcal{G}}^* \simeq \bar{\mathcal{G}}$ and arbitrary $f \in D(\bar{\mathcal{G}}^*)$ the adjoint mapping

$$ad^*_{\nabla f(\bar{l},\bar{a})}(\bar{a} \ltimes \bar{l}) = ([\nabla h_{\bar{l}}, \bar{a}] \ltimes (ad^*_{\nabla h_{\bar{l}}} \bar{l} - ad^*_{\bar{a}} \nabla h_{\bar{a}}),$$

the condition (40) can be rewritten as

$$[\nabla h_{\bar{l}}, \bar{a}] = 0, \qquad ad^*_{\nabla h_{\bar{l}}} \bar{l} - ad^*_{\bar{a}} \nabla h_{\bar{a}} = 0,$$

from which one easily obtains that the Casimir functional $h \in I(\bar{\mathcal{G}}^*)$ satisfies the system of determining equations

$$\begin{aligned} &\langle \nabla h_l, \partial/\partial \mathrm{x} \rangle\, a - \langle a, \partial/\partial \mathrm{x} \rangle\, \nabla h_l = 0, \\ &\langle \partial/\partial \mathrm{x}, \nabla h_l \rangle\, l + \langle l, (\partial/\partial \mathrm{x} \nabla h_l) \rangle - \langle \partial/\partial \mathrm{x}, a \rangle\, \nabla h_a - \langle a, (\partial/\partial \mathrm{x} \nabla h_a) \rangle = 0. \end{aligned} \tag{41}$$

For the Casimir functional $h \in D(\bar{\mathcal{G}}^*)$ the equations (41) should be solved analytically. In the case when an element $\bar{l} \ltimes \bar{a} \in \bar{\mathcal{G}}^*$ is singular as $|\lambda| \to \infty$, one can consider the general asymptotic expansion

$$\nabla h^{(p)}(l,a) \sim \lambda^p \sum_{j \in \mathbb{Z}_+} (\nabla h^{(p)}_{l,j}; \nabla h^{(p)}_{a,j}) \lambda^{-j} \tag{42}$$

for some suitably chosen $p \in \mathbb{Z}_+$, which is substituted into the equations (41). The latter is then solved recurrently giving rise to a set of gradient expressions for the Casimir functionals $h^{(p)} \in D(\bar{\mathcal{G}}^*)$ at the specially found integers $p \in \mathbb{Z}_+$.

Assume now that $h^{(y)}, h^{(t)} \in I(\bar{\mathcal{G}}^*)$ are such Casimir functionals for which the Hamiltonian vector field generators

$$\nabla h^{(y)}(\bar{l},\bar{a})_+ := (\nabla h^{(p_y)}(\bar{l},\bar{a}))_+, \qquad \nabla h^{(t)}(\bar{l},\bar{a})_+ := (\nabla h^{(p_t)}(\bar{l},\bar{a}))_+, \tag{43}$$

where $\nabla h^{(y)}(\bar{l},\bar{a})_+ := (\nabla h^{(y)}_{\bar{l},+} \ltimes \nabla h^{(y)}_{\bar{a},+}) \in \bar{\mathcal{G}}_+$ and $\nabla h^{(t)}(\bar{l},\bar{a})_+ := (\nabla h^{(t)}_{\bar{l},+} \ltimes \nabla h^{(t)}_{\bar{a},+}) \in \bar{\mathcal{G}}_+$, are, respectively, defined at some specially found integers $p_y, p_t \in \mathbb{Z}_+$. These invariants generate owing to the Lie-Poisson bracket (39) the following commuting to each other Hamiltonian flows:

$$\begin{aligned} \frac{\partial}{\partial y}(\bar{a} \ltimes \bar{l}) &= -ad^*_{\nabla h^{(y)}(\bar{l},\bar{a})_+}(\bar{a} \ltimes \bar{l}), \\ \frac{\partial}{\partial t}(\bar{a} \ltimes \bar{l}) &= -ad^*_{\nabla h^{(t)}(\bar{l},\bar{a})_+}(\bar{a} \ltimes \bar{l}) \end{aligned}$$

of an element $\bar{a} \ltimes \bar{l} \in \bar{\mathcal{G}}^* \simeq \bar{\mathcal{G}}$ with respect to the corresponding evolution parameters $t, y \in R$. The flows (43) can be rewritten as

$$\begin{aligned} \partial a/\partial y &= -\left\langle \nabla h_l^{(p_y)}, \frac{\partial}{\partial \mathrm{x}} \right\rangle a + \left\langle a, \frac{\partial}{\partial \mathrm{x}} \right\rangle \nabla h_l^{(p_y)}, \\ \partial a/\partial t &= -\left\langle \nabla h_l^{(p_t)}, \frac{\partial}{\partial \mathrm{x}} \right\rangle a + \left\langle a, \frac{\partial}{\partial \mathrm{x}} \right\rangle \nabla h_l^{(p_t)}, \end{aligned} \tag{44}$$

and

$$\partial l/\partial y = -\left\langle \frac{\partial}{\partial x}, \nabla h_l^{(p_y)} \right\rangle l - \left\langle l, (\frac{\partial}{\partial x}\nabla h_l^{(p_y)}) \right\rangle + \left\langle \frac{\partial}{\partial x}, a \right\rangle \nabla h_a^{(p_y)} + \left\langle a, (\frac{\partial}{\partial x}\nabla h_a^{(p_y)}) \right\rangle,$$
$$\partial l/\partial t = -\left\langle \frac{\partial}{\partial x}, \nabla h_l^{(p_t)} \right\rangle l - \left\langle l, (\frac{\partial}{\partial x}\nabla h_l^{(p_t)}) \right\rangle + \left\langle \frac{\partial}{\partial x}, a \right\rangle \nabla h_a^{(p_t)} + \left\langle a, (\frac{\partial}{\partial x}\nabla h_a^{(p_t)}) \right\rangle,$$

where $y, t \in \mathbb{R}$ are the corresponding evolution parameters. Since the invariants $h^{(y)}, h^{(t)} \in I(\bar{\mathcal{G}}^*)$ are commuting to each other with respect to the Lie-Poisson bracket (39), the flows (44) are commuting too. This is equivalent that the following equalities

$$[\nabla h_{\bar{l},+}^{(y)}, \nabla h_{\bar{l},+}^{(t)}] - \frac{\partial}{\partial t}\nabla h_{\bar{l},+}^{(y)} + \frac{\partial}{\partial y}\nabla h_{\bar{l},+}^{(t)} = 0, \tag{45}$$

and

$$ad_{\bar{a}}^* \bar{P} = 0,$$
$$\bar{P} = ad^*_{\nabla h_{\bar{l},+}^{(y)}}(\nabla h_{\bar{a},+}^{(t)}) - ad^*_{\nabla h_{\bar{l},+}^{(t)}}(\nabla h_{\bar{a},+}^{(y)}) - \frac{\partial}{\partial t}\nabla h_{\bar{a},+}^{(y)} + \frac{\partial}{\partial y}\nabla h_{\bar{a},+}^{(t)}$$

hold for any $\bar{a} \ltimes \bar{l} \in \bar{\mathcal{G}}$. On the other hand, the equation (45) is equivalent to the compatibility condition of three linear equations

$$\frac{\partial \psi}{\partial y} + \nabla h_{\bar{l},+}^{(y)}\psi = 0, \qquad \langle a, \partial/\partial x\rangle \psi = 0, \qquad \frac{\partial \psi}{\partial t} + \nabla h_{\bar{l},+}^{(t)}\psi = 0 \tag{46}$$

for a function $\psi \in C^2(\mathbb{R}^2 \times \mathbb{C} \times \mathbb{T}^n; \mathbb{C})$, all $y, t \in \mathbb{R}$ and any $x \in \mathbb{T}^n$. The obtained above results can be formulated as the following proposition.

Proposition 6. *Let a seed element $\bar{a} \ltimes \bar{l} \in \bar{\mathcal{G}}^*$ and $h^{(y)}, h^{(t)} \in I(\bar{\mathcal{G}}^*)$ are some Casimir functionals subject to the product $(\cdot|\cdot)$ on the holomorphic Lie algebra $\bar{\mathcal{G}}$ and the natural coadjoint action on the co-algebra $\bar{\mathcal{G}}^* \simeq \bar{\mathcal{G}}$. Then the following dynamical systems*

$$\frac{\partial}{\partial y}(\bar{a} \ltimes \bar{l}) = -ad^*_{\nabla h^{(y)}(\bar{l},\bar{a})_+}(\bar{a} \ltimes \bar{l}), \qquad \frac{\partial}{\partial t}(\bar{a} \ltimes \bar{l}) = -ad^*_{\nabla h^{(t)}(\bar{l},\bar{a})_+}(\bar{a} \ltimes \bar{l})$$

are commuting to each other Hamiltonian flows for evolution parameters $y, t \in R$. Moreover, the compatibility condition of these flows leads to the vector field representation (46).

Remark 2. *As it was mentioned above, the expansion* (42) *is effective if a chosen seed element $\bar{a} \ltimes \bar{l} \in \bar{\mathcal{G}}^*$ is singular as $|\lambda| \to \infty$. In the case when it is singular as $|\lambda| \to 0$, the expression* (42) *should be respectively replaced by the expansion*

$$\nabla h^{(p)}(\bar{l}, \bar{a}) \sim \lambda^{-p} \sum_{j \in \mathbb{Z}_+} \nabla h_j^{(p)}(\bar{l}, \bar{a})\lambda^j$$

for suitably chosen integers $p \in \mathbb{Z}_+$, and the reduced Casimir function gradients then are given by the Hamiltonian vector field generators

$$\nabla h^{(y)}(\bar{l}, \bar{a})_- := \lambda(\lambda^{-p_y-1}\nabla h^{(p_y)}(\bar{l}, \bar{a}))_-, \qquad \nabla h^{(t)}(\bar{l}, \bar{a})_- := \lambda(\lambda^{-p_t-1}\nabla h^{(p_t)}(\bar{l}, \bar{a}))_-$$

for suitably chosen positive integers $p_y, p_t \in \mathbb{Z}_+$ and the corresponding Hamiltonian flows are, respectively, written as

$$\frac{\partial}{\partial t}(\bar{a} \ltimes \bar{l}) = ad^*_{\nabla h^{(t)}(\bar{l},\bar{a})_-}(\bar{a} \ltimes \bar{l}), \quad \frac{\partial}{\partial y}(\bar{a} \ltimes \bar{l}) = ad^*_{\nabla h^{(y)}(\bar{l},\bar{a})_-}(\bar{a} \ltimes \bar{l})$$

for evolution parameters $y, t \in R$.

As in Section 3 the presented above construction of Hamiltonian flows on the adjoint space $\bar{\mathcal{G}}^*$ can be generalized proceeding to the point product $\bar{\mathcal{G}}^{\mathbb{S}^1} := \prod_{z\in\mathbb{S}^1} \bar{\mathcal{G}}$ of the holomorphic Lie algebra $\bar{\mathcal{G}}$ endowed with the central extension, generated by a two-cocycle $\omega_2 : \bar{\mathcal{G}} \times \bar{\mathcal{G}} \to \mathbb{C}$, where

$$\omega_2(\bar{a}_1 \ltimes \bar{l}_1, \bar{a}_2 \ltimes \bar{l}_2) := \int_{\mathbb{S}^1} [(\bar{l}_1, \partial\bar{a}_2/\partial z)_1 - (\bar{l}_2, \partial\bar{a}_1/\partial z)_1]$$

for any pair of elements $\bar{a}_1 \ltimes \bar{l}_1, \bar{a}_2 \ltimes \bar{l}_2 \in \bar{\mathcal{G}}$. The resulting R-deformed Lie-Poisson bracket (18) for any smooth functionals $h, f \in D(\bar{\mathcal{G}}^*)$ on the adjoint space $\bar{\mathcal{G}}^*$ to the centrally extended loop Lie algebra $\bar{\mathcal{G}} := \bar{\mathcal{G}} \oplus \mathbb{C}$ becomes equal to

$$\begin{aligned}\{h, f\}_{\mathcal{R}} := {} & (\bar{a} \ltimes \bar{l}, [\nabla h(\bar{l}, \bar{a}), \nabla f(\bar{l}, \bar{a})]_{\mathcal{R}}) \\ & + \omega_2(\mathcal{R}\nabla h(\bar{l}, \bar{a}), \nabla f(\bar{l}, \bar{a})) + \omega_2(\nabla h(\bar{l}, \bar{a}), \mathcal{R}\nabla f(\bar{l}, \bar{a})).\end{aligned} \tag{47}$$

The corresponding Casimir functionals $h^{(p)} \in I(\bar{\mathcal{G}}^*)$ for specially chosen $p \in \mathbb{Z}_+$, are defined with respect to the standard Lie-Poisson bracket as

$$\{h^{(p)}, f\} := (\bar{a} \ltimes \bar{l}, [\nabla h^{(p)}(\bar{l}, \bar{a}), \nabla f(\bar{l}, \bar{a})]) + \omega_2(\nabla h^{(p)}(\bar{l}, \bar{a}), \nabla f(\bar{l}, \bar{a})) = 0$$

for all smooth functionals $f \in D(\bar{\mathcal{G}}^*)$. Based on the equality (21) one easily finds that the gradients $\nabla h^{(p)} \in \bar{\mathcal{G}}$ of the Casimir functionals $h^{(p)} \in I(\bar{\mathcal{G}}^*), p \in \mathbb{Z}_+$, satisfy the following equations:

$$[\nabla h_{\bar{l}}, \bar{a}] - \frac{\partial}{\partial z}\nabla h_{\bar{l}} = 0, \quad ad^*_{\nabla h_{\bar{l}}}\bar{l} - ad^*_{\bar{a}}\nabla h_{\bar{a}} - \frac{\partial}{\partial z}\nabla h_{\bar{a}} = 0$$

for a chosen element $\bar{a} \ltimes \bar{l} \in \bar{\mathcal{G}}^*$. Making use of the suitable Casimir functionals $h^{(y)}, h^{(t)} \in I(\bar{\mathcal{G}}^*)$, one can construct, making use of (47), the following commuting Hamiltonian flows on the adjoint space $\bar{\mathcal{G}}^*$:

$$\frac{\partial}{\partial y}(\bar{a} \ltimes \bar{l}) = \{\bar{a} \ltimes \bar{l}, h^{(y)}\}_{\mathcal{R}}, \qquad \frac{\partial}{\partial t}(\bar{a} \ltimes \bar{l}) = \{\bar{a} \ltimes \bar{l}, h^{(t)}\}_{\mathcal{R}}, \tag{48}$$

which are equivalent to the evolution equations

$$\frac{\partial}{\partial y}\bar{a} = -[\nabla h^{(y)}_{\bar{l},+}, \bar{a}] + \frac{\partial}{\partial z}\nabla h^{(y)}_{\bar{l},+}, \qquad \frac{\partial}{\partial t}\bar{a} = -[\nabla h^{(t)}_{\bar{l},+}, \bar{a}] + \frac{\partial}{\partial z}\nabla h^{(t)}_{\bar{l},+} \tag{49}$$

and

$$\begin{aligned}\frac{\partial}{\partial y}\bar{l} &= -ad^*_{\nabla h^{(y)}_{\bar{l},+}}\bar{l} + ad^*_{\bar{a}}(\nabla h^{(y)}_{\bar{a},+}) + \frac{\partial}{\partial z}\nabla h^{(y)}_{\bar{a},+}, \\ \frac{\partial}{\partial t}\bar{l} &= -ad^*_{\nabla h^{(t)}_{\bar{l},+}}\bar{l} + ad^*_{\bar{a}}(\nabla h^{(t)}_{\bar{a},+}) + \frac{\partial}{\partial z}\nabla h^{(t)}_{\bar{a},+}.\end{aligned} \tag{50}$$

The commutativity condition for these flows is split into two equations

$$[\nabla h^{(y)}_{\bar{l},+}, \nabla h^{(t)}_{\bar{l},+}] - \frac{\partial}{\partial t}\nabla h^{(y)}_{\bar{l},+} + \frac{\partial}{\partial y}\nabla h^{(t)}_{\bar{l},+} = 0, \tag{51}$$

and

$$\frac{\partial \bar{P}}{\partial z} + ad^*_{\bar{a}}\bar{P} = 0,$$

$$\bar{P} = ad^*_{\nabla h^{(y)}_{\bar{l},+}}(\nabla h^{(t)}_{\bar{a},+}) - ad^*_{\nabla h^{(t)}_{\bar{l},+}}(\nabla h^{(y)}_{\bar{a},+}) - \frac{\partial}{\partial t}\nabla h^{(y)}_{\bar{a},+} + \frac{\partial}{\partial y}\nabla h^{(t)}_{\bar{a},+}$$

for any $\bar{a} \ltimes \bar{l} \in \bar{G}$. The obtained above results one can be formulated as the following proposition.

Proposition 7. *The Hamiltonian flows* (48) *on the adjoint space* $\bar{\mathcal{G}}^*$ *generate the separately commuting evolution equations* (49) *and* (50). *The evolution equations* (49) *give rise to the Lax type compatibility condition* (51), *being equivalent to some system of nonlinear heavenly type equations in partial derivatives. Moreover, the system of evolution equations* (49) *can be considered as the compatibility condition for the following set of linear vector equations*

$$\frac{\partial\psi}{\partial y}+\nabla h^{(y)}_{\tilde{l},+}\psi=0,\quad \frac{\partial\psi}{\partial z}+\langle a,\partial/\partial x\rangle\psi=0,\quad \frac{\partial\psi}{\partial t}+\nabla h^{(t)}_{\tilde{l},+}\psi=0$$

for all $(y,t,z;x)\in(\mathbb{R}^2\times\mathbb{S}^1)\times\mathbb{T}^n$ *and a function* $\psi\in C^2((\mathbb{R}^2\times\mathbb{C}\times\mathbb{S}^1)\times\mathbb{T}^n;\mathbb{C})$.

4.1 Example: the generalized Mikhalev-Pavlov heavenly type system

Let a seed element $\bar{a}\ltimes\bar{l}\in\bar{\mathcal{G}}^*$ be chosen as

$$\bar{a}\ltimes\bar{l}=((u_x-\lambda)\partial/\partial x+v_x\partial/\partial\lambda)\ltimes(w_x dx+\eta_x d\lambda), \tag{52}$$

where $u,v,w,\eta\in C^2(\mathbb{R}^2\times(\mathbb{S}^1\times\mathbb{T}^1);\mathbb{R})$. The asymptotic splits for the components of the gradients of the corresponding Casimir functionals $h^{(p)}\in I(\bar{\mathcal{G}}^*)$, $p\in\mathbb{Z}_+$, as $|\lambda|\to\infty$ have the following forms:

$$\nabla h_{\tilde{l}}\simeq\lambda^p\left(\begin{array}{c}1-u_x\lambda^{-1}+(-u_z+(p-1)v)\lambda^{-2}+(u_y+(p-2)(-u_xv+\varkappa))\lambda^{-3}+\dots\\ -v_x\lambda^{-1}-v_z\lambda^{-2}+(v_y-(p-2)v_xv)\lambda^{-3}+\dots\end{array}\right),$$

$$\nabla h_{\tilde{a}}\simeq\lambda^p\left(\begin{array}{c}-w_x\lambda^{-1}-w_z\lambda^{-2}+(w_y-(p-2)(wv)_x)\lambda^{-3}+\dots\\ -\eta_x\lambda^{-1}-(\eta_z+(p-1)w)\lambda^{-2}+(\eta_y-(p-2)(-u_xw+v\eta_x+\omega))\lambda^{-3}+\dots\end{array}\right),$$

where $p\in\mathbb{Z}_+$ and

$$\varkappa_x=v_z+u_xv_x,\qquad \omega_x=w_z-u_xw_x-v_x\eta_x. \tag{53}$$

In the case when

$$\nabla h^{(y)}_{\tilde{l},+}:=\left(\begin{array}{c}\lambda^2-u_x\lambda+(-u_z+v)\\ -v_x\lambda-v_z\end{array}\right),$$
$$\nabla h^{(y)}_{\tilde{a},+}:=\left(\begin{array}{c}-w_x\lambda-w_z\\ -\eta_x\lambda-(\eta_z+w)\end{array}\right),$$

and

$$\nabla h^{(t)}_{\tilde{l},+}:=\left(\begin{array}{c}\lambda^3-u_x\lambda^2+(-u_z+2v)\lambda+(u_y-u_xv+\varkappa)\\ -v_x\lambda^2-v_z\lambda+(v_y-v_xv)\end{array}\right),$$
$$\nabla h^{(t)}_{\tilde{a},+}:=\left(\begin{array}{c}-w_x\lambda^2-w_z\lambda+(w_y-(wv)_x)\\ -\eta_x\lambda^2-(\eta_z+2w)\lambda+(\eta_y+u_xw-v\eta_x-\omega)\end{array}\right),$$

the compatibility condition of the Hamiltonian vector flows (48) leads to the system of evolution equations:

$$
\begin{aligned}
u_{zt} + u_{yy} &= -u_y u_{zx} + u_z u_{xy} - u_{xy} v - u_{zz} v - \varkappa u_{xz}, \\
v_{zt} + v_{yy} &= v v_x^2 - v_z^2 - v v_{xy} - v v_{zz} - u_y v_{xz} + u_z v_{xy} - u_z v_x^2 - \varkappa v_{xz}, \\
-u_{xy} - u_{zz} &= u_x u_{xz} - u_z u_{xx} + u_{xx} v, \\
-v_{xy} - v_{zz} &= v_x^2 + v_{xx} v + u_x v_{xz} - u_z v_{xx}, \\
-u_{xt} + u_{yz} &= -u_x u_{xy} + u_y u_{xx} + u_{xz} v + u_{xx} \varkappa, \\
-v_{xt} + v_{yz} &= -u_x v_{xy} + u_y v_{xx} + u_x v_x^2 + v_{xz} v + \varkappa v_{xx} + 2 v_x v_z.
\end{aligned} \tag{54}
$$

Under the constraint $v = 0$ one obtains the set of equations (29)–(31). Thus, the following proposition holds.

Proposition 8. *The constructed system of heavenly type equations* (54) *and* (53) *has the Lax-Sato vector field representation* (51) *with the "spectral" parameter $\lambda \in \mathbb{C}$, which is related with element $\bar{a} \ltimes \bar{l} \in \bar{\mathcal{G}}^*$ in the form* (52).

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Гентош О.Є., Балінський О.А., Прикарпатський А.К. *Узагальнені центрально розширені Лі-алгебраїчні структури та асоційовані інтегровні рівняння небесного типу* // Карпатські матем. публ. — 2020. — Т.12, №1. — С. 242–264.

Вивчаються центрально розширені Лі-алгебраїчні структури та аоційовані інтегровні рівняння небесного типу як потоків на орбітах коприєднаної дії півпрямої суми алгебри векторних полів на торі та її спряженого простору. Показано, що ц потоки породжують сумісні векторні поля типу Лакса-Сато, з якими тісно пов'язана нескінченна ієрархія законів збереження, породжених відповідними інваріантами Казіміра. Наводено типові приклади таких рівнянь і детально продемонстрована їх інтегровність в межах запропоновоної схеми. Як приклади ми отримали та описали нові багатовимірні інтегровні узагальнення бездисперсійних рівнянь Михальова-Павлова та Алонсо-Шабата, для котрих генераторні елементи мають особливу факторизовану структуру, що дозволяє поширити їх на випадок довільного виміру.

Ключові слова і фрази: рівняння небесного типу, інтегровність за Лаксом, динамічна система Гамільтона, дифеоморфізми тора, алгебра Лі петель, центральне розширення, Лі-алгебраїчна схема, інваріанти Казіміра, структура Лі-Пуассона, *R*-структура, рівняння Міхальова-Павлова.